Zero Cancellation

To Damir Z. Arov, valued friend and colleague, on the occasion of his 70th birthday

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Abstract

The problem of eliminating the right half plane zeros of an rmvf (rational matrix valued function) \( G(z) \) with minimal realization \( G(z) = D + C(zI_n - A)^{-1}B \) by multiplication on the right by a suitably chosen \( J \)-inner rmvf \( \Theta(z) \) is studied. The analysis exploits the theory of Smith-McMillan forms to extend the method of \( J \)-lossless conjugators that was introduced by Kimura to more general settings.

\textit{Key words:} pole cancellation, zero cancellation, \( J \)-lossless conjugators, \( J \)-inner matrix valued functions, Riccati equations, Smith-McMillan forms, stability

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1 Introduction

In this paper we deal with the problem of cancelling the zeros of a \( p \times m \) rational matrix valued function (rmvf) \( G(z) \) in the open right half plane \( \Pi_+ \), by multiplying it on the right by an \( m \times m \) rmvf \( \Theta(z) \) that is \( J \)-inner with respect to \( \Pi_+ \), where \( J \) is a given fixed \( m \times m \) constant signature matrix. This problem turns out to be equivalent to the problem of cancelling the poles of

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an appropriately constructed rmvf $\tilde{G}(z)$ by multiplication on the right by a $J$-inner rmvf $\tilde{\Theta}(z)$ that is still $J$-inner w.r.t. $\Pi_+$. If $p = m$, then $\tilde{G}(z) = G^{-\#}(z)$ and $\tilde{\Theta}(z) = \Theta^{-\#}(z)$, where $f^\#(z) = f(-\bar{z})^*$ for every rmvf $f(z)$. However, if $p \neq m$, then the situation is more delicate, as will be explained in due course.

The analysis depends heavily upon results that were obtained earlier for the analogous problem of cancelling the poles of an rmvf $G(z)$ in $\Pi_+$. The pole cancellation problem was handled in [7] by $H_\infty$-control techniques, and subsequently in [3] by reproducing kernel Krein (Hilbert) space methods. A more comprehensive study that exploits the notions of the local Smith-McMillan (SM-form) of rmvf and the theory of null-pole structure that is developed in [1] is presented in [4]. (Additional information on the null-pole structure of nonsquare rmvf’s may be found in [9].) The conclusions of [4] that are needed for the present development will be summarized in Section 3, after first reviewing a number of preliminary definitions and facts connected with the definition of poles and zeros by the local SM-form, Kalman’s Theorem, $J$-unitary rmvf’s and Riccati equations, in Section 2. In Section 4.1 we define a minimal zero conjugator in a way that is analogous to the definition of a minimal pole conjugator that is given in Section 3 and discuss some of its features. Section 4.2 develops a necessary and sufficient condition for the existence of a minimal zero conjugator of a full-rank $p \times m$ rmvf when $p \geq m$. In Section 4.3 we find a sufficient condition for the existence of minimal zero conjugators of a full-rank $p \times m$ $G(z)$, $p < m$. Section 4.4 deals with necessary conditions for this case and Section 4.5 reduces the case where $G(z)$ is not of full-rank to the cases dealt with in Sections 4.3 and 4.4. In all cases, the results are obtained by finding either a left or a right inverse to $G(z)$ (or to an rmvf that is obtained from $G(z)$ in Sections 4.4, 4.5), then reducing this rmvf to a proper rmvf $\tilde{G}(z)$ and then finding an appropriate minimal pole conjugator for $\tilde{G}(z)$, using Theorem 3.3 to see when this is possible. The conditions for the existence of conjugators will be formulated in terms of solutions to Riccati equations.

Analogous zero cancellation problems that are formulated for the open unit disc instead of the open half plane $\Pi_+$ will be considered in a separate publication.

2 Preliminaries

2.1 Notations

- A $p \times m$ rmf (matrix valued function) $G(z)$ is said to be an rmvf if all of its entries are rational functions. An rmvf and a meromorphic matrix valued function have the same local behavior. However, we treat only
rmvf’s because the rmvf’s $G(z)$ that are considered in this paper are all rational.

- The $ij$ entry of a matrix $A(z), B(z), \ldots, \hat{A}(z), \hat{B}(z)$ will be denoted by $a_{ij}, b_{ij}(z), \hat{a}_{ij}(z), \hat{b}_{ij}(z)$, etc., with or without the variable $z$.

- The normal-rank of a $p \times m$ rmvf $G(z)$ is equal to the order of the largest square submatrix that is invertible except for at most a finite number of points in $\mathbb{C}$. A $p \times m$ rmvf $G(z)$ is said to be of full-rank if its normal rank is equal to $\min\{p, m\}$. A square rmvf of full rank is said to be regular.

- A rmvf $G(z)$ is called proper or analytic at infinity if $\lim_{z \to \infty} G(z) = G(\infty)$ is finite. It is said to be invertible at infinity if $G(\infty)$ is invertible.

- A $p \times m$ mvf (matrix valued function) $G(z)$ is said to be unimodular if $p = m$, $G(z)$ is holomorphic in $\mathbb{C}$ and $\det G(z)$ is a nonzero constant. The simplest examples of unimodular matrices are holomorphic triangular rmvf’s with constant nonzero entries on the diagonal.

- The symbol $R_i(G)$ stands for the $i$th row of a rmvf $G(z)$, whereas the symbol $C_j(G)$ stands for the $j$th column of $G(z)$. If the dependence on $G(z)$ is clear from the context, then we may drop the dependence on $G(z)$ from the notation and simply write $R_i$ and $C_j$.

2.2 The local Smith-McMillan form, zeros and poles

Definition 2.1 Let $G(z)$ be a $p \times m$ rmvf of normal rank $r$ and let $z_0 \in \mathbb{C}$. Then $G(z)$ admits a factorization of the form

$$G(z) = E(z)\Lambda(z)F(z),$$  

(2.1)

where $E(z)$ and $F(z)$ are rmvf’s of sizes $p \times p$ and $m \times m$, respectively, that are holomorphic and invertible in a neighborhood of $z_0$,

$$\Lambda(z) = \begin{bmatrix} 
D(z) & 0_{r \times (m-r)} \\
0_{(p-r) \times r} & 0_{(p-r) \times (m-r)} 
\end{bmatrix},$$  

(2.2)

$$D(z) = \text{diag}\{ (z - z_0)^{k_1}, \ldots, (z - z_0)^{k_r} \},$$  

(2.3)

and the numbers $k_j$ are integers that are listed in non-decreasing order:

$$k_1 \leq \cdots \leq k_r.$$

The numbers $k_1, \ldots, k_r$ are uniquely determined by $G(z)$. We shall refer to these numbers as the indices of the local SM-form of $G(z)$ at $z_0$ and to $\Lambda(z)$, the central term in the factorization (2.1), as the local SM-form of $G(z)$ at $z_0$. If $G_1(z)$ and $G_2(z)$ are two $p \times m$ mvf’s such that the indices of the local
SM-form of $G_1(z)$ at $z_0$ coincide with the indices of the local SM-form of $G_2(z)$ at $z_0$, then we shall write

$$G_1(z) \equiv G_2(z).$$

If $H_1(z)$ and $H_2(z)$ are rmvfs that are holomorphic and invertible at $z_0$ of sizes $p \times p$ and $m \times m$, respectively, then

$$G(z) \equiv H_1(z)G(z)H_2(z).$$

**Definition 2.2** Given a rmvf $G(z)$ and a point $z_0 \in \mathbb{C}$, the following operations will be called elementary operations:

(a) Interchanging two rows (columns) of $G(z)$, $R_j \leftrightarrow R_k$ ($C_j \leftrightarrow C_k$).

(b) Adding a multiple of one row (column) of $G(z)$ by a function $\alpha(z)$ that is holomorphic at $z_0$ to another row (column) of $G(z)$. This operation will be denoted by $R_j \leftrightarrow R_j + \alpha(z)R_k$ ($C_j \leftrightarrow C_j + \alpha(z)C_k$).

These two operations on rows (columns) are achieved by multiplication on the left (right) of $G(z)$ by either a permutation matrix or by a triangular matrix with ones on the diagonal and hence does not change the local SM-form at $z_0$.

The local SM-form of $G(z)$ at $z_0$ [1, pp.10,69] is obtained by performing these operations in the right order.

**Definition 2.3** Let $f(z)$ be a scalar rational function and, for $z_0 \in \mathbb{C}$, set

$$n(f; z_0) = \max\{\ell : f(z)(z - z_0)^{-\ell} \text{ is analytic at } z_0\},$$

i.e.,

$$n(f; z_0) = \begin{cases} 
  k & \text{if } f \neq 0 \text{ and } z_0 \text{ is a zero of order } k \text{ of } f \\
  -t & \text{if } f \neq 0 \text{ and } z_0 \text{ is a pole of order } t \text{ of } f \\
  0 & \text{if } f \neq 0 \text{ and } z_0 \text{ is neither a pole nor a zero of } f \\
  \infty & \text{if } f \equiv 0.
\end{cases}$$

In other words, $n(f; z)$ is the index of first nonzero entry in the Laurent expansion of $f$ about $z_0$.

**2.3 The Smith-McMillan form**

Let $G(z)$ be a $p \times m$ rmvf with normal rank $r$. Then $G(z)$ admits a factorization of the form

$$G(z) = U(z)\Lambda(z)V(z),$$

(2.4)
where $U(z)$ and $V(z)$ are unimodular mvf’s (i.e., polynomial mvf’s),

$$
\Lambda(z) = \begin{bmatrix}
D(z) & O_{r \times (m-r)} \\
O_{(p-r) \times r} & O_{(p-r) \times (m-r)}
\end{bmatrix},
$$

(2.5)

$$
D(z) = \text{diag}\{\varphi_1(z), \ldots, \varphi_r(z)\}
$$

(2.6)

and

$$
\varphi_i(z) = p_i(z)/q_i(z)
$$

is the ratio of two monic polynomials $p_i(z)$ and $q_i(z)$ that have no common factors, i.e., g.c.d. $(p_i, q_i) = 1$. Moreover, $\varphi_i(z)/\varphi_i(z)$ is a polynomial for $i = 1, \ldots, r$. $D(z)$ is called the Smith-McMillan form of $G(z)$. It is uniquely determined by $G(z)$.

