ON THEOREMS OF YANG AND SCHWICK

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Abstract. Let $D$ be a plane domain, $\psi \not\equiv 0$ a meromorphic function on $D$, and $k$ a fixed positive integer. Let $\mathcal{F}$ be a collection of functions meromorphic on $D$, none of which have poles in common with $\psi$. According to a result of Schwick [6] (cf. Yang [9]), if each $f \in \mathcal{F}$ satisfies $f(z) \neq 0$ and $f^{(k)}(z) \neq \psi(z)$ for $z \in D$, then $\mathcal{F}$ is a normal family. We give a very simple proof of this result, based on applying a suitable refinement of Zalcman’s Lemma.

1. Introduction.

The goal of this paper is to give a simple proof of the following result.

Theorem 1. Let $\mathcal{F}$ be a family of functions meromorphic on a plane domain $D$ and $k$ a positive integer. Let $\psi$ be a function meromorphic on $D$ such that the following conditions hold:

(a) $\psi \not\equiv 0$;
(b) $f(z) \neq 0$, $z \in D$, $f \in \mathcal{F}$;
(c) $f^{(k)} \neq \psi(z)$, $z \in D$, $f \in \mathcal{F}$;
(d) no $f \in \mathcal{F}$ has poles in common with $\psi$ in $D$.

Then $\mathcal{F}$ is a normal family on $D$.

This theorem was first proved by Yang Lo [9] (cf. [7]) for $\psi$ analytic and later generalized by Schwick [6] to meromorphic $\psi$. Their proofs are based on Nevanlinna theory.

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2. Notation.

We shall write $\chi$ for the spherical (chordal) metric on the extended complex plane $\hat{\mathbb{C}}$ and for $z_0 \in \mathbb{C}$, $r > 0$ set $\Delta(z_0, r) = \{ z : |z - z_0| < r \}$, $\overline{\Delta}(z_0, r) = \{ z : |z - z_0| \leq r \}$, $\Delta'(z_0, r) = \{ z : 0 < |z - z_0| < r \}$.

Let $\{ g_n \}$ be a sequence of meromorphic functions on a domain $D \subset \mathbb{C}$. If $\{ g_n \}$ converges uniformly on compact subsets of $D$ to $g$ (where $g$ is a meromorphic function on $D$ or the constant $\infty$) with respect to the spherical metric $\chi$ on $\hat{\mathbb{C}}$, we say that $\{ g_n \}$ converges to $g$ locally $\chi$-uniformly on $D$ and write $g_n \chi \Rightarrow g$ on $D$.

In case the functions $g_n$ are holomorphic in $D$, then either the convergence is locally uniform with respect to the Euclidean metric, in which case the limit function $g$ is holomorphic on $D$, or $\{ g_n \}$ diverges uniformly to $\infty$ on compacta. In this case, we write $g_n \Rightarrow g$ on $D$ or $g_n \Rightarrow \infty$ on $D$, respectively.

Likewise, if $A \subset D$ and $\{ g_n \}$ converges uniformly with respect to $\chi$ to $g$ (which may be $\infty$) on $A$, we shall write $g_n \chi \rightarrow g$ on $A$ for meromorphic functions, $g_n$, $n \geq 1$; and $g_n \rightarrow g$ on $A$ in the case of holomorphic functions.

3. Preliminary results.

We shall use the following sharpening of Zalcman’s Lemma [10], [3, p.74] due to Chen and Gu [2] (cf. [11]).

**Lemma 1.** Let $k$ be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions on a domain $D$ with the property that each function in $\mathcal{F}$ has only zeros of degree at least $k$. If $\mathcal{F}$ is not normal at the point $z_0 \in D$, then for each $\alpha$ such that $0 \leq \alpha < k$ there exist

1. points $z_n \in D$, $z_n \rightarrow z_0$;
2. numbers $\rho_n \rightarrow 0^+$, and
3. functions $f_n \in \mathcal{F}$,
such that

$$
\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on} \quad \mathbb{C},
$$

where $g$ is a nonconstant meromorphic function on $\mathbb{C}$.

In particular, if the function in $\mathcal{F}$ do not vanish on $D$ (so that the multiplicity condition is satisfied vacuously for every $\ell$), one may choose $\alpha \geq 0$ arbitrarily so that

(1) $$
\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on} \quad \mathbb{C}
$$

will be satisfied for appropriate $\{f_n\}, \{\rho_n\}, \text{and} \{z_n\}$.

Remark. Our reference to the result of Chen and Gu is more a matter of convenience than necessity. In our application, the functions on $\mathcal{F}$ do not vanish; thus, it would suffice to invoke the (earlier and simpler) result of Xue and Pang [8] (suitably localized to require $z_n \to z_0$) instead. However, as this reference is relatively inaccessible in the West, we have chosen to cite the result of [2] instead.

For a self-contained, elementary proof of (the nonlocal version of) Lemma 1, see [5].

