DISCREPANCY ESTIMATE OF NORMAL VECTORS

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ABSTRACT. Let $A$ be an $s \times s$ invertible matrix with integer entries and with eigenvalues $|\lambda_i| > 1$, $i = 1, \ldots, s$. In this paper we prove explicitly that there exists a vector $\alpha$, such that the discrepancy of the sequence $\{\alpha A^n\}_{n=1}^{N}$ is equal to $O(N^{-1} (\log N)^{1/2} + 3)$ for $N \rightarrow \infty$. This estimate can be improved no more than on the logarithmic factor.

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1. Introduction

Let $(x_n)_{n\geq 0}$ be an infinite sequence of points in an $s$-dimensional unit cube $[0,1)^s$; $v = [0,\gamma_1] \times \cdots \times [0,\gamma_s]$ a box in $[0,1)^s$; and $J_v(N)$ a number of indexes $n \in [1,N]$ such that $x_n$ lies in $v$. The sequence $(x_n)_{n\geq 0}$ is said to be uniformly distributed in $[0,1)^s$ if for every box $v$, $J_v(N)/N \rightarrow \gamma_1 \ldots \gamma_s$. The quantity

$$D((x_n)_{n=1}^{N}) = \sup_{v \in [0,1]^s} \left| \frac{1}{N} J_v(N) - \gamma_1 \ldots \gamma_s \right|$$

is called the discrepancy of $(x_n)_{n=1}^{N}$.

In 1954 Roth (see [DrTi], [KN]) proved that for any sequence in $[0,1)^s$

$$\lim_{N \rightarrow \infty} N D(N)/\log^{s/2} N > 0.$$ (2)

Let $A$ be an $s \times s$ invertible matrix with integer entries. A matrix $A$ is said to be ergodic if for almost all $\alpha \in \mathbb{R}^s$ the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

A vector $\alpha \in \mathbb{R}^s$ is said to be normal ($A$ normal) if the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

Let $\lambda_i$ ($1 \leq i \leq s$) denote the eigenvalues of a matrix $A$. For the case of $|\lambda_i| > 1$, $i = 1, \ldots, s$ normal vectors were constructed by Postnikov ($s = 2$) and by Polosuev ($s \geq 2$) (see [Po]). Normal vectors were constructed for the general

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case of an ergodic matrix in [Le1]. The author [Le1] obtained also the following discrepancy estimate
\[ D \left( \{ \alpha A^n \}_{n=1}^N \right) = O \left( N^{-\frac{1}{2}} (\log N)^{s+3} \right). \]

In [Ko1], Korobov posed the problem of finding a function \( \psi(N) \) with maximum decay, such that there exists \( \alpha \) with
\[ D \left( \{ \alpha A^n \}_{n=1}^N \right) = O \left( \psi(N) \right), \quad \text{for} \quad N \to \infty. \]

The author [Le2] proved that \( \psi(N) = N^{-1} (\log N)^{2s+3} \) for the case of a diagonal ergodic matrix. In this paper we extend this result to the general case of an integer matrix with \( |\lambda_i| > 1, \quad i = 1, \ldots, s \). By (2) this result can be improved no more than on the logarithmic factor.

2. Construction and Auxiliary results

Let \( s \geq 2, \ p \geq 3 \) be a prime number, \( A \) an \( s \times s \) invertible matrix with integer entries, \( \lambda_1, \ldots, \lambda_s \) eigenvalues of the matrix \( A \), where \( |\lambda_i| > 1, \quad i = 1, \ldots, s \), \( q = \det A \). Let \( F_m \subset \mathbb{Z}^s \) be any complete set of coset representatives for the group \( \mathbb{Z}^s/A^{2k_0m}\mathbb{Z}^s, \ m = 1, 2, \ldots \). It is easy to see that \( \#F_m = q^{2k_0m} \). Let us take \( k_0 \) such that
\[ \min_{1 \leq i \leq s} \left( \frac{1 + |\lambda_i|}{2} \right)^{k_0} > p^{2s}. \]  

(3)

Now let
\[ n_1 = 0, \quad n_m = n_{m-1} + 3k_0(m-1)p^{m-1}, \quad m = 2, 3, \ldots, \]  

(4)

\[ \alpha = \sum_{m=1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \left\{ nb_{\nu,m} A^{-2k_0m} \right\} A^{-(n_{m} + k_0 m (3n+\nu))}, \]  

(5)

where \( b_{\nu,m} \in F_m \).

**Theorem.** There exists \( b_{\nu,m} \in F_m \) \( (m = 1, 2, \ldots \nu = 0, 1, 2) \) such that
\[ D \left( \{ \alpha A^n \}_{n=1}^N \right) = O \left( \log^{2s+3} N \right), \quad \text{for} \quad N \to \infty. \]

We prove this result in Section 3.