The local SM-form of $G(z)$ at a point $z_0 \in \mathbb{C}$ may be obtained from the SM-form by extracting factors of $z - z_0$ from the $\varphi_j(z)$. Thus, if $\varphi_j(z) = (z - z_0)^{k_j} \psi_j(z)$, $j = 1, \ldots, r$, where $\psi_j(z)$ is holomorphic in a neighborhood of $z_0$ and $\psi_j(z_0) \neq 0$, then the numbers $t_1, \ldots, t_r$ are the indices of the local SM-form of $G(z)$ at $z_0$. In particular, $t_1 \leq \cdots \leq t_r$.

**Definition 2.4** Let $G(z)$ be a $p \times m$ mvf of normal rank $r$ and let $k_1 \leq \cdots \leq k_r$ denote the indices in the local SM-form of $G(z)$ at $z_0$. Then $z_0$ is said to be a zero (pole) of $G(z)$ if at least one of these indices is positive (negative). In this case, $\{k_i : k_i > 0\}$ ($\{-k_i : k_i < 0\}$) are called the zero (pole) multiplicities of $G$ at $z_0$. The total multiplicity of $z_0$ as a zero (pole) of $G(z)$ is defined by the formula

$$
M_\zeta(G; z_0) = \sum_{j=1}^{r} \max(k_j, 0) \quad (M_\pi(G; z_0) = \sum_{j=1}^{r} \max(-k_j, 0)).
$$

If $p = m$ and $G(z)$ is regular, and if $k_1, \ldots, k_p$ are the indices of the local SM-form of $G(z)$ at $z_0$, then $-k_p, \ldots, -k_1$ are the indices of the local SM-form of $G(z)^{-1}$ at $z_0$. In particular,

$$
G(z) \text{ regular } \Rightarrow M_\zeta(G; z_0) = M_\pi(G^{-1}; z_0) \text{ and }
M_\pi(G; z_0) = M_\zeta(G^{-1}; z_0).
$$

We remark that if $A(z)$ and $B(z)$ are mvf’s (not necessarily square) and

$$
C(z) = \begin{bmatrix}
A(z) & 0 \\
0 & B(z)
\end{bmatrix}
$$
is block diagonal, then

\[ M_\zeta(C; z_0) = M_\zeta(A; z_0) + M_\zeta(B; z_0) \]

and

\[ M_\pi(C; z_0) = M_\pi(A; z_0) + M_\pi(B; z_0). \]

A rmvf can have both a zero and a pole at the same point. The classical example is

\[ G(z) = \begin{bmatrix} 1 & (z - z_0)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (z - z_0)^{-1} & 0 \\ 0 & (z - z_0) \end{bmatrix}, \]

since

\[ \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix}. \]

This factorization is achieved by successive applications of the rules \( R_2 \rightarrow R_2 - zR_1 \), \( C_1 \rightarrow C_1 - zC_2 \), followed by a permutation and then a change of sign in one row.

**J-inner rmvfs.** Let \( J \) be an \( m \times m \) signature matrix, i.e., \( J \in \mathbb{C}^{m \times m} \) and \( J = J^* = J^{-1} \). A square rmvf \( \Theta(z) \) is said to be \( J \)-unitary on \( i \mathbb{R} \) if \( \Theta^*(z)J\Theta(z) = J \) for every point of analyticity \( z \in i \mathbb{R} \). Then, by analytic continuation,

\[ \Theta^*(z)J\Theta(z) = J \quad \text{on} \quad \mathbb{C}, \quad (2.7) \]

where

\[ \Theta^*(z) = \Theta(-z)^*. \quad (2.8) \]

If, in addition, \( \Theta^*(z)J\Theta(z) \leq J \) (\( \Theta^*(z)J\Theta(z) \geq J \)) at every point of analyticity in \( \Pi_+ \), then \( \Theta \) is said to be \( J \)-inner (\( -J \)-inner) w.r.t. \( \Pi_+ \). Observe that \( \Theta \) is \( J \)-unitary on \( i \mathbb{R} \) if and only if \( \Theta^{-1} \) is such, and \( \Theta \) is \( J \)-inner w.r.t. \( \Pi_+ \) if and only if \( \Theta^{-1} \) is \( -J \)-inner w.r.t. \( \Pi_+ \).

**Stability and similarity.** We assume a basic knowledge of realization theory. However, for the convenience of the reader, we shall briefly review some of the definitions and results that play a key role in the sequel.

**Definition 2.5** A rmvf \( G(z) \) is said to be **stable** if \( M_\pi(G; \Pi_+) = 0 \); it is said to be **anti-stable** if \( M_\pi(G; \Pi_-) = 0 \).

**Theorem 2.6** Let \( G(z) = C(zI_n - A)^{-1}B + D \).
(a) If \( z_0 \in \mathbb{C} \) is a pole of \( G(z) \), then \( z_0 \in \sigma(A) \). The converse statement is valid if the realization is minimal.

(b) If \( D \) is invertible (and hence \( G(z) \) is square) and \( z_0 \) is a zero of \( G \), then \( z_0 \in \sigma(A - BD^{-1}C) \). In the other direction, if the realization is minimal and \( D \) is invertible and \( z_0 \in \sigma(A - BD^{-1}C) \), then \( z_0 \) is a zero of \( G(z) \).

**Definition 2.7** An \( n \times n \) matrix \( A \) is stable (anti-stable) if \( \sigma(A) \subset \Pi_- (\sigma(A) \subset \Pi_+) \). The pair \((A,B)\) is stabilizable (anti-stabilizable) if there is \( F \) such that \( A + BF \) is stable (anti-stable).

**Kalman’s Theorem** [6]. The following theorem plays a central role in the problem of cancelling poles.

**Theorem 2.8** [6] Let \( G(z) \) be a proper rnvf with minimal realization \( G(z) = C(zI_n - A)^{-1}B + D \). Then the order of \( A \) is equal to \( M_\pi(G; \mathbb{C}) \).

Moreover, for every pole \( z_0 \) of \( G(z) \),

\[
M_\pi(G; z_0) = \text{the algebraic multiplicity of } z_0 \text{ as an eigenvalue of } A.
\]

This justifies the use of the term \( M_{\pi}(G) \) (which is equal to \( M_\pi(G; \mathbb{C}) \)) for the order of the matrix \( A \) in the minimal realization of \( G(z) \).

**Corollary 2.9** If \( H(z) \) is proper and invertible at \( \infty \), then \( M_\pi(H; \mathbb{C}) = M_\zeta(H; \mathbb{C}) \).

**Proof.** This follows from Theorem 2.8 and from the inversion formula for realizations. If \( G(z) = C(zI_n - A)^{-1}B + D \) and \( D \) is invertible, then

\[
G(z)^{-1} = -D^{-1}C(zI_n - (A - BD^{-1}C))^{-1}BD^{-1} + D^{-1}. \quad \Box \quad (2.9)
\]

### 3 Cancelling poles

**Definition 3.1** Let \( G(z) \) be a \( p \times m \) rnvf. A proper \( m \times m \) rnvf \( \Theta(z) \), that is \( J \)-inner w.r.t. \( \Pi_+ \) is called a minimal pole conjugator of \( G(z) \) w.r.t. a subset \( \Omega \) of \( \mathbb{C} \) if

(a) \( M_\pi(G\Theta; \Omega) = 0 \) and
(b) \( M_{\text{deg}}(\Theta) = M_\pi(G; \Omega) \).

In this article, the set \( \Omega \) referred to in the preceding definition will usually be chosen equal to either \( \Pi_+ \) or \( \Pi_- \), or to the closure of one of these sets.
Remark 3.2 If $G(z)$ is a proper rnvf with minimal realization
\[ G(z) = C(zI_n - A)^{-1}B + D \quad \text{and} \quad \sigma(A) \cap i\mathbb{R} = \emptyset, \quad (3.1) \]
then the existence of a conjugator $\Theta(z)$ does not depend on $C$ or on $D$; see Subsection 4.1 of [4]. In other words, $\Theta(z)$ is a minimal pole conjugator of the rnvf $G(z)$ with minimal realization (3.1) w.r.t. $\Omega$ if and only if
\begin{align*}
& (a) \quad M_r ((zI_n - A)^{-1}B\Theta; \Omega) = 0. \\
& (b) \quad \text{Modeg} (\Theta) = \#(\sigma(A) \cap \Omega), \text{ counting multiplicities.} \\
\end{align*}

The main results on pole cancellation are summarized in the following theorem, which is discussed in detail in [4]:

**Theorem 3.3** Let $(A, B)$ be a controllable pair such that
\[ \sigma(A) \cap i\mathbb{R} = \emptyset. \quad (3.2) \]
Then:

1. There exists at most one Hermitian solution $X$ to the Riccati equation
\[ XA + A^*X - XBJB^*X = 0 \quad (3.3) \]
such that $\hat{A} = A - BJB^*X$ is stable (anti-stable). If such a solution exists, then it will be denoted by the symbol $R_{st}(A, B)$ ($R_{ast}(A, B)$).
2. There exists a minimal pole conjugator $\Theta(z)$ of the pair $(A, B)$ w.r.t. $\Pi_+$ ($\Pi_-$) if and only if there exists a positive semidefinite matrix $X = R_{st}(A, B)$ ($X = R_{ast}(A, B)$).