Lemma 2. Let $S = \{f_n\}$ be a sequence of meromorphic functions in a domain $D$, and assume that $f_n \overset{\chi}{\Rightarrow} f$ on $D$, where $f$ is a meromorphic function on $D$. Let $E$ be a compact subset of $D$. Assume that $f$ is finite on $E$. Then there exists a positive integer $N$ such that $f_n(z)$ is finite on $E$ for $n \geq N$, and $f_n \overset{n \to \infty}{\to} f$ on $E$.

Lemma 3. Let $\{f_n\}$ be a sequence of meromorphic functions in a domain $D$ which converges there locally $\chi$-uniformly, and let $\{a_n\}$ be a sequence of holomorphic functions in $D$, converging locally uniformly to a holomorphic function in $D$. Then the sequence of functions $g_n = f_n + a_n \ (n = 1, 2, \ldots) \text{ converges locally }\chi\text{-uniformly on } D$.

The proofs are almost obvious; for details, see [3].

Finally, we need the following important result due to W.K. Hayman.
Theorem H. Suppose that $f$ is meromorphic and transcendental in the plane and $\ell \in \mathbb{N}$. Then either $f(z)$ assumes every finite value infinitely often, or $f^{(\ell)}(z)$ assumes every finite value except possibly zero infinitely often.

3. Proof of Theorem 1 and an extension.

Proof of Theorem 1. Normality is a local property; hence it is enough to show that $\mathcal{F}$ is normal at each $z_0 \in D$. We distinguish two cases.

Case (1). $\psi(z_0) \neq 0, \infty$. Suppose, to the contrary, that $\mathcal{F}$ is not normal at $z_0 \in D$. Then condition (b) and (1) (Lemma 1 is applicable) imply the existence of sequences $\rho_n \to 0^+, z_n \to z_0, \{f_n\}, f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \xrightarrow{\mathcal{C}} g(\zeta)$$

on $\mathbb{C}$, where $g$ is a nonconstant meromorphic function in $\mathbb{C}$. By condition (b) and Hurwitz’s Theorem,

$$g(\zeta) \neq 0, \quad \zeta \in \mathbb{C}. \tag{2}$$

For every compact subset $K$ of $\mathbb{C}$ which contains no pole of $g$, one has $g_n^{(k)}(\zeta) \to g^{(k)}(\zeta)$ in $K$, so that

$$f_n^{(k)}(z_n + \rho_n \zeta) \to g^{(k)}(\zeta) \quad \text{on} \quad K. \tag{3}$$

Since $\psi(z)$ is holomorphic at $z_0$ and $z_n + \rho_n \zeta \to z_0$ on $\mathbb{C}$, (3) gives rise to

$$f_n^{(k)}(z_n + \rho_n \zeta) - \psi(z_n + \rho_n \zeta) \to g^{(k)}(\zeta) - \psi(z_0) \quad \text{in} \quad K.$$

Thus, by condition (c) and Hurwitz’s theorem, we have for any $\zeta \in \mathbb{C}$ that is not a pole of $g$

$$g^{(k)}(\zeta) \neq \psi(z_0). \tag{4}$$
(The second alternative, \( g^{(k)}(\zeta) \equiv \psi(z_0) \) for any \( \zeta \) which is not a pole of \( g \), would force \( g \) to be a polynomial, violating (2)).

Since (4) certainly holds for any pole of \( g \), we have (4) for each \( \zeta \in \mathbb{C} \). Thus, together with (2) and Theorem H, we deduce that \( g \) is constant, a contradiction. So \( \mathcal{F} \) must be normal at \( z_0 \).

**Case (2).** \( \psi(z_0) = 0 \) or \( \psi(z_0) = \infty \). According to condition (a), there exists \( r > 0 \) such that \( \psi(z) \not\equiv 0, \infty \) in \( \Delta'(z_0, r) \); and by Case (1) \( \mathcal{F} \) is normal there. Now if \( S = \{ f_n \} \) is a sequence of functions of \( \mathcal{F} \), then \( S \) has a subsequence (which, without loss of generality, we may take to be \( S \) itself) with \( f_n \Rightarrow h \) on \( \Delta'(z_0, r) \).

Here \( h \) is a meromorphic function in \( \Delta'(z_0, r) \) or the constant \( \infty \). We consider three possibilities.

- **(A)** \( h \equiv \infty \). Then for each \( 0 < r' < r \), \( \frac{1}{f_n(z)} \rightarrow 0 \) on \( \{|z - z_0| = r'\} \); and since the functions \( \frac{1}{f_n} \) are holomorphic, we get by the maximum principle that \( \frac{1}{f_n} \rightarrow 0 \) on \( \Delta(z_0, r') \). Since this is true for any \( 0 < r' < r \), we conclude that \( f_n \Rightarrow \infty \) on \( \Delta(z_0, r) \), and \( \{ f_n \} \) is normal at \( z_0 \).