**Remark.** We will prove a similar result for the case of a hyperbolic matrix \( (|\lambda_i| \neq 1, \ i = 1, \ldots, s) \) in a forthcoming paper.
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Let \( f(x) = x^s - b_1x^{s-1} - \ldots - b_{s-1}x - b_s \) be a polynomial with roots \( \lambda_i, \lambda_i > 1, 1 \leq i \leq s \). Consider the following recurrence sequence

\[
\psi(n) = b_1\psi(n-1) + b_2\psi(n-2) + \ldots + b_s\psi(n-s),
\]

where \( \psi(i) = \alpha_i \) \( (i = 1, \ldots, s) \) and \( b_1, \ldots, b_s \) are integers.

**Corollary.** There exists \( \alpha = (\alpha_1, \ldots, \alpha_s) \) such that

\[
D\left(\left(\psi(n), \ldots, \psi(n + s - 1)\right)_{n=1}^N\right) = O\left(N^{-1}(\log N)^{2s+3}\right).
\]

**Proof.** Using the Theorem it follows instantly, if we will denote

\[
(\psi(n), \ldots, \psi(n + s - 1)) = \alpha A^{n-1},
\]

where \( A \) is the companion matrix of \( f(x) \).

We will need the following inequalities:

The Erdős-Turán-Koksma inequality (see [DrTi], p. 15):

\[
D\left(\{x_n\}_{n=0}^{N-1}\right) \leq \left(\frac{3}{2}\right)^s \left(\frac{2}{M+1} + \frac{1}{N} \sum_{0 < \max|m_i| \leq M} \frac{\left|\sum_{n=0}^{N-1} e(\langle m, x_n \rangle)\right|}{m_1 \ldots m_s}\right), \quad (6)
\]

where \( e(y) = \exp(2\pi iy) \), \( x_n = (x_{n,1}, \ldots, x_{n,s}) \), \( m = (m_1, \ldots, m_s) \), \( \overline{m} = \max(1,|m_i|) \), and \( < (a_1, \ldots, a_s), (b_1, \ldots, b_s) >= a_1b_1 + \ldots + a_sb_s \).

**Lemma A** (see [Ko2, p. 1]). Let \( \beta \) be a real number, \( M \) and \( N \) natural, then

\[
\left|\sum_{n=M}^{M+N-1} e(n\beta)\right| \leq \min\left(N, \frac{1}{2\|\beta\|}\right),
\]

where \( \|\beta\| = \min(\{\beta\}, 1 - \{\beta\}) \) and \( \{\beta\} \) is the fractional part of \( \beta \).

**Lemma B** (see [Ko2, p. 72]). Let \( P \geq 2, (a, P) = 1 \), then for any real \( \varphi \)

\[
\sum_{n=1}^{P} \min\left(P, \frac{1}{an/P + \varphi}\right) \leq 8P(1 + \log P).
\]

**Lemma C** (see [Ko2, p. 2]). Let

\[
\delta_q(a) = \begin{cases} 
1, & \text{if } a \equiv 0 \pmod{q} \\
0, & \text{else} 
\end{cases}
\]

where \( q \geq 1, a \in \mathbb{Z} \). Then

\[
\delta_q(a) = \frac{1}{q} \sum_{x=1}^{q} e\left(\frac{ax}{q}\right).
\]
3. Proof of the Theorem

**Lemma 1.** Let $0 < |m| \leq p^j$, $0 \leq l < k_0j$. Then there exists $j_0 > 0$, such that for all $j \geq j_0$, we have

\[
\langle m, b_0A^{l-2k_0j} \rangle > \max_{1 \leq i \leq s} \frac{|b_{0i}|}{p^j},
\]

where $b_0 = (b_{01}, \ldots, b_{0s})$, $m = (m_1, \ldots, m_s)$.

**Proof.** Let us denote

\[
\lambda_{\min} = \min_{1 \leq i \leq s} |\lambda_i|, \quad \lambda_0 = \frac{\lambda_{\min} + 1}{2}, \quad a_{ik} = a_{ik}(t) = (A^{-t})_{ik}, \quad 1 \leq i, k \leq s.
\]

Using Jordan’s form of the matrix $A$, we obtain

\[
a_{ik}(t) = o \left( \lambda_0^{-t} \right),
\]

(7)

Bearing in mind (3), we obtain

\[
\lambda_0^{-j_0} < p^{-2j}, \quad a_{ik}(-l + 2k_0j) = o(p^{-2j}), \quad i, k = 1, \ldots, s.
\]

(8)

Hence

\[
\langle m, b_0A^{l-2k_0j} \rangle = \left| \sum_{i,k=1}^{s} m_i b_{0k} (A^{l-2k_0j})_{ki} \right| = o \left( p^{-j} \max_{1 \leq k \leq s} |b_{0k}| \right).
\]

Lemma 1 is proved. □

**Lemma 2.** Let $0 < |m| \leq p^j$, $m = (m_1, \ldots, m_s)$, $j \geq j_0$, $0 \leq l < k_0j$.

\[G(m, j) = \{ \langle m, bA^{l-2k_0j} \rangle \mid b \in F_j \} \]

and $v_0 = \#G(m, j)$.