In this case every such $\Theta$ is of the form
\[ \Theta(z) = -JB^*(zI_n + A^*)^{-1}XB + I_m = -JB^*X(zI_n - \hat{A})^{-1}B + I_m \quad (3.4) \]
up to multiplication on the right by a $J$-unitary constant matrix. Moreover, if
\[ G(z) = C(zI_n - A)^{-1}B + D, \]
then
\[ G(z)\Theta(z) = (C - DJB^*X)(zI_n - \hat{A})^{-1}B + D. \quad (3.5) \]

**Lemma 3.4** If $X = R_{st}(A, B)$ ($R_{ast}(A, B)$) and $T$ is any invertible matrix of the same size as $X$, then $T^*XT = R_{st}(T^{-1}AT, T^{-1}B)$ ($R_{ast}(T^{-1}AT, T^{-1}B)$).
4 Cancelling zeros

The principle for cancelling zeros is that if $F(z)$ is a $p \times m$ full-rank rmvf and \( \tilde{F}(z) \) is either a right or a left inverse of $F(z)$ (i.e., $F(z)\tilde{F}(z) = I_p$ if $p \leq m$, $\tilde{F}(z)F(z) = I_m$ if $p \geq m$), then $z_0 \in \mathbb{C}$ is a pole of $\tilde{F}(z)$ if $z_0$ is a zero of $F(z)$. Roughly speaking, this principle enables us to obtain a conjugator for zeros from a conjugator of poles.

4.1 Minimal J-inner stabilizing conjugator

Definition 4.1 Let $G(z)$ be a $p \times m$ rmvf. A proper $J$-inner $m \times m$ rmvf $\Theta(z)$ is said to be a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$ if

(a) $M_\epsilon(G; \Pi_+) = 0$
(b) $\text{Mdeg}(\Theta) = M_\epsilon(G; \Pi_+)$.

If $G(z)$ has no zeros on $i\mathbb{R}$, then $\Pi_+$ can be replaced by $\Pi_-$ in (a) and (b) of the preceding definition.

A pole can be easily located since $z_0$ is a pole of $G(z)$ if and only if $z_0$ is a pole of some entry of $G$. The problem of locating zeros is more delicate. The next lemma provides a useful condition.

Lemma 4.2 Let $G(z)$ be a full-rank $p \times m$ rmvf.

(a) If $p \geq m$ and $\lim_{z \rightarrow z_0} G(z)u = 0$ for some nonzero vector $u \in \mathbb{C}^m$, then $z_0$ is a zero of $G(z)$.
(b) If $p \leq m$ and $\lim_{z \rightarrow z_0} v^*G(z) = 0$ for some nonzero vector $v \in \mathbb{C}^p$, then $z_0$ is a zero of $G(z)$.

Proof. In setting (a), the local SM-form ((2.1)–(2.3)) of $G(z)$ at $z_0$ is

$$\Lambda(z) = \begin{bmatrix} D(z) \\ O_{(p-m) \times m} \end{bmatrix},$$

where

$$D(z) = \text{diag}\{(z - z_0)^{k_1}, \ldots, (z - z_0)^{k_m}\}$$

and $k_1 \leq \cdots \leq k_m$. Thus, with the help of formula (2.1), it is readily shown that

$$\lim_{z \rightarrow z_0} G(z)u = 0 \iff \lim_{z \rightarrow z_0} D(z)F(z)u = 0.$$

Moreover, if $k_m \leq 0$, then $D(z)^{-1}$ is a polynomial matrix that tends to a finite limit as $z \rightarrow z_0$. Thus,
\[
F(z_0)u = \lim_{z \to z_0} D(z)^{-1} D(z) F(z) u
= \lim_{z \to z_0} D(z)^{-1} \lim_{z \to z_0} D(z) F(z) u = 0,
\]
which contradicts the assumption that the vector \( u \) is nonzero (since \( F(z_0) \) is invertible). Therefore, (a) holds. The proof of (b) is similar. \( \Box \)

The converse to Lemma 4.2 is true if \( G(z) \) is holomorphic at the point \( z_0 \), but may fail if \( z_0 \) is both a zero and a pole of \( G(z) \). If, for example,
\[
G(z) = \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \\ z^2 & z \end{bmatrix}
\quad \text{and} \quad u = \begin{bmatrix} a \\ b \end{bmatrix},
\]
then
\[
G(z)u = \begin{bmatrix} az^{-1} + b \\ az^2 + bz \end{bmatrix}.
\]
Consequently,
\[
\lim_{z \to 0} G(z)u = 0 \Rightarrow u = 0,
\]
i.e., 0 is a zero of \( G(z) \) but there does not exist a nonzero vector \( u \) such that \( \lim_{z \to 0} G(z)u = 0 \).

In particular, Lemma 4.2 implies that if \( G(z) \) is a full-rank \( p \times m \) rmvf such that every entry in a certain column (row) of \( G(z) \) vanishes at \( z_0 \) when \( p \geq m \) \((p \leq m)\), then \( z_0 \) is a zero of \( G(z) \). The next lemma generalizes this conclusion.

**Lemma 4.3** Let \( G(z) \) be a full-rank \( p \times m \) rmvf and let \( z_0 \in \mathbb{C} \).

(a) If \( p \geq m \) and if for some \( 0 \leq i \leq m - 1 \) there is a \((p - i) \times (1 + i)\) sub-matrix \( \tilde{G}(z) \) of \( G(z) \), all of whose entries vanish at \( z_0 \), then \( z_0 \) is a zero of \( G(z) \).

(b) If \( p \leq m \) and if for some \( 0 \leq i \leq p - 1 \) there is a \((1 + i) \times (m - i)\) sub-matrix \( \tilde{G}(z) \) of \( G(z) \), all of whose entries vanish at \( z_0 \), then \( z_0 \) is a zero of \( G(z) \).

**Proof.** Suppose that \( p \geq m \). By interchanging rows and columns, we can assume that \( G = \begin{bmatrix} \tilde{G} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \). Then, by performing elementary operations on the last \( i \) rows and on the first \( i + 1 \) columns, we obtain a new rmvf
\[
\tilde{G}(z) \tilde{z} \begin{bmatrix} \tilde{G}_{11}(z) & G_{12}(z) \\ \tilde{G}_{21}(z) & \tilde{G}_{22}(z) \end{bmatrix},
\]

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where $\tilde{G}_{11}(z_0) = 0_{(p-d) \times (1+i)}$ and $\tilde{G}_{21}(z)$ is the local SM-form of the $i \times (i+1)$ rmv G21(z) at $z_0$. Therefore, the $(i + 1)$’th column of $\tilde{G}(z)$ also vanishes at $z_0$. Thus, by Lemma 4.2, $z_0$ is a zero of $\tilde{G}(z)$ and so too of $G(z)$.

The case $p \leq m$ follows similarly or by passing to transposes. □

**Lemma 4.4** Let $G(z)$ be a $p \times m$ rmv of full-rank $r$, let $H(z)$ be an $m \times m$ rmv of full-rank and assume that $z_0$ is a zero of $G(z)$ with zero multiplicities $k_1, \ldots, k_\ell$ but that $z_0$ is not a zero of $G(z)H(z)$. Then $z_0$ is a pole of $H(z)$ and the indices $t_1 \leq \cdots \leq t_m$ in the local SM-form of $H(z) = E_2(z)\Lambda_2(z)$ at $z_0$ are subject to the constraints

$$t_j + k_{t-j+1} \leq 0 \text{ for } j = 1, \ldots, \ell .$$

**Proof.** Suppose first that $p \leq m$, let $e_1, \ldots, e_p$ denote the standard basis for $\mathbb{C}^p$ and let $s_1 \leq \cdots \leq s_p$ denote the indices in the local SM-form at $z_0$ of $G(z) = E_1(z)\Lambda_1(z)F_1(z)$. Then

$$s_j \leq 0 \text{ for } j = 1, \ldots, p - \ell$$

and

$$s_{p-j} = k_{t-j} > 0 \text{ for } j = 0, \ldots, \ell - 1 .$$

Therefore, if

$$s_p + t_1 > 0 ,$$

then the bottom row

$$\lim_{z \to z_0} e_p^* \Lambda_1(z)F_1(z)E_2(z)\Lambda_2(z) = 0$$

and hence, in view of Lemma 4.2, $z_0$ is a zero of $\Lambda_1(z)F_1(z)E_2(z)\Lambda_2(z)$ and so too of $G(z)H(z)$, which contradicts the hypothesis of the lemma. Thus, we have established the inequality

$$s_{p-j+1} + t_j \leq 0 \text{ for } j = 1 .$$

If the inequality also holds for $j = 2, \ldots, \ell$, then we are finished. If not, then there exists a smallest positive integer $i$ between 2 and $\ell$ for which it fails. Let

$$\Lambda_1 F_1 E_2 = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} ,$$

where $X_{21}$ is a block of size $i \times (i-1)$. Then there exists an $i \times i$ unimodular matrix $V(z)$ such that the top row of $V(z)X_{21}(z)$ is identically equal to zero. Moreover, since every non identically vanishing entry in $V(z)X_{22}(z)$ has zero
of order at least $s_{p-i+1}$ at $\zeta_0$, it follows that

$$\lim_{z \to \zeta_0} e^{*}_{r-i+1} \begin{bmatrix} I_{p-i} & 0 \\ 0 & V(z) \end{bmatrix} \Lambda_1(z)F_1(z)E_2(z)\Lambda_2(z) = 0.$$ 

Therefore, by Lemma 4.2, $\zeta_0$ is a zero of

$$\begin{bmatrix} I_{p-i} & 0 \\ 0 & V(z) \end{bmatrix} \Lambda_1(z)F_1(z)E_2(z)\Lambda_2(z)$$

and so too of $\Lambda_1(z)F_2(z)E_2(z)\Lambda_2(z)$ and of $G(z)H(z)$. But this contradicts the given hypotheses of the lemma. This completes the proof for $p \leq m$.