- **(B)** \( h \not\equiv 0, \infty \). In this case, Hurwitz’s Theorem implies that \( h(z) \not\equiv 0 \) in \( \Delta'(z_0, r) \); hence for each \( 0 < r' < r \), \( \frac{1}{f_n} \Rightarrow \frac{1}{h} \) on \( \Delta'(z_0, r) \). Set \( M = \max_{|z - z_0| = r'} \frac{1}{|h(z)|} > 0 \).

Then for large enough \( n \), \( \frac{1}{|f_n(z)|} \leq M + 1 \) for \( z \in \{|z - z_0| = r'\} \), and by the maximum principle \( \frac{1}{|f_n|} \leq M + 1 \) in \( \Delta(z_0, r') \). Hence \( \frac{1}{h} \) is holomorphic and bounded in \( \Delta'(z_0, r') \). Thus, \( \frac{1}{h} \) extends to be holomorphic in \( \Delta(z_0, r') \); and, again by the maximum principle, it follows that \( \left| \frac{1}{f_n} - \frac{1}{h} \right| \rightarrow 0 \) on \( \Delta(z_0, r') \). Thus \( f_n \Rightarrow h \) on \( \Delta(z_0, r) \).

- **(C)** \( h \equiv 0 \). Here we borrow an idea from Yang [7]. For a meromorphic function \( f \) defined in a domain \( D \), denote by \( n(f, w_0, R) \) the number of poles (counting multiplicity) of \( f \) in \( \Delta(w_0, R) \subset D \). By Lemma 2, for sufficiently large \( n \), \( f_n \) is holomorphic in \( E = \{ \frac{r}{4} \leq |z - z_0| \leq \frac{3}{4}r \} \). Since \( f_n \rightarrow 0 \) on \( E \), we get that \( f_n^{(k)} \rightarrow 0 \) on \( E \). Now there exists \( C > 0 \) such that \( |\psi(z)| \geq C, z \in E \). Thus \( f_n^{(k)} \psi \rightarrow 0 \) there,
and so \( \left[ \frac{f^{(k)}}{\psi} \right]' \rightarrow 0 \) on \( E \).

By conditions (c) and (d), \( \frac{f^{(k)}}{\psi}(z) - 1 \neq 0 \) (in \( D \)), whence \( n \left( \frac{1}{\frac{f^{(k)}}{\psi} - 1}, z_0, \frac{r}{2} \right) = 0 \). Then by the argument principle

\[
(5) \quad \left| n \left( \frac{f^{(k)}}{\psi} - 1, z_0, \frac{r}{2} \right) \right| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=\frac{r}{2}} \frac{\left( \frac{f^{(k)}}{\psi} - 1 \right)'(z)}{\frac{f^{(k)}}{\psi}(z) - 1} \, dz \right| \rightarrow 0, \quad n \rightarrow \infty
\]

(since the denominator of the integrand converges uniformly to \(-1\) on \( \{ |z-z_0|=\frac{r}{2} \} \), and the numerator tends uniformly to \( 0 \) there). The left-hand side of (5) is an integer, which implies that \( n \left( \frac{f^{(k)}}{\psi} - 1, z_0, \frac{r}{2} \right) = 0 \) for large enough \( n \). Thus \( f^{(k)}_n \) is holomorphic in \( \Delta(z_0, \frac{r}{2}) \) for sufficiently large \( n \), and hence so is \( f_n \). By the maximum principle, it follows that \( f_n \Rightarrow 0 \) on \( \Delta(z_0, \frac{r}{2}) \); and the proof is completed.

It is possible to extend Theorem 1 in the following way.

**Theorem 2.** Let \( F \) be a family of meromorphic functions in a domain \( D \), and let \( k \) be a positive integer. Suppose that \( \varphi \) is a holomorphic function on \( D \) and \( \psi \) is a meromorphic function on \( D \) such that the following conditions hold:

(a') \( \psi \neq \varphi^{(k)} \);
(b') \( f(z) - \varphi(z) \neq 0, \quad z \in D, \quad f \in F \);
(c') \( f^{(k)} \neq \psi(z), \quad z \in D, \quad f \in F \);
(d') For each \( f \in F \), \( f \) and \( \psi \) have no common poles in \( D \).

Then \( F \) is a normal family on \( D \).

Indeed, set \( \psi^* = \psi - \varphi^{(k)} \); then from conditions (a') - (d') one obtains conditions (a) - (d) of Theorem 1 for the family \( F^* = \{ f - \varphi : f \in F \} \). Since \( F \) is normal if and only if \( F^* \) is normal (see Lemma 3), it follows from Theorem 1 that \( F \) is a normal family on \( D \).

**Remark.** Theorem 2 does not hold for meromorphic \( \varphi \). An instructive example is
given by taking $D = \Delta'(0,1)$, $k = 2$, and

$$f_n(z) = nz + \frac{1}{z - \frac{1}{2}}, \quad \varphi(z) = \frac{1}{z - \frac{1}{2}}, \quad \psi \equiv 0.$$ 

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References