Then

\[G(m, j) = \left\{ \frac{\mu}{v_0}, \quad 0 \leq \mu < v_0 \right\}
\]

with

\[\#G(m, j) > p^j,
\]

where $\#G(m, j)$ is the number of elements of $G(m, j)$.

**Proof.** Bearing in mind that $A$ is an integer matrix, we get

\[
\{ \langle m, bA^{l-2k_0j} \rangle \} = \frac{\mu}{v_1(b)},
\]

where $\mu = \mu(b) \geq 0$, $v_1(b)$ are integer numbers. Let $v_2 = \max_{b \in F_j} v_1(b)$ and let

\[
\{ \langle m, b_0A^{l-2k_0j} \rangle \} = \frac{\mu}{v_2}, \quad (\mu, v_2) = 1
\]

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for some \( b_0 \in F_j \). Taking into account that \( G(m,j) \) is the group, we obtain

\[
\{ (nm, b_0 A^{l-2k\sigma_j}) \} = \left\{ \frac{n\mu}{v_2} \right\} \in G(m,j), \quad 0 \leq n \leq v_2 - 1.
\]

Hence, there exists an integer \( n_0 \) with \( \{ n_0 m, b_0 A^{l-2k\sigma_j} \} = 1/v_2 \) and also \( \{ n/v_2 \} \in G(m,j) \) for all \( n \in [0, v_2) \). Suppose that there exists \( b \in F_j \) with \( \mu(b)/v_1(b) \notin \{ 0, 1/v_2, \ldots, (v_2 - 1)/v_2 \} \). This means that \( v_1(b) \nmid v_2 \) and \( v_1(b) < v_2 \).

Therefore there exists \( d \geq 1 \) such that \( d/v_1(b) \) and \( (d, v_2) = 1 \). Bearing in mind that \( \{ h\mu/v_1(b) \} \in G(m,j) \) \( (0 \leq h < v_1(b)) \), we have \( \{ h/d \} \in G(m,j) \) for \( h \in [0, v_1(b)) \).

Hence

\[
\left\{ \frac{n}{v_2} + \frac{h}{d} \right\} \in G(m,j) \quad (0 \leq n < v_2, \ 0 \leq h < d) \quad \text{and} \quad \left\{ \frac{l}{v_2d} \right\} \in G(m,j)
\]

for \( l \in [0, v_2d - 1] \). But \( v_2 = \max_{b \in F_j} v_1(b) \). We have the contradiction. Then, \( \gamma = (\gamma_1, \ldots, \gamma_s) = mA^{l-2k\sigma_j} \).

We have that there exist integers \( c_1, \ldots, c_s \geq 0 \) with

\[
|\langle m, b_0, A^{l-2k\sigma_j} \rangle| = |\langle \gamma, b_0, i \rangle| = |\gamma_i| = \frac{c_i}{v_2}.
\]

According to Lemma 1, \( c_i/v_2 < 1/p^j \), \( i = 1, \ldots, s \). Taking into account that \( |m| > 0 \), we obtain \( \gamma \neq 0 \). Therefore there exists \( i_0 \) with \( c_{i_0} > 0 \). Thus \( v_2 > p^j \).

Lemma 2 is proved.

**Lemma 3.** Let \( \varphi \) be a real number, \( 0 < |m| \leq p^j \), \( m = (m_1, \ldots, m_s) \), \( j \geq j_0 \). Then

\[
\sigma_1(j) := \frac{1}{q^{2k\sigma_j}} \sum_{b \in F_j} \min \left( p^j \frac{1}{2 \| (m, bA^{l-2k\sigma_j}) + \varphi \|} \right) = O(j),
\]

where the \( O \)-constant does not depend on \( \varphi, m, \) and \( l \).

**Proof.** Bearing in mind that \( b \rightarrow \{ (m, b A^{l-2k\sigma_j}) \} \), where \( b \in F_j \), is a group homomorphism, we get

\[
\# \{ b \in F_j \mid \{ (m, b A^{l-2k\sigma_j}) \} = g \} = \# \{ b \in F_j \mid \{ (m, b A^{l-2k\sigma_j}) \} = 0 \}
\]

for all \( g \in G(m,j) \). Hence, using Lemma 2, we obtain

\[
\frac{1}{q^{2k\sigma_j}} \sum_{b \in F_j} \min \left( p^j, \frac{1}{2 \| (m, b A^{l-2k\sigma_j}) + \varphi \|} \right) = \frac{1}{v_0} \sum_{\mu=1}^{v_0} \min \left( p^j, \frac{1}{2 \| \mu/v_0 + \varphi \|} \right),
\]

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Applying Lemma B, we have
\[
\sigma_1(j) \leq \frac{1}{v_0} \sum_{\mu=1}^{m_0} \min \left\{ v_0, \frac{1}{2\|\mu/v_0 + \varphi\|} \right\} \leq 8(1 + \log v_0).
\]

Taking into account that \( v_0 = \#G(m, j) \leq \#F_j = q^{2k_0j} \), we obtain the assertion of the lemma.