If $p > m$ and if the local SM-form of $G(z)$ at $\zeta_0$ is $G(z) = E(z)\Lambda(z)F(z)$, then

$$E(z)^{-1}G(z) = \begin{bmatrix} \hat{G}(z) \\ 0 \end{bmatrix},$$

where $\hat{G}(z)$ is an $m \times m$ regular rmvf. It is easy to verify that $G(z)$ and $\hat{G}(z)$ have the same indices in their local SM-form at $\zeta_0$. The same holds for $GH$ and $\hat{G}H$ at $\zeta_0$. The lemma now follows from the case $p = m$. \(\square\)

**Remark 4.5** Lemma 4.4 implies that if $\Theta(z)$ is a minimal zero conjugator of a full-rank rmvf $G(z)$ w.r.t. $\Pi_+$, then every zero $\zeta_0$ of $G(z)$ in $\Pi_+$ is cancelled by a pole of $\Theta(z)$ at $\zeta_0$. Moreover, if $k_1 \leq \cdots \leq k_r$ are the zero multiplicities of $G$ at $\zeta_0$, then $\Theta(z)$ has at least $r$ pole multiplicities at $\zeta_0$, i.e., if $t_1 \leq \cdots \leq t_r$ are the first $r$ indices in the local SM-form of $\Theta$ at $\zeta_0$, then $-t_i \geq k_i$, $1 \leq i \leq r$. In particular,

$$M_\pi(\Theta; \zeta_0) \geq M_\zeta(G; \zeta_0)$$

and consequently,

$$M_\pi(\Theta; \Pi_+) \geq M_\zeta(G; \Pi_+).$$

Therefore,

$$\text{Mcdeg}(\Theta) = M_\pi(\Theta; \mathbb{C}) \geq M_\pi(\Theta; \Pi_+) \geq M_\zeta(G; \Pi_+). \quad (4.1)$$

Thus, we see that the McMillan degree of a minimal zero conjugator $\Theta(z)$ of a rmvf $G(z)$ w.r.t. $\Pi_+$ is indeed minimal and that the poles of $\Theta(z)$ coincide with the zeros $G(z)$ in $\Pi_+$ and the multiplicity of each pole of $\Theta(z)$ is equal to the multiplicity of the corresponding zero of $G(z)$ in $\Pi_+$. 

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**Remark 4.6** The requirement in Definition 4.1 that $G$ have no zeros on $i\mathbb{R}$ is needed in general for the existence of a conjugator. Indeed, since

$$\Theta^{-1}(z) = J\Theta^\#(z)J = J\Theta(-z^*)^*J$$

by (2.7), a point $z_0$ is a zero (pole) of $\Theta(z)$ if and only if $-z_0$ is a pole (zero) of $\Theta(z)$ (and if $s_1 \leq \cdots \leq s_m$ are the indices in the local SM-form of $\Theta(z)$ at $z_0$, then $-s_m \leq \cdots \leq -s_1$ are the indices in the local SM-form of $\Theta(z)$ at $-z_0$). Thus, if $G(z)$ has zero $z_0 \in i\mathbb{R}$, then, by Lemma 4.4, $z_0$ is a pole of $\Theta(z)$ and $z_0 = -z_0$ is also a zero of $\Theta(z)$. So, in general, we get a new zero for $(G\Theta)(z)$, instead of the one that was cancelled (see, e.g., [4, Remark 3.5]). The requirement (3.2) is imposed in Theorem 3.3 for analogous reasons.

### 4.2 A necessary and sufficient condition for existence of minimal zero conjugator for regular full-rank tall rmvf

Let $p \geq m$ and let $G(z)$ be a $p \times m$ full-rank rmvf that does not have zeros on $i\mathbb{R}$. Then there is a unimodular $p \times p$ rmvf $W(z)$ such that

$$WG = \begin{bmatrix} \hat{G} \\ 0 \end{bmatrix}, \quad (4.2)$$

where $\hat{G}(z)$ is a regular $m \times m$ rmvf without zeros on $i\mathbb{R}$. Since $W$ is unimodular, $G$ and $\hat{G}$ share the same minimal zero conjugators w.r.t. $\Pi_+$. Moreover, since $\hat{G}(z)$ is regular, the zeros of $\hat{G}(z)$ correspond to the poles of $\hat{G}(z)^{-1}$ and the right half plane zeros of $\hat{G}(z)$ are in one to one correspondence with the left half plane poles of $\hat{G}^\#(z)^{-1}$. Therefore, since

$$\Theta(z) \quad \text{is } J\text{-inner w.r.t. } \Pi_+ \iff \Theta^{-\#}(z) \quad \text{is } J\text{-inner w.r.t. } \Pi_+$$

and (as $\Theta(z)$ is proper)

$$\text{Modeg } \Theta(z) = \text{Modeg } \Theta^{-\#}(z),$$

it is readily seen that $\Theta(z)$ is a minimal zero conjugator of $\hat{G}(z)$ w.r.t. $\Pi_+$ if and only if $\Theta^{-\#}(z)$ is a minimal pole conjugator of $\hat{G}^\#(z)^{-1}$ w.r.t. $\Pi_-$. Observe that the operations $\Theta(z) \mapsto \Theta^{-1}(z)$ and $\Theta(z) \mapsto \Theta^\#(z)$ commute, so we can freely write $\Theta^{-\#}(z)$ instead of $(\Theta(z)^{-1})^\#$ or $(\Theta^\#(z))^{-1}$. The same remark applies to the regular rmvf $\hat{G}(z)$. Note also that $\hat{G}^{-\#}(z)$ has no poles on $i\mathbb{R}$ since $G$ has no zeros there.

Our next objective is to find a proper rmvf that has the same pole structure as $\hat{G}^{-\#}(z)$ in $\Pi_\perp$. If $\hat{G}^{-\#}(z)$ is not proper, and if $k$ is the largest order of a
pole at infinity of any entry in $\hat{G}^{-\#}(z)$, then we apply one of the following schemes:

(I) Cut the polynomial part of $\hat{G}^{-\#}(z)$ to obtain a proper rmvf $G_1(z)$.
(II) Set $G_2(z) = (z - 1)^{-k} \hat{G}^{-\#}(z)$.

The minimality property of a minimal zero conjugator $\Theta(z)$ of $G(z)$ w.r.t. $\Pi_+$, (Definition 4.1) implies that $\Theta^{-\#}(z)$ is analytic in $\Pi_-$, and thus, for $i = 1, 2$, $G_i(z)\Theta^{-\#}(z)$ has a pole at $z_0 \in \Pi_-$ if and only if $\hat{G}^{-\#}(z)\Theta^{-\#}(z)$ has a pole at $z_0$. Moreover, by Lemma 4.7, which is given below, neither of the operations (I) or (II) changes the pole multiplicities in $\Pi_-$.

**Lemma 4.7** Let $G(z)$ be a $p \times m$ rmvf and let $z_0 \in \mathbb{C}$. If $A(z)$ is a $p \times m$ rmvf that is analytic at $z_0$ and $T(z)$ is a $p \times p$ rmvf that is analytic and invertible at $z_0$, then the three rmvf’s $G(z), G(z) + A(z)$, and $T(z)G(z)$ have the same pole multiplicities in their local SM-form at $z_0$.

**Proof.** The fact that $T(z)G(z)$ and $G(z)$ have the same pole (and zero) multiplicities at $z_0$ follows from the uniqueness of the local SM-form at $z_0$. The fact that $G(z)$ and $G(z) + A(z)$ have the same pole multiplicities at $z_0$ is established by comparing the local SM-form of $G(z) + A(z)$ with that of $G(z)$ at $z_0$. It suffices to verify the assertion for the case that only one entry in $A(z)$ is not identically zero. □

The last lemma can also be proved by the theory of pole structure.

Theorem 4.3 yields the following result:

**Theorem 4.8** Let $p \geq m$ and let $G(z)$ be a full-rank $p \times m$ rmvf that does not have zeros on $i\mathbb{R}$. Let $G_i(zI_m - A_i)^{-1}B_i + D_i$ be a minimal realization of the proper rmvf $G_i(z)$, $i = 1, 2$, that is constructed by (I) and (II) above, respectively. Then there exists a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$ if and only if there exists a positive semi-definite matrix $X_i = R_{aut}(A_i, B_i)$. If this condition is met, then the stable parts of $(A_1, B_1)$ and $(A_2, B_2)$ are left similar and the minimal zero conjugator of $G(z)$ is uniquely specified by the formulas

$$
\Theta(z) = \Theta_i(z) = -B_i^*X_i(zI_m - A_i)^{-1}B_iJ + I_m
= -B_i^*(zI_m + A_i^*)^{-1}X_iB_iJ + I_m
$$

(4.3)

up to multiplication on the right by a $J$-unitary constant matrix, where $\hat{A}_i = A_i - B_iJ B_i^*X_i$ is anti-stable.

**Proof.** The formula for the rmvf $\Theta_i(z)$ is obtained from the relation $\Theta_i(z) = \hat{A}_i - B_iJ B_i^*X_i$
\[ \tilde{\Theta}^{-\#}(z), \] where \( \tilde{\Theta}(z) \) is a minimal pole conjugator of \( G(z) \) w.r.t. \( \Pi_\perp \) and the well-known identity

\[
(G(zI - E)^{-1} F + H)^{-\#} = H^{-\#} F^*(zI - (G^* H^{-\#} F^* - E^*))^{-1} G^* H^{-\#} + H^{-\#}
\]

(4.4)

where \( E, F, G \) are of appropriate sizes and \( H \) is invertible, and we get the (unique) conjugator \( \Theta \) with \( \Theta(\infty) = I_m \). Thus, in this case, \( \Theta_1(z) = \Theta_2(z) \). This means that it does not matter whether we invoke (I) or (II) above. Indeed, by Remark 3.2, \( C_i \) and \( D_i \) have no effect on the existence of (or the formula for) the conjugator \( \Theta_i \) and thus \( \Theta_i \) may be called a minimal pole conjugator of \( (A_i, B_i) \) w.r.t. \( \Pi_\perp \). Moreover, in \([4, \text{Theorem } 4.5]\), it was shown (as expected) that \( \Theta_i \) depends only on the stable part of \( (A_i, B_i) \), i.e., if

\[
S_i^{-1} A_i S_i = \begin{bmatrix} A_{t+}^{(i)} \\ A_{t+}^{(i)} \end{bmatrix} \quad \text{and} \quad S_i^{-1} B_i = \begin{bmatrix} B_{t+}^{(i)} \\ B_{t+}^{(i)} \end{bmatrix} \quad \text{for} \quad i = 1, 2,
\]

where \( \sigma(A_{t+}^{(i)}) \subset \Pi_\perp \), \( \sigma(A_{t+}^{(i)}) \subset \Pi_+ \) and the last decomposition is conformal w.r.t. the block decomposition of \( S_i^{-1} A_i S_i \), then \( \Theta_i \) is a minimal pole conjugator of the pair \( (A_i, B_i) \) w.r.t. \( \Pi_\perp \) if and only if \( \Theta_i \) is a minimal pole conjugator of the pair \( (A_{t+}^{(i)}, B_{t+}^{(i)}) \) w.r.t. \( \Pi_\perp \). By pole structure theory \([1, \text{Ch.3}]\), it can be shown directly that the stable parts of \( (A_1, B_1) \) and \( (A_2, B_2) \) are left similar, i.e., there is a unique \( T \) such that \( T^{-1} A_{t+}^{(1)} T = A_{t+}^{(2)}, T^{-1} B_{t+}^{(1)} = B_{t+}^{(2)} \). Lemma 3.4 implies that formula (4.3) is invariant under (left) similarity of \( (A_i, B_i) \) and hence, as \( \Theta_i(z) \) depends only on the stable part of \( (A_i, B_i) \), it is invariant under (left) similarity of \( (A_{t+}^{(i)}, B_{t+}^{(i)}) \). Since \( (A_{t+}^{(1)}, B_{t+}^{(1)}) \) is similar to \( (A_{t+}^{(2)}, B_{t+}^{(2)}) \), we have obtained another proof that \( \Theta_1(z) = \Theta_2(z) \). (The similarity between \( (A_{t+}^{(1)}, B_{t+}^{(1)}) \) and \( (A_{t+}^{(2)}, B_{t+}^{(2)}) \) can also be deduced by invoking null structure theory (see \([4, \text{Theorem } 4.10]\)).)

\( \square \)

In the special case that the rmvf \( G(z) \) is proper and invertible at infinity, the previous theorem is applicable directly:

**Corollary 4.9** Let \( G(z) = C(zI_n - A)^{-1} B + D \) be a minimal realization, and assume that \( D \) is invertible and that \( \sigma(A - BD^{-1} C) \cap i\mathbb{R} = \emptyset \). Then there exists a minimal zero conjugator of \( G(z) \) w.r.t. \( \Pi_+ \) if and only if there exists a positive semi-definite matrix \( X = R_{ast}(C^* D^{-\ast} B^* - A^* C^* D^{-\ast}) \). In this case the normalized conjugator \( \Theta \) is uniquely specified by the formula

\[
\Theta(z) = -D^{-1} C X (zI_n - \hat{A})^{-1} C^* D^{-\ast} J + I_m
\]

\[
= -D^{-1} C (zI_n - (A - BD^{-1} C))^{-1} X C^* D^{-\ast} J + I_m,
\]

where \( \hat{A} = C^* D^{-\ast} B^* - A^* - C^* D^{-\ast} JD^{-1} C \) is antistable. Moreover,

\[
G(z) \Theta(z) = C(zI_n - A)^{-1} (B - XD^* J) + D.
\]

(4.5)
Proof. This corollary follows directly from Theorem 4.8. One has just to observe that in this case \( G(z) = \tilde{G}(z) \) in (4.2) and to use formula (4.4) both to calculate \( G^{-\#}(z) \) and \( \Theta(z) = \tilde{\Theta}^{-\#}(z) \), where \( \tilde{\Theta}(z) \) is the minimal pole conjugator of \( G^{-\#}(z) \) w.r.t \( \Pi \) given by (3.4). Formula (4.5) for \( G\Theta \) follows from the identities

\[
G(z)\Theta(z) = \left( [G^{-\#}\Theta^{-\#}]^\#(z) \right)^{-1} = \left( [G^{-\#}\tilde{\Theta}]^\#(z) \right)^{-1},
\]

and \( G^{-\#}(z)\tilde{\Theta}(z) \) is obtained from (3.5). \( \Box \)

Our next goal is to count the number of poles of \( G\Theta \). But first we need some preparation.

Lemma 4.10 Let \( G(z) \) be a \( p \times m \) rmuf of full-rank \( r = \min\{p, m\} \), and let \( k_1 \leq \cdots \leq k_r \) denote the indices in the local SM-form of \( G(z) \) at \( z_0 \). Let \( \tilde{G}(z) \) denote the rmuf that is obtained from \( G(z) \) by either multiplying one column of \( G(z) \) by \( z - z_0 \) if \( p \geq m \) or by multiplying one row of \( G(z) \) by \( z - z_0 \) if \( p \leq m \) (but not both), and let \( \tilde{k}_1 \leq \cdots \leq \tilde{k}_r \) denote the indices in the local SM-form of \( \tilde{G}(z) \) at \( z_0 \). Then

\[
\tilde{k}_j \geq k_j \quad \text{and} \quad \sum_{j=1}^r \tilde{k}_j = 1 + \sum_{j=1}^r k_j ,
\]
i.e., only one index is changed, and it increases by 1.

Proof. Suppose first that \( p \geq m \). If \( m = 1 \), then the lemma is obvious, so assume \( m > 1 \). Let \( \tilde{G}(z) = [\tilde{g}_{ij}(z)] \) denote the matrix that is obtained from \( G(z) = [g_{ij}(z)] \) by multiplying each entry in the \( \nu \)th column by \( z - z_0 \) and let \( \tilde{g}_{\mu \nu}(z) \) denote an entry in \( \tilde{G}(z) \) for which

\[
n(\tilde{g}_{\mu \nu}; z_0) \leq n(\tilde{g}_{ij}; z_0) \quad \text{for} \quad i = 1, \ldots, p, \ j = 1, \ldots, m .
\]

In this selection we shall always choose \( \nu = \ell \) if possible. Thus,

\[
\nu \neq \ell \Rightarrow n(\tilde{g}_{\mu \nu}; z_0) < n(\tilde{g}_{ij}; z_0) \quad \text{for} \quad i = 1, \ldots, p .
\]

Starting from \( \tilde{G} \), successive application of the rules

\[
R_i \rightarrow R_i - \frac{\tilde{g}_{i \nu}}{\tilde{g}_{\mu \nu}} R_{\mu} , \quad i = 1, \ldots, p , \ i \neq \mu ,
\]

followed by a permutation of the first row with the \( \mu \)th row and a permutation of the \( \nu \)th column with the first column produces a new matrix of the form

\[
\begin{bmatrix}
\tilde{g}_{\mu \nu} & * \\
0 & \tilde{G}^{(1)}
\end{bmatrix}
\begin{bmatrix}
z \circ \\
\tilde{g}_{\mu \nu} & 0 \\
0 & \tilde{G}^{(1)}
\end{bmatrix}.
\]
The last equivalence holds because $n(\hat{g}_{i\nu}; z_0) \leq n(f, z_0)$ for every entry $f(z)$ in the top row of the matrix on the left.

In the next step, there are two cases to consider: $\nu = \ell$ or $\nu \neq \ell$.

If $\nu = \ell$, then, since

$$R_i(\hat{G}) - \frac{\hat{g}_{i\nu}}{\hat{g}_{j\nu}} R_j(\hat{G}) = R_i(G) - \frac{g_{i\nu}}{g_{j\nu}} R_j(G) \quad \text{for} \quad i = 1, \ldots, p,$$

and

$$n(g_{i\nu}; z_0) \leq n(g_{j\nu}; z_0) \quad \text{for} \quad i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, m,$$

it is readily checked by applying the same set of transformations to $G(z)$ that were applied to $\hat{G}(z)$ that the local SM-form of $G(z)$ at $z_0$ is the same as the local SM-form of

$$\begin{bmatrix}
g_{i\nu}(z) & 0 \\
0 & \hat{G}^{(1)}(z)
\end{bmatrix}
$$

at $z_0$. Therefore, since $\hat{g}_{i\nu}(z) = (z - z_0)g_{i\nu}(z)$, the proof is complete if $\nu = \ell$.

If $\nu \neq \ell$, then $\hat{g}_{i\nu}(z) = g_{i\nu}(z)$,

$$\hat{G}(z) \overset{\approx}{=} \begin{bmatrix}
g_{i\nu}(z) & 0 \\
0 & \hat{G}^{(1)}(z)
\end{bmatrix} \quad \text{and} \quad G(z) \overset{\approx}{=} \begin{bmatrix}
g_{i\nu}(z) & 0 \\
0 & G^{(1)}(z)
\end{bmatrix},$$

where $\hat{G}^{(1)}(z)$ may be obtained from $G^{(1)}(z)$ by multiplying one of its columns by $(z - z_0)$. Thus, we have returned to the starting point, but with full-rank matrices $\hat{G}^{(1)}(z)$ and $\hat{G}^{(1)}(z)$ of size $(p - 1) \times (m - 1)$. Therefore, the previous procedure can be iterated until there exists an entry in $\hat{G}^{(t)}(z) = \left[ \hat{g}_{ij}^{(t)} \right]$, $i = 1, \ldots, p - t$, $j = 1, \ldots, m - t$ in the column that was multiplied by $(z - z_0)$ that minimizes $n(\hat{g}_{ij}^{(t)}; z_0)$. If this happens for $t < m - 1$, then

$$\hat{G}(z) \overset{\approx}{=} \begin{bmatrix}
\hat{a}_1 & 0 \\
\vdots & \\
\hat{a}_t & 0 \\
0 & \hat{G}^{(t)}
\end{bmatrix} \quad \text{and} \quad G(z) \overset{\approx}{=} \begin{bmatrix}
a_1 & 0 \\
\vdots & \\
a_t & 0 \\
0 & G^{(t)}
\end{bmatrix},$$