Let us denote
\[
S_0(m, l, R, b_{0,j}, b_{1,j}, b_{2,j}) = \sum_{n=0}^{R-1} e \left( \langle m, b_{2,j}(n-1)A^{l-k_0j} + b_{0,j}nA^{l-2k_0j} + \{b_{1,j}nA^{-2k_0j}\}A^{l-k_0j}\rangle \right),
\]

\[
S_1(m, l, R, b_{0,j}, b_{1,j}, b_{2,j}) = \sum_{n=0}^{R-1} e \left( \langle m, b_{0,j}nA^{l-k_0j} + b_{1,j}nA^{l-2k_0j} + \{b_{2,j}nA^{-2k_0j}\}A^{l-k_0j}\rangle \right),
\]

\[
S_2(m, l, R, b_{0,j}, b_{1,j}, b_{2,j}) = \sum_{n=0}^{R-1} e \left( \langle m, b_{1,j}nA^{l-k_0j} + b_{2,j}nA^{l-2k_0j} + \{b_{0,j}(n+1)A^{-2k_0j}\}A^{l-k_0j}\rangle \right),
\]

\[
\beta_0 = \langle m, b_{2,j}A^{l-k_0j}\rangle + \langle m, b_{0,j}A^{l-2k_0j}\rangle - \frac{1}{q^{2k_0j}}(z, b_{1,j}q^{2k_0j}A^{-2k_0j}),
\]

\[
\beta_1 = \langle m, b_{0,j}A^{l-k_0j}\rangle + \langle m, b_{1,j}A^{l-2k_0j}\rangle - \frac{1}{q^{2k_0j}}(z, b_{2,j}q^{2k_0j}A^{-2k_0j}),
\]

\[
\beta_2 = \langle m, b_{1,j}A^{l-k_0j}\rangle + \langle m, b_{2,j}A^{l-2k_0j}\rangle - \frac{1}{q^{2k_0j}}(z, b_{0,j}q^{2k_0j}A^{-2k_0j}).
\]

**Lemma 4.** For \( \nu = 0, 1, 2 \), we have
\[
|S_{\nu}(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})| \leq \sum_{z_1, \ldots, z_{\nu}=0}^{q^{2k_0j}-1} \min \left( \frac{1}{2\|\beta\|}, \frac{1}{2\|\beta\|} \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0j}\|m_1a_{i1} + \ldots + m_s a_{is} + z_i\|} \right),
\]

where \( (a_{ij})_{i,j=1}^{s} = A^{l-2k_0j} \).

**Proof.** It is easy to see that \( A^{-1} = B_0/\det A \), where \( B_0 \) is an integer matrix, so \( A^{-2k_0j} = B_1/q^{2k_0j} \), where \( B_1 \) is an integer matrix. Let \( b_1 \in F_j, n \in \mathbb{Z} \) and
$k \equiv nq^{2k_0j}b_1A^{-2k_0j} \ (mod q^{2k_0j}Z^*)$ with $k = (k_1, \ldots, k_s) \in Z^* \cap [0, q^{2k_0j})^s$. Hence
\[ \{nb_1A^{-2k_0j} \} = k/q^{2k_0j}. \]

Let $\nu = 0$. Removing a fractional part, we get
\[
S_0(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j}) = \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} \sum_{n=0}^{p^j - 1} \sum_{z_1, \ldots, z_s = 0} \nu(m, b_{2,j}(n-1)A^{l-k_0j} + b_{0,j}nA^{l-2k_0j} + \frac{k}{q^{2k_0j}}A^{l-k_0j})
\]
\[ \times \prod_{i=1}^s \delta_{q^{2k_0j}}(k_i - (b_{1,j}q^{2k_0j}A^{-2k_0j})n). \]