where $\hat{a}_j(z) = a_j(z)$ for $j = 1, \ldots, t - 1$; $\hat{a}_t(z) = (z - z_0)a_t(z)$ and $\hat{G}^{(t)}(z) = G^{(t)}(z)$. If this happens for $t = m - 1$, then $\hat{a}_j(z) = a_j(z)$ for $j = 1, \ldots, m - 1$ and $\hat{G}^{(m-1)}(z) = (z - z_0)G^{(m-1)}(z)$. In either case it is easily seen that the desired conclusion prevails. This completes the proof for $p \geq m$. The proof for $p < m$ can be handled in much the same way, or by passing to transposes. $\square$
Remark 4.11 If a column (or row) of a full-rank rmvf $G(z)$ is multiplied by $(z - z_0)^s$, where $s$ is an integer and $s > 1$ then the sum of the indices in the local SM-form of $G(z)$ will increase by $s$. However, in contrast to the case $s = 1$, more than one index may change as a result of this multiplication. If, for example,

$$G(z) = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{G}(z) = G(z) \begin{bmatrix} 1 & 0 \\ 0 & z^2 \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & z^2 \end{bmatrix},$$

then the local SM-forms of $G(z)$ and $\hat{G}(z)$ at $z = 0$ are

$$\Lambda(z) = \begin{bmatrix} 0 & 0 \\ z^{-1} \end{bmatrix} \quad \text{and} \quad \hat{\Lambda}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^2 \end{bmatrix},$$

respectively.

As a by-product of the preceding analysis, we obtain the following conclusions:

**Corollary 4.12** If $\Theta(z)$ is a minimal zero conjugator of a $p \times m$ rmvf $G(z)$ w.r.t. $\Pi_+$, and if $G(z)$ has no zeros on $i\mathbb{R}$ and $p \geq m$, then:

1. $M_\pi(G\Theta; \mathbb{C}) \leq M_\pi(G; \mathbb{C})$ with equality if no pole of $G(z)$ in $\Pi_-$ is a zero of $\Theta(z)$.
2. If $G(z)$ is a proper rmvf, and if $\sigma(A \cap \Pi_-) \cap \sigma(-(A - BD^{-1}C)^*) = \emptyset$,

then equality holds in (1) and the realization (4.5) for $G(z)\Theta(z)$ is minimal.

4.3 Sufficient conditions for the existence of a conjugator for full-rank wide rmvf’s

The main aim of this section is to find sufficient conditions for the existence of a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$ when $G(z)$ is a full-rank $p \times m$ rmvf without zeros on $i\mathbb{R}$ and $p < m$. We shall need the following preliminary result.

**Lemma 4.13** Let $G_1(z)$ and $G_2(z)$ be rmvf’s of sizes $p \times m$ and $m \times p$, respectively, such that either $G_1(z)G_2(z) = I_p$ or $G_2(z)G_1(z) = I_m$. Let $r = \min(p, m)$, let $s_1, \ldots, s_r$ and $t_1, \ldots, t_r$ denote the indices of the local SM-forms of $G_1(z)$ and $G_2(z)$ at $z_0 \in \mathbb{C}$, respectively, and suppose that $s_{r-\ell} \leq 0$ and $s_{r-\ell+1} > 0$ for some integer $\ell$, $1 \leq \ell \leq r$. Then

$$t_j + s_{r-j+1} \leq 0 \quad \text{for} \quad j = 1, \ldots, \ell.$$
Consequently, the zeros of $G_1(z)$ at $z_0$ are compensated by the poles of $G_2(z)$ at $z_0$ and
\[ M_c(G_1; z_0) \leq M_r(G_2;z_0) . \]

**Proof.** Suppose first that $p \geq m$. Then for $j = 1, 2$
\[ G_j(z) = E_j(z)\Lambda_j(z)F_j(z), \]
where
\[ \Lambda_1(z) = \begin{bmatrix} D_1(z) \\ 0 \end{bmatrix}, \quad \Lambda_2(z) = \begin{bmatrix} D_2(z) \\ 0 \end{bmatrix}, \]
\[ D_1(z) = \text{diag}\{(z - z_0)^{s_1}, \ldots, (z - z_0)^{s_m}\} \]
and
\[ D_2(z) = \text{diag}\{(z - z_0)^{t_1}, \ldots, (z - z_0)^{t_m}\}. \]
By assumption
\[ \Lambda_2(z)F_2(z)E_1(z)\Lambda_1(z) = E_2(z)^{-1}F_1(z)^{-1} \]
is holomorphic and invertible in a neighborhood of $z_0$. Let
\[ W(z) = \begin{bmatrix} I_m \\ 0_{m \times (p-m)} \end{bmatrix} F_2(z)E_1(z) \begin{bmatrix} I_m \\ 0_{(p-m) \times m} \end{bmatrix} \]
denote the upper left hand $m \times m$ corner of $F_2(z)E_1(z)$. Then the $m \times m$ mvf $D_2(z)W(z)D_1(z) = \Lambda_2(z)F_2(z)E_1(z)\Lambda_1(z)$ is holomorphic and invertible in a neighborhood of $z_0$. By assumption $s_m > 0$. Therefore, if $t_1 + s_m > 0$, then $t_j + s_m > 0$ for $j = 1, \ldots, m$ and hence every entry in the last column of $D_2(z)W(z)D_1(z)$ has a zero at $z = z_0$. Thus, by Lemma 4.3 (or Lemma 4.2), $D_2(z)W(z)D_1(z)$ has a zero at $z_0$, contrary to assumption. Suppose next that $t_1 + s_m \leq 0$ but $t_2 + s_{m-1} > 0$. Then, all non identically vanishing entries in the lower right hand $(m-1) \times 2$ block of $D_2(z)W(z)D_1(z)$ will be equal to zero at $z_0$. Another application of Lemma 4.3 shows that this too leads to a contradiction. Continuing this way, we see that $t_j + s_{m-j+1} \leq 0$ for $j = 1, \ldots, \ell$, as claimed.

The proof for $p \leq m$ is similar. \( \Box \)

**Remark 4.14** The role of poles and zeros in the last lemma cannot be interchanged. A pole of $G_1$ at $z_0$ can sometimes be cancelled in the product of $G_2G_1$ even though $G_2$ does not have a zero at $z_0$, as in the product of the mvfs
\[ G_1(z) = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \quad \text{and} \quad G_2(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
considered at the point $z_0 = 0$. 

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Theorem 4.15 Let $p \leq m$, and let $G(z)$ be a full-rank $p \times m$ rmvf without zeros on $i\mathbb{R}$. Let $G_1(z) = C_1(zI - A_1)^{-1}B_1$ be the rmvf that is obtained from $G(z)$ by method I and assume that the indicated realization is minimal. Suppose there exists a positive semi-definite matrix $X = R_{\text{asf}}(A_1, B_1)$. Then
\[
\Theta(z) = -B_1^T X(zI_n - \tilde{A}_1)^{-1}B_1J + I_m = -B_1^T(zI_n + A_1^*)^{-1}XB_1J + I_m \quad (4.6)
\]
is a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$. 

Proof. Every $p \times m$ rmvf $G(z)$ of rank $p$ admits a right inverse $\tilde{G}(z)$, such that $M_c(G, z_0) = M_{\pi}(\tilde{G}, z_0)$ at every point $z_0 \in \mathbb{C}$. Moreover, by Lemma 4.13, the zero multiplicities of every zero $z_0$ of $G$ in $\Pi_+$ are equal to the pole multiplicities of $z_0$ as a pole of $\tilde{G}$. One way to construct such a right inverse $\tilde{G}(z)$ is by the global Smith-McMillan form of $G(z)$. Thus, if $G(z) = U(z)\Lambda(z)V(z)$ as in (2.4)–(2.6), define
\[
\tilde{G}(z) = V(z)^{-1} \begin{bmatrix}
\frac{1}{\varphi_1(z)} & & \\
& \ddots & \ & \\
& & \frac{1}{\varphi_p(z)} & \\
O_{(m-p)\times p}
\end{bmatrix} U(z)^{-1}. \quad (4.7)
\]

Now let $(\tilde{G}^\#(z))_{\text{pol}}$ be the polynomial part of $\tilde{G}^\#(z)$ and set
\[
\tilde{G}(z) = \tilde{G}^\#(z) - (\tilde{G}^\#(z))_{\text{pol}}.
\]
The rmvf $\tilde{G}(z)$ is strictly proper. Let a minimal realization of $\tilde{G}$ be
\[
\tilde{G}(z) = [C_1 \ C_2 \left( zI - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right)]^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]
where $\sigma(A_2) \subset i\mathbb{R}$, and $\sigma(A_1) \cap i\mathbb{R} = \emptyset$. Then
\[
\tilde{G}^\#(z) = (\tilde{G}^\#(z))_{\text{pol}} + G_1(z) + G_2(z),
\]
where $G_1(z) = C_1(zI-A_1)^{-1}B_1$ and $G_2(z) = C_2(zI-A_2)^{-1}B_2$. By Lemma 4.7, the pole multiplicities of every $z_0 \in \mathbb{C} \setminus i\mathbb{R}$ are the same for $G_1(z)$ and for $\tilde{G}^\#(z)$. Suppose now that $\Theta(z)$ is a minimal pole conjugator of $G_1(z)$ w.r.t. $\Pi_-$. By the analyticity of $\Theta(z)$ in $\Pi_-$, it follows that $\tilde{G}^\#(z)\Theta(z)$ has no poles in $\Pi_-$, and therefore $\Theta^\#(z)\tilde{G}(z)$ has no poles in $\Pi_+$, and $G(z)\Theta^\#(z)^{-1}$ has no zeros in $\Pi_+$. Thus, as $G(z)\Theta^\#(z)^{-1}G(z) = I_p$, Lemma 4.13 guarantees that $G(z)\Theta^\#(z)^{-1}$ has no zeros in $\Pi_+$. Moreover, since $\Theta^{-\#}(z)$ is also $J$-inner w.r.t. $\Pi_+$ and
\[
\text{Mdeg}(\Theta^{-\#}(z)) = \text{Mdeg}(\Theta(z)) = M_{\pi}(\tilde{G}^\#; \Pi_-) = M_{\xi}(G; \Pi_+),
\]

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it is a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$. The desired conclusion now follows easily from Theorem 3.3 and formula (4.4). The latter is used in order to calculate $\Theta(z)$ from formula (3.4). □

Remark 4.16 In the preceding analysis a formula for $\Theta(z)$ was obtained from the polynomial part of $\hat{G}^\#(z)$. The same formula will be obtained if one divides $\hat{G}^\#(z)$ by a sufficiently high power of $z - 1$ in order to make it proper, i.e., if method II (which is described after Lemma 4.7) is invoked instead of method I.