By Lemma C, we have
\[
|S_0(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})| = \frac{1}{q^{2k_0j}s} \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} e \left( \nu(m, b_{2,j}(n-1)A^{l-k_0j}) \right)
\]
\[
+ b_{0,j}nA^{l-2k_0j} + \frac{k}{q^{2k_0j}}A^{l-k_0j})e((z, \frac{k - b_{1,j}q^{2k_0j}A^{-2k_0j}n}{q^{2k_0j}}))
\]
\[ \times \prod_{n=0}^{p^j - 1} \nu \left( m, b_{2,j}A^{l-k_0j} + m, b_{0,j}A^{l-2k_0j} - \frac{1}{q^{2k_0j}}(z, b_{1,j}q^{2k_0j}A^{-2k_0j}) \right) \]
\[ = \frac{1}{q^{2k_0j}s} \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} \sum_{z_1, \ldots, z_s = 0} \nu \left( \nu(n, b_{2,j}A^{l-k_0j}) + \nu(m, b_{0,j}A^{l-2k_0j}) - \frac{1}{q^{2k_0j}}(z, b_{1,j}q^{2k_0j}A^{-2k_0j}) \right) \]
\[ \leq \frac{1}{q^{2k_0j}s} \sum_{z_1, \ldots, z_s = 0} \nu(\nu(n, b_{0,j})) \sum_{n=0}^{p^j - 1} \nu(\nu(n, b_{2,j})), \quad (12) \]

where
\[ \nu = \nu(m, kA^{l-k_0j}) - \frac{1}{q^{2k_0j}}(z, b_{2,j}A^{l-k_0j}) + \frac{1}{q^{2k_0j}}(z, k) \]
and $\beta_0$ defined in (10).

Let us approximate the following

$$\sigma := \left| \frac{1}{q^{2k_0}j} \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} e(\varphi_0) \right| = \left| \frac{1}{q^{2k_0}j} \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} e \left( \frac{1}{q^{2k_0}j} \left( \langle m, kA^{l-k_0} \rangle + \langle z, k \rangle \right) \right) \right|.$$  

We have for $(a_{ij})_{1 \leq i, j \leq s}$ that

$$\sigma = \left| \frac{1}{q^{2k_0}j} \sum_{k_1, \ldots, k_s = 0}^{q^{2k_0j} - 1} e \left( \frac{1}{q^{2k_0}j} \sum_{i=1}^{s} \left( k_i(m_1a_{i1} + \ldots + m_sa_{is} + z_i) \right) \right) \right|.$$  

Using Lemma A, we get

$$\sigma \leq \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0}j \| \frac{m_1a_{i1} + \ldots + m_sa_{is} + z_i} {q^{2k_0}j} \|} \right)$$  

and

$$\left| p^{l-1} \sum_{n=0}^{p^l} e(n\beta_0) \right| \leq \min \left( p^l, \frac{1}{2 \| \beta_0 \|} \right).$$

By (12) we obtain

$$|S_0(m, l, p^l - 1, b_0, j, b_1, j, b_2, j)| \leq \sum_{z_1, \ldots, z_s = 0}^{q^{2k_0j} - 1} \min \left( p^l, \frac{1}{2 \| \beta_0 \|} \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0}j \| \frac{m_1a_{i1} + \ldots + m_sa_{is} + z_i} {q^{2k_0}j} \|} \right).$$

So, (11) is proved. In the same way we will get the inequalities for $\nu = 1$ and $\nu = 2$. Lemma 4 is proved.

**Lemma 5.** For $\nu = 0, 1, 2$ and $R \in [0, p^l)$, we have

$$|S_\nu(m, l, R, b_0, j, b_1, j, b_2, j)| \leq \sum_{m_{s+1} = -[p^l/2]}^{[p^l/2]} \frac{1}{m_{s+1}} \times \sum_{z_1, \ldots, z_s = 0}^{q^{2k_0j} - 1} \min \left( p^l, \frac{1}{\| \beta_0 \|} \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0}j \| \frac{m_1a_{i1} + \ldots + m_sa_{is} + z_i} {q^{2k_0}j} \|} \right).$$  

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Proof. By the same way, as in Lemma 4 we get

\[ |S_0(m, l, R, b_{0,j}, b_{1,j}, b_{2,j})| = \frac{1}{q^{2k_0j}} \left| \sum_{k_1, \ldots, k_s=0}^{q^{2k_0j} - 1} \sum_{z_1, \ldots, z_s=0}^{q^{2k_0j} - 1} e(\varphi_0) \sum_{n=0}^{R-1} e(n\beta_0) \right| , \]

where \( \varphi_0 \) defined in (13) and \( \beta_0 \) defined in (10).

Applying Lemma C, we have

\[ R - \sum_{n=0}^{R-1} e(pj) = \frac{1}{p^j} \sum_{m_{s+1}=[p^j/2]}^{[p^j/2]} R - \sum_{n=0}^{R-1} e(n\beta_0 + \varphi_0) \frac{m_{s+1}n}{p^j} \]

According to [Ni, p. 35]

\[ \frac{1}{p^j} \sum_{n=0}^{R-1} e \left( \frac{m_{s+1}n}{p^j} \right) \leq \frac{1}{m_{s+1}}, \quad 1 \leq R \leq p^j. \]

Now the proof of the following inequality is the same as that of [Ko2, p. 13].

\[ R - \sum_{n=0}^{R-1} e \left( n\beta_0 + \varphi_0 \right) \frac{m_{s+1}n}{p^j} \]

Therefore

\[ |S_0(m, l, R, b_{0,j}, b_{1,j}, b_{2,j})| \leq \frac{1}{q^{2k_0j}} \times \frac{1}{m_{s+1}} \left| \sum_{k_1, \ldots, k_s=0}^{q^{2k_0j} - 1} \sum_{z_1, \ldots, z_s=0}^{q^{2k_0j} - 1} e(\varphi_0) \sum_{n=0}^{R-1} e(n\beta_0 + \varphi_0) \frac{m_{s+1}n}{p^j} \right| . \]

Using Lemma A, (13), and (14), we get
\[ |S_0(m, l, R, b_{0,j}, b_{1,j}, b_{2,j})| \leq \sum_{m+s = -\lfloor \frac{p^j}{2} \rfloor}^{\lfloor \frac{p^j}{2} \rfloor} \frac{1}{m+s+1} \]

\[ \times \sum_{z_1, \ldots, z_s=0}^{q^{2k_0j}-1} \min \left( \frac{p^j}{2} \left\| \hat{\beta}_0 + \frac{m_{s+1}}{m_s} \right\| \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0j} \left\| m_{s+1} + \ldots + m_1 + z_i \right\|} \right). \]

Hence (15) is proved for \( \nu = 0 \). In the same way, we will get (15) for \( \nu = 1 \) and \( \nu = 2 \). Lemma 5 is proved.

Let

\[ S^{(1)}(m, j) = \sum_{m=0}^{2} \frac{1}{m_1 \ldots m_{s}} \]

\[ \times \sum_{z_1, \ldots, z_s=0}^{q^{2k_0j}-1} \min \left( \frac{p^j}{2} \left\| \hat{\beta}_0 + \frac{m_{s+1}}{m_s} \right\| \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0j} \left\| m_{s+1} + \ldots + m_1 + z_i \right\|} \right), \]

\[ S^{(2)}(m, j) = \sum_{m+s = -\lfloor \frac{p^j}{2} \rfloor}^{\lfloor \frac{p^j}{2} \rfloor} \frac{1}{m_1 \ldots m_{s+1}} \]

\[ \times \sum_{z_1, \ldots, z_s=0}^{q^{2k_0j}-1} \min \left( \frac{p^j}{2} \left\| \hat{\beta}_0 + \frac{m_{s+1}}{m_s} \right\| \right) \prod_{i=1}^{s} \min \left( 1, \frac{1}{2q^{2k_0j} \left\| m_{s+1} + \ldots + m_1 + z_i \right\|} \right), \]

\[ T_i(b_{0,j}, b_{1,j}, b_{2,j}) = \sum_{l=0}^{k_0j-1} S^{(i)}(m, j), \]

where \( M = \lfloor \frac{p^j}{2} \rfloor, i = 1, 2. \)

**Lemma 6.** Let us take \( b_{0,j}, b_{1,j}, b_{2,j} \) so that

\[ T_1(b_{0,j}, b_{1,j}, b_{2,j}) + T_2(b_{0,j}, b_{1,j}, b_{2,j})/j \]

will be minimal. Then

\[ T_1(b_{0,j}, b_{1,j}, b_{2,j}) = O(j^{2s+2}) \]

and

\[ T_2(b_{0,j}, b_{1,j}, b_{2,j}) = O(j^{2s+3}). \]

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**Proof.** Consider the mean values
\[
\tilde{T}_i = \frac{1}{q^{k_{0,j}} b_0, b_1, b_2 \in F_j} T_i(b_0, b_1, b_2), \quad i = 1, 2. \tag{18}
\]

It is easy to see that Lemma 6 goes after from the following assertion
\[
\tilde{T}_1 + \tilde{T}_2/j = O(j^{2s+2}). \tag{19}
\]

Let
\[
\sigma_1 = \frac{1}{q^{k_{0,j}} b_0, b_1, b_2 \in F_j} \sum \min \left( p', \frac{1}{2\|\beta\|} \right).
\]

According to (10) and Lemma 3, we have
\[
\sigma_1 = \frac{1}{q^{k_{0,j}} b_0, b_1, b_2 \in F_j} \sum \min \left( p', \frac{1}{2\|m, b_0 A l - 2k_{0,j}^2 + \varphi(z, b_0, b_2) \|} \right) = O(j),
\]
where the \(O\)-constant does not depend on \(z, l\).