4.4 Zeros of submatrices

Our next main objective is to obtain necessary conditions for the existence of minimal zero conjugators for wide full-rank mvf’s. First, however, in this subsection, we shall establish some inequalities between the number of zeros of certain submatrices $S(z)$ of a given mvf $T(z)$ at a given point $z_0$ and the number of zeros of $T(z)$ at $z_0$.

We begin with the following preliminary result:

**Lemma 4.17** Let $a_1(z), \ldots, a_r(z)$ and $b_1(z), \ldots, b_r(z)$ be scalar rational functions, such that $a_i(z) \neq 0$ for $1 \leq i \leq r$. Set

$$A_r = \text{diag}\{a_1, \ldots, a_r\} \quad \text{and} \quad B_r = [A_r \quad b_r], \quad \text{where} \quad b_r = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix}.$$  

Then for any $z_0 \in \mathbb{C}$, $M_\zeta(B_r; z_0) \leq M_\zeta(A_r; z_0)$.

**Proof.** The proof proceeds by induction. For $r = 1$ the lemma is obvious. Assume the lemma is true for $r = k$ and consider the $(k + 1) \times (k + 2)$ matrix $B_{k+1}$:

$$B_{k+1} = \begin{bmatrix} a_1 & b_1 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \ddots & \ddots & 0 \\ a_k & b_k \\ a_{k+1} & b_{k+1} \end{bmatrix}.$$
If for some \(i, 1 \leq i \leq k + 1\), \(n(b_i; z_0) \geq n(a_i; z_0)\), then \(b_i/a_i\) is analytic at \(z_0\) and the elementary column operation \(C_{k+2} \leftrightarrow C_{k+2} - (b_i/a_i)C_i\) forces the \((i, k+2)\) entry to be identically zero. Therefore, we can assume that for every \(1 \leq i \leq k + 1\)

\[
n(b_i; z_0) < n(a_i; z_0).
\]  

(Of course, we can assume that not every \(b_i\) is identically zero, since in this case the lemma is immediate.) Without loss of generality, we may assume that \(n(b_{k+1}; z_0) \leq n(b_i; z_0), 1 \leq i \leq k\). Next, the elementary column operation \(C_{k+1} \leftarrow C_{k+1} - (a_{k+1}/b_{k+1})C_{k+2}\) followed by the elementary row operations \(R_i \leftarrow R_i - (b_i/b_{k+1})R_{k+1}\) for every \(1 \leq i \leq k\), yield the matrix

\[
\hat{B}_{k+1} = \begin{bmatrix}
  a_1 & \hat{b}_1 & 0 \\
  0 & \ddots & \vdots \\
  0 & \ldots & a_k \hat{b}_k & 0 \\
  0 & \ldots & 0 & 0 & b_{k+1}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  \vdots \\
  0 \\
  0 \ldots 0 & b_{k+1}
\end{bmatrix},
\]

where \(\hat{B}_k\) denotes the upper left hand \(k\times(k+1)\) block of \(\hat{B}_{k+1}\). Therefore, since \(M_\zeta(\hat{B}_k; z_0) \leq M_\zeta(A_k; z_0)\) and \(M_\zeta(b_{k+1}; z_0) \leq M_\zeta(a_{k+1}; z_0)\), by the induction assumption and (4.8) for \(i = k + 1\), respectively, it is readily seen that

\[
M_\zeta(\hat{B}_{k+1}; z_0) = M_\zeta(\hat{B}_k; z_0) + M_\zeta(b_{k+1}; z_0) \leq M_\zeta(A_k; z_0) + M_\zeta(a_{k+1}; z_0)
\]

as needed to complete the proof. \(\square\)

**Theorem 4.18** Let \(T(z)\) be a full-rank \(n \times \ell\) rmvf, and let \(S(z)\) be a sub-matrix of \(T(z)\) with

- \(n\) rows if \(n > \ell\)
- \(\ell\) columns if \(n < \ell\)
- either \(n\) columns or \(n\) rows if \(n = \ell\) (and hence \(S(z) = T(z)\)),

i.e., up to permutation of columns or rows, respectively,

\[
T = \begin{bmatrix} S & R \end{bmatrix} \text{ if } n \geq \ell \text{ and } T = \begin{bmatrix} S \\ R \end{bmatrix} \text{ if } n \leq \ell.
\]
Then for any point \( z_0 \in \mathbb{C} \),
\[
M_{\zeta}(S; z_0) \leq M_{\zeta}(T; z_0).
\]

**Proof.** Suppose first that \( T = \begin{bmatrix} S \\ R \end{bmatrix} \), where \( R \) is \( 1 \times \ell \). Then, by performing elementary column operations on \( T \), we see that
\[
T = \begin{bmatrix} S \\ R \end{bmatrix} \begin{bmatrix} \hat{S} \\ \begin{bmatrix} 0 & \ldots & 0 & f \end{bmatrix} \end{bmatrix} \quad \text{and} \quad S \approx \hat{S}.
\]
Next, elementary operations on rows 1, 2, \ldots, \( r \) and columns 1, \ldots, \( \ell - 1 \), yield the equivalence
\[
\begin{bmatrix} \hat{S} \\ \begin{bmatrix} 0 & \ldots & 0 & f \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 & A_r & b \\ 0 & 0 & f \end{bmatrix} \quad \text{and} \quad \hat{S} \begin{bmatrix} 0 & A_r & b \end{bmatrix},
\]
where
\[
A_r = \text{diag}\{a_1, \ldots, a_r\} \quad \text{is of rank } r \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix}.
\]
There are two cases to consider.

(a) If \( n(f; z_0) \leq n(b_i; z_0) \) for every \( i, \ 1 \leq i \leq r \), then
\[
\begin{bmatrix} 0 & A_r & b \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} 0 & A_r & 0 \\ 0 & 0 & f \end{bmatrix}.
\]
Therefore,
\[
M_{\zeta} \left( \begin{bmatrix} S \\ R \end{bmatrix} ; z_0 \right) = M_{\zeta}(A_r; z_0) + M_{\zeta}(f; z_0) \geq M_{\zeta}(A_r; z_0)
\]
and, by Lemma 4.17,
\[
M_{\zeta}(A_r; z_0) \geq M_{\zeta}(A_r b; z_0) = M_{\zeta}(S; z_0).
\]

(b) If \( n(f; z_0) > n(b_{i_0}; z_0) \) for some \( i_0, \ 1 \leq i_0 \leq r \), then there exists an integer
\[ t > 0 \text{ such that } n((z - z_0)^{-t}f; z_0) \leq n(b_i; z_0) \text{ for } i = 1, \ldots, r. \] Thus, as

\[
M_{\zeta}(T; z_0) = M_{\zeta}\left(\begin{bmatrix} A_r & b \\ 0 & f \end{bmatrix}; z_0 \right) \geq M_{\zeta}\left(\begin{bmatrix} A_r & b \\ 0 & (z - z_0)^{-t}f \end{bmatrix}; z_0 \right)
\]

by Lemma 4.10, the desired conclusion now follows from (a).

The previous analysis shows that if one row is removed from a full-rank wide matrix \( T \), then the new matrix \( T' \) is also a full-rank wide matrix and

\[
M_{\zeta}(T'; z_0) \leq M_{\zeta}(T; z_0).
\]

Therefore, the procedure can be iterated to yield the asserted conclusions for wide full-rank matrices. The case of full-rank tall matrices can be handled by passing to transposes. \( \square \)

**Remark 4.19** Lemma 4.17 played a key role in the proof of Theorem 4.18. It can also be used to prove Lemma 4.13.

### 4.5 Necessary conditions for the existence of minimal zero conjugators for full-rank wide rmvf’s

**Lemma 4.20** Let \( G(z) \) be a full-rank \( p \times m \) rmvf with \( p < m \) that has no zeros on \( \mathbb{C}^+ \), and assume that \( \Theta(z) \) is a minimal zero conjugator of \( G(z) \) w.r.t. \( \Pi^+ \). Then there exists a regular \( m \times m \) rmvf \( G_0(z) \) such that:

1. \( M_{\zeta}(G_0; z_0) = M_{\zeta}(G; z_0) \) for every point \( z_0 \in \Pi^+ \).
2. \( \Theta(z) \) is a minimal zero conjugator of \( G_0(z) \) w.r.t. \( \Pi^+ \).

**Proof.** The first step is to find an \((m - p) \times m\) rmvf \( H(z) \) such that the \( m \times m \) rmvf

\[
\tilde{G}(z) = \begin{bmatrix} (G\Theta)(z) \\ H(z) \end{bmatrix}
\]

is regular and has no zeros in \( \Pi^+ \). Such a rmvf \( H(z) \) can be found by first using the global SM-form of \((G\Theta)(z)\) to write

\[ G\Theta = U[\Phi \quad 0]V, \]

with

\[ \Phi = \text{diag}\{\varphi_1, \ldots, \varphi_p\} \]

as in (2.4)–(2.6), and then setting

\[ H(z) = [0 \quad I_{m-p}]V \]
to get

\[ \hat{G} = \begin{bmatrix} U \Phi & 0 \\ 0 & I_{m-p} \end{bmatrix} V = \begin{bmatrix} U & 0 \\ 0 & I_{m-p} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & I_{m-p} \end{bmatrix} V. \]

Clearly, \( \hat{G}(z) \) has no zeros in \( \Pi_+ \), since \( \Phi(z) \) has no zeros in \( \Pi_+ \).