By (16) and (17) we obtain
\[
\tilde{T}_1 = O \left( \sum_{0 < \max |m_i| \leq M} \frac{j^2}{m_1 \ldots m_s} \sum_{z_1, \ldots, z_s = 0} q^{2k_{0,j}} \prod_{i=1}^s \min \left( 1, \frac{j^2}{q^{2k_{0,j}}} \| \frac{1}{q^{2k_{0,j}}} m_1 a_1 + \ldots + m_s a_s + z_i \| \right) \right).
\]

Using Lemma B, we get:
\[
\sum_{z_1, \ldots, z_s = 0} q^{2k_{0,j}} \prod_{i=1}^s \min \left( 1, \frac{j^2}{q^{2k_{0,j}}} \| \frac{1}{q^{2k_{0,j}}} m_1 a_1 + \ldots + m_s a_s + z_i \| \right) \leq (8(1 + 2 \log(q^{2k_{0,j}}))^s = O \left( (\log q^{2k_{0,j}})^s \right) = O(j^s),
\]

and we have
\[
\sum_{0 < \max |m_i| \leq p'/2} \frac{1}{m_1 \ldots m_s} \leq (3 + 2 \log p')^s = O(j^s).
\]

Thus
\[
\tilde{T}_1 = O(j^{2s+2}). \tag{20}
\]

Approximation for \(\tilde{T}_2\) is the same as for \(\tilde{T}_1\). So we get
\[
\tilde{T}_2 = O(j^{2s+3}). \tag{21}
\]

Now from (18), (20) and (21), we get (19) and the assertion of the lemma. \(\square\)

We will use vectors \(b_{0,j}, b_{1,j}, b_{2,j} (j = 1, 2, \ldots)\) in (5).
Completion of the proof of the Theorem

Let us decompose our interval \([1, N]\) into subintervals:

\[[1, n_2), [n_2, n_3), \ldots, [n_{r-1}, n_r), [n_r, N],\]

where \(n_{r+1} > N \geq n_r\) (see (4)). Hence by (4)

\[4k_0rp^r > N \geq (r - 1)p^{r-1}.\]  \hspace{1cm} (22)

Let us take a full interval, i.e., \(k \in [n_j, n_{j+1})\), where \(j = 2, \ldots, r - 1,\)
\[k = n_j + k_0j(3n^* + \nu^*) + l^*, \hspace{0.5cm} 0 \leq l^* < k_0j, \hspace{0.5cm} 0 \leq n^* \leq p^j - 1, \hspace{0.5cm} 0 \leq \nu^* \leq 2.\]  \hspace{1cm} (23)

By (5) we have:
\[
\alpha A^k = \sum_{m=1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \{nb_{v,m}A^{-2k_0m}\} A^{k-(n_m+k_m(3n+\nu))} \\
= \sum_{m=1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \{nb_{v,m}A^{-2k_0m}\} A^{n_j+k_0j(3n^*+\nu^*)+l^*-(n_m+k_m(3n+\nu))}.
\]

Let
\[R_{\nu^*, n^*} = \sum_{m=j}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \{nb_{v,m}A^{-2k_0m}\} A^{n_j+k_0j(3n^*+\nu^*)+l^*-(n_m+k_m(3n+\nu))}.
\]

We have, for example, for \(\nu^* = 0\)
\[
R_{0, n^*} = \sum_{n=n^*+1}^{p^j-1} \{nb_{0,j}A^{-2k_0j}\} A^{3k_0jn^*+l^*-3k_0jn} \\
+ \{nb_{1,j}A^{-2k_0j}\} A^{3k_0jn^*+l^*-(3k_0jn+k_0j)} \\
+ \sum_{n=n^*}^{p^j-1} \{nb_{2,j}A^{-2k_0j}\} A^{3k_0jn^*+l^*-(3k_0jn+2k_0j)} \\
+ \sum_{m=j+1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \{nb_{v,m}A^{-2k_0m}\} A^{n_j+3k_0jn^*+l^*-(n_m+k_m(3n+\nu))}.
\]

Bearing in mind that
\[
\{nb_{v,m}A^{-2k_0m}\} A^{n_j+k_0j(3n^*+\nu^*)+l^*-(n_m+k_m(3n+\nu))} = 0
\]
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for \(n_j + k_0j(3n^* + \nu^*) + l^* - (n_m + k_0m(3n + \nu)) \geq 2k_0m\), we obtain

\[
\{\alpha A^k\} = \{f_{\nu^*,n^*} + R_{\nu^*,n^*}\},
\]

(24)

for \(n^* \neq 0\) and \(n^* \neq p^j - 1\), where

\[
f_{0,n^*} = \{b_2(n^* - 1)A^{-2k_0j}\}A^{l^*+k_0j} + \{b_0n^*A^{-2k_0j}\}A^{l^*-k_0j},
\]

\[
f_{1,n^*} = \{b_0n^*A^{-2k_0j}\}A^{l^*+k_0j} + \{b_1n^*A^{-2k_0j}\}A^{l^*-k_0j},
\]

\[
f_{2,n^*} = \{b_1(n^* - 1)A^{-2k_0j}\}A^{l^*+k_0j} + \{b_2n^*A^{-2k_0j}\}A^{l^*-k_0j}.
\]