Our next objective is to show that \( M_\zeta(G; z_0) = M_\zeta(\hat{G}\Theta^{-1}; z_0) \) for every point \( z_0 \in \Pi_+ \).

Let \( z_0 \in \Pi_+ \). If \( z_0 \) is not a zero of \( G \), then by the minimality of \( \Theta \), \( z_0 \) is not a pole of \( \Theta \). Moreover, since \( \hat{G} = \hat{G}\Theta^{-1}\Theta \), Lemma 4.4 guarantees that \( z_0 \) is not a zero of \( \hat{G}\Theta^{-1} \). Thus, the zeros of \( \hat{G}\Theta^{-1} \) in \( \Pi_+ \) are a subset of the zeros of \( G \) in \( \Pi_+ \), and, in particular, they do not belong to \( i\mathbb{R} \). Let \( z_0 \in \Pi_+ \) be a zero of \( G \). By Theorem 4.18, \( M_\zeta(\hat{G}\Theta^{-1}; z_0) \geq M_\zeta(G; z_0) \). But as \( z_0 \) is also a pole of \( \Theta \) with \( M_\zeta(\Theta; z_0) = M_\zeta(G; z_0) \), Lemma 4.4 implies that \( M_\zeta(\hat{G}\Theta^{-1}; z_0) = M_\zeta(G; z_0) \).

Therefore, by Definition 4.1, \( \Theta \) is a minimal zero conjugator w.r.t. \( \Pi_+ \) of the regular \( m \times m \) rmvf \( G_0 = \hat{G}\Theta^{-1} \). □

Next, following much the same procedure that was used in Subsection 4.2 (see Theorem 4.8), set

\[ G_1(z) = C(zI_n - A)^{-1}B \]

(4.9)

is minimal, then, in view of Theorem 3.3, there exists a positive semidefinite matrix \( X = \ast_{\text{ast}}(A, B) \) and the rmvf \( \Theta^{-\#}(z) \), normalized to be \( I_m \) at \( z = \infty \), is given by the formula

\[ \Theta^{-\#}(z) = -JB^*(zI_n + A^*)^{-1}XB + I_m = -JB^*X(zI_n - \hat{A})^{-1}B + I_m, \]

where \( \hat{A} = A - BJ^* B \) is anti-stable. Thus, by (4.4), we obtain the following conclusion:

**Theorem 4.21** Let \( G(z) \) be a full-rank \( p \times m \) rmvf with \( p < m \) that has no zeros on \( i\mathbb{R} \), and assume that \( \Theta(z) \) is a minimal zero conjugator of \( G(z) \) w.r.t. \( \Pi_+ \). Then, there exists a positive semidefinite matrix \( X = \ast_{\text{ast}}(A, B) \) such that

\[ \Theta(z) = -B^*X(zI_n - \hat{A})^{-1}BJ + I_m = -B^*(zI_n + A^*)^{-1}XBJ + I_m, \]

where \( A \) and \( B \) are taken from the realization (4.9) of \( G_1(z) \) and \( \hat{A} = A - BJ^* B \).

**Remark 4.22** In contrast to the case \( p \geq m \), a full-rank rmvf \( G \) with \( p < m \)
sometimes admits more than one minimal zero conjugator \( \Theta(z) \) w.r.t. \( \Pi_+ \) with \( \Theta(\infty) = I \). The next lemma and the example which follows it explain why.

**Lemma 4.23** Let \( G(z) \) be a \( p \times m \) rmv of full-rank \( r = \min(p, m) \). Then \( G(z) \) has a zero at \( z_0 \in \mathbb{C} \) if and only if every \( r \times r \) regular sub-matrix of \( G(z) \) has a zero at \( z_0 \).

**Proof.** Suppose first that \( p \leq m \), and let \( \hat{G}(z) \) be any \( p \times p \) regular sub-matrix of \( G(z) \). Then, since the columns of \( G(z) \) can be permuted without affecting the zero structure, we can assume that

\[
G(z) = \begin{bmatrix} \hat{G}(z) & V_1(z) & \cdots & V_k(z) \end{bmatrix}, \quad k = m - p.
\]

Since \( \hat{G}(z) \) is regular, every sub-matrix of \( G(z) \) that contains \( \hat{G}(z) \) is full-rank. Therefore, by Lemma 4.17,

\[
M_\zeta(G; z_0) \leq M_\zeta([\hat{G} & V_1 \cdots V_{k-1}]; z_0) \\
\leq \cdots \leq M_\zeta([\hat{G} & V_1]; z_0) \leq M_\zeta(\hat{G}; z_0).
\]

Thus, if \( G(z) \) has a zero at \( z_0 \), then so does \( \hat{G}(z) \).

In order to prove the converse, we use induction on \( p \). The case \( p = 1 \) is obvious. Suppose we have proved the claim for \( p = k \), and let \( G(z) \) be a full-rank \( (k + 1) \times m \) rmv, (with \( k + 1 \leq m \)) such that every \( (k + 1) \times (k + 1) \) regular sub-matrix of \( G(z) \) has a zero at \( z_0 \). Without loss of generality, assume that \( n(g_{ij}; z_0) \) attains its minimal value at \( g_{11} \). If \( n(g_{11}; z_0) > 0 \), we are done. If not, then by performing \( k \) elementary row operations \( R_i \leftrightarrow R_i - \frac{g_{ii}}{g_{i1}} R_1 \), \( 2 \leq i \leq k + 1 \) starting from \( G \), we get

\[
G(z) \bigotimes \hat{G}(z) = \begin{bmatrix}
    g_{11} & g_{12} & \cdots & g_{1m} \\
    0 & \ddots & & \\
    \vdots & & & \hat{G} \\
    0 & & & 
\end{bmatrix},
\]

where \( \hat{G}(z) \) is a full rank \( k \times (m - 1) \) rmv. Since every \( (k + 1) \times (k + 1) \) regular sub-matrix of \( \hat{G} \) has zero at \( z_0 \) and since \( n(g_{11}; z_0) \leq 0 \), it is not hard to show that every \( k \times k \) regular sub-matrix of \( \hat{G} \) has zero at \( z_0 \), and thus, by the induction assumption, \( \hat{G} \) has zero at \( z_0 \). By performing the elementary
column operations $C_j \to C_j - (g_{1j}/g_{11})C_1$ starting from $\tilde{G}$, we get that

$$
\tilde{G} \approx \begin{bmatrix}
g_{11} & 0 & \ldots & 0 \\
0 & \ddots & \cdots & \vdots \\
0 & \cdots & \ddots & 0 \\
\vdots & \cdots & \ddots & \tilde{G} \\
0 & \cdots & \ddots & 0
\end{bmatrix}
$$

and thus, $z_0$ is also a zero of $\tilde{G}$, i.e., a zero of $G$.

The case where $p \geq m$ follows by applying transposes. $\square$

In view of Lemma 4.23, a zero of a full-rank wide rmvf $G(z)$ can be removed by constructing a $\Theta(z)$ that removes it from only one regular $p \times p$ subblock. This, as the following example illustrates, may permit the existence of more than one normalized minimal zero conjugator in this setting.

**Example 4.24** Let $G(z) = \begin{bmatrix} 0 & \frac{z-1}{z+1} \\ \frac{z+1}{z-1} & \frac{1}{2} \end{bmatrix}$ and $J = \text{diag}\{1, -1, -1\}$. Then the proper $J$-inner rmvf's $\Theta_1(z) = \text{diag}\{1, \frac{z-1}{z+1}, 1\}$ and $\Theta_2(z) = \text{diag}\{1, 1, \frac{z+1}{z-1}\}$ are both minimal zero conjugators of $G(z)$ w.r.t. $\Pi_+$. 

**Remark 4.25** The nonuniqueness of the conjugator of full-rank wide $G$ may also be understood as follows: For regular rmvf's $G$ the minimal pole conjugators of $G$ w.r.t. $\Pi_+$ are in 1-1 correspondence with the minimal pole conjugators of $G^{-\#}(z)$ w.r.t. $\Pi_-$ and the latter are uniquely determined (up to normalization) by $G^{-\#}(z)$. For full-rank tall $G$ we made a reduction to the regular case, without changing the conjugators (see subsection 4.2). For full-rank wide $G$, we produce (see subsection 4.3) minimal zero conjugators of $G$ w.r.t. $\Pi_+$ from minimal pole conjugators of $G^{\#}(z)$ w.r.t. $\Pi_-$, where $\tilde{G}$ is a right inverse to $G$. Such right inverses are in general not unique.

### 4.6 Conjugators for rmvf not of full-rank

Let $G(z)$ be a $p \times m$ rmvf not of full-rank without zeros on $i\mathbb{R}$. There is a unimodular $W$ such that $WG = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}$ where $\tilde{G}$ is a full-rank wide rmvf.

Clearly, $\Theta(z)$ is a minimal zero conjugator of $G(z)$ w.r.t. $\Pi_+$ if and only if $\Theta(z)$ is a minimal zero conjugator of $\tilde{G}(z)$ w.r.t. $\Pi_+$, and this case can be analyzed by the results of subsections 4.3 and 4.4.

The distinction between the conclusions of tall and wide full-rank rmvf's $G(z)$
stems from the fact that the conjugator \( \Theta(z) \) multiplies \( G(z) \) on the right. If conjugation is defined by multiplying \( G(z) \) on the left by \( \Theta(z) \), then the conclusions for tall and wide mvf's would be reversed.

References