Let us approximate \(R_{0,n^*}\). By (7) and (23) we have

\[
R_{0,n^*} = O\left(\sum_{n=n^*+1}^{p^j-1} \lambda^j_0 n^{j+2l} - 3k_0n\right)
\]

\[+
\sum_{n=n^*}^{p^j-1} \lambda^j_0 n^{j+2l} - 3k_0n + \sum_{m=j+1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^2 \lambda_a n_j + 3k_0n^{j+2l} - (n_m + k_0m(3n + \nu))\right).
\]

According to (4) and (8), we obtain

\[
R_{\nu^*,n^*} = O(\lambda_0^{-k_0j} + \sum_{n \geq 0} \lambda_0^{-k_0j-n}) = O(\lambda_0^{-k_0j}) = O(p^{-2j})
\]

for \(\nu^* = 0\). We have the same estimate for \(\nu^* = 1, 2\).

Using the inequality

\[
|e(x) - 1| = |2\sin(\pi x)| \leq 2\pi|x|,
\]

we get

\[
|e(< m, f_{\nu^*,n^*} + R_{\nu^*,n^*} >) - e(< m, f_{\nu^*,n^*} >)| = |e(< m, R_{\nu^*,n^*} > - 1)|
\]

\[\leq 2\pi |< m, R_{\nu^*,n^*} >| \leq 2\pi |m||R_{\nu^*,n^*}| \leq 2\pi sp^j |R_{\nu^*,n^*}| = O(p^{-j}).
\]

By (1) we have the trivial estimate

\[
LD\left(\left(\frac{x_n}{h_n}\right)^{j+L-1}\right) \leq L, \quad L = 1, 2, \ldots
\]

Hence by (23), (24) and (6), we get

\[
(n_j + 1 - n_j)D(\{\alpha A^k\})^{n_j+1-1}_{k=n_j}
\]

\[\leq \sum_{l^*=0}^{k_0j-1} \sum_{\nu^*=0}^2 \left(p^j - 2\right)D((f_{\nu^*,n^*} + R_{\nu^*,n^*}))^{p^j-2}_{n^*=1 + 2}\]

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\[
\leq \sum_{l^* = 0}^{k_0 j - 1} \sum_{\nu^* = 0}^{2} \left( p^j D((f_{\nu^* n^*} + R_{\nu^* n^*})|_{n^* = 0}^{n^* - 1} + 4) \right) \leq \left( \frac{3}{2} \right)^s
\]

\[
\times \left( \frac{p^j}{M} + 12k_0 j + \sum_{\nu^* = 0}^{k_0 j - 1} \sum_{l^* = 0}^{2} \sum_{0 < \max|m_i| \leq M} \left| \sum_{n^* = 0}^{p^j - 1} e(< m, (f_{\nu^* n^*} + R_{\nu^* n^*})>) \right| \right)
\]

\[
= O \left( j + \sum_{0 < \max|m_i| \leq M} \sum_{l^* = 0}^{k_0 j - 1} \sum_{\nu^* = 0}^{2} \left| \sum_{n^* = 0}^{p^j - 1} e(< m, f_{\nu^* n^*}>) \right| + 1 \right)
\]

with \( M = \lfloor p^j/2 \rfloor \). According to (9), we obtain

\[
(n_{j+1} - n_j) D({\alpha A^k}_{k = n_j}^{n_{j+1} - 1}) = O(j^{2s+2})
\]

Applying Lemma 4, (16), (17) and Lemma 6, we get

\[
(n_{j+1} - n_j) D({\alpha A^k}_{k = n_j}^{n_{j+1} - 1}) = O(j^{2s+2})
\]

for any full interval \([n_j, n_{j+1}], 2 \leq j \leq r - 1\).

Consider the not full interval \([n_r, N]\). Using Lemma 5 and Lemma 6 we get, similarly, that

\[
(N - n_r + 1) D({\alpha A^k}_{k = n_r}^{N}) = O(r^{2s+3}).
\]

So, finally, we have the following:

\[
ND \left( {\alpha A^k}_{k = 1}^{N} \right) \leq (n_2 - 1) D({\alpha A^k}_{k = 1}^{n_2}) + \sum_{j=2}^{r-1} (n_{j+1} - n_j) D({\alpha A^k}_{k = n_j}^{n_{j+1}}) + (N - n_r + 1) D({\alpha A^k}_{k = n_r}^{N})
\]

\[
= O(1) + \sum_{j=2}^{r-1} O(j^{2s+2}) + O(r^{2s+3}) = O(r^{2s+3}).
\]

Now by (22), we obtain

\[
D({\alpha A^k}_{k = 1}^{N}) = O \left( N^{-1} (\log N)^{2s+3} \right).
\]

The Theorem is proved. \(\square\)
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