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THE UNIFORM DISTRIBUTION OF FRACTIONAL PARTS OF RECURRENT SEQUENCES

M. B. Levin and I. E. Shparlinskii

Let \( f(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - \ldots - a_0 \) be a polynomial with integer coefficients of degree \( n > 1 \) and reducible over \( \mathbb{Z} \), let \( \lambda_1, \ldots, \lambda_n \) be the roots of \( f(\lambda) \) and \( D \) the discriminant.

We consider the recurrence equation with the characteristic polynomial \( f(\lambda) \):

\[
\Psi(x + n) = a_{n-1}\Psi(x + n - 1) + \ldots + a_0\Psi(x) \quad (x = 1, 2, \ldots).
\]

Let \( (\omega(x)) \) (for \( x = 1, 2, \ldots \)) be some sequence of real numbers satisfying (1), with the initial conditions \( \omega(v) = \omega_v \) for \( v = 1, \ldots, n \).

If \( 0 < \sigma < 1 \) and \( \rho \geq 1 \) is an integer, let \( N_\sigma(P) \) be the number of solutions of the inequality

\[
\{\omega(x)\} < \sigma, \quad \text{for} \quad x = 1, \ldots, P.
\]

The quantity \( N_\sigma(P) \) has been most extensively studied in the case \( n = 1 \), that is, when \( \omega(x) = \omega_1 \chi^x \) is an exponential function with an integer base. For example, Korobov [1] has constructed, for prime \( \lambda \), a \( \omega_1 \) such that

\[
N_\sigma(P) = aP + O(P^{1/3} \log P^{4/3}).
\]

In [2] this result was extended to any integer \( \lambda_1 > 1 \). For \( n > 1 \) examples of uniformly distributed sequences were given in [3], [4], and [5]. It follows from [6] that for almost all initial conditions

\[
\limsup_{P \to \infty} \max \{ N_\sigma(P) - aP / P^{1/3} \} > 0.
\]

A more detailed survey can be found in [7].

In this note we construct, for any \( n > 1 \) and \( \lambda_1, \ldots, \lambda_n \) (for \( x = 1, 2, \ldots, n \)), initial conditions \( \omega_1, \ldots, \omega_n \) such that the sequence \( (\omega(x)) \) (for \( x = 1, 2, \ldots \)) satisfies (2).

We denote by \( \Gamma_k \) the exponent of \( q_k \) modulo \( p_k \) (for \( k = 1, 2, \ldots \)). Because the polynomials \( f(\lambda) \) and \( \lambda^{T_1} - 1 \) are relatively prime, it follows that there is a \( \beta \) such that \( T_1 = \ldots = T_\beta \neq T_{\beta+1} \).

LEMMA 1. For an integer \( k > \beta \), \( \Gamma_k = \Gamma \cdot \chi^{p_k - \beta} \).

The proof follows from Lemma 1 of [8].

Let \( n_0 = 0, n_{k+1} = n_k + [p_k^{k/2} k^2] \cdot \Gamma_k \) (for \( k = 1, 2, \ldots \)), and suppose that only the first \( m \) of the roots \( \lambda_1, \ldots, \lambda_n \) are of modulus greater than 1. For integers \( a_{i, r} \in \{0, p^r\} \) we set

\[
A_{\nu, k} = \sum_{r=1}^{k} \lambda_\nu^{-n_r} \chi^{p_k - r} \sum_{i=1}^{n_r} a_i k^{\lambda_\nu^{-1}} \quad (\nu = 1, \ldots, m; \quad k = 1, 2, \ldots).
\]

LEMMA 2. There exist integers \( a_{i, r} \in \{0, p^r\} \) such that

\[
A_{1, 1} + \ldots + A_{n, 1} \equiv q_1^1 \left( \text{mod } p^r \right) \quad (i = 1, \ldots, n; \quad r = 1, 2, \ldots).
\]

If the integers \( a_{i, j} \) (for \( i = 1, \ldots, n \) and \( j = 1, \ldots, r - 1 \)) are defined, then for the choice of \( a_{i, r} \) (for \( i = 1, \ldots, n \)) we obtain a system of \( n \) linear congruences with the determinant \( a_0 D = 0 \) (mod \( p \)).

Suppose that the \( a_{i, r} \) are chosen as indicated in Lemma 2. We set

\[
\theta_v = \sum_{r=1}^{\infty} \lambda_v^{-n_r} \chi^{p_r - r} \sum_{i=1}^{n_r} a_i k^{\lambda_v^{-1}} \quad (\nu = 1, \ldots, m),
\]

\[
\omega_v = \theta_1 \lambda_1^{-1} + \ldots + \theta_m \lambda_m^{-1} \quad (\nu = 1, \ldots, n).
\]

LEMMA 3. Suppose that \( \lambda_v \neq 1 \) for \( \nu = 1, \ldots, n \); let \( \omega(x) \) (for \( x = 1, 2, \ldots \)) be a sequence satisfying (1) with the initial conditions (4). Then
\{ \omega (n r + x) \} = \{ q r^2 + o (r^2) \},
\quad x = r^2, \ldots, \lceil p^2 r^2 \rceil - r^2 \quad (r = 1, 2, \ldots).

The lemma follows from Lemma 2, from (4), and the relation
\[ \theta_v = A_v h^{-n_2} p^{-h} + O (h^{-n_2+1}) \quad (v = 1, \ldots, m). \]

**THEOREM.** Suppose that \(| \lambda_v | \neq 1 \) for \( v = 1, \ldots, n \); let \( \omega(x) \) for \( x = 1, 2, \ldots \) be a sequence satisfying (1) with initial conditions determined by (4). Then the following asymptotic formula holds:
\[ N_\sigma (P) = a P + O (P^{1/3} (\log P)^{1/2}). \]

The proof follows immediately from Lemma 3 and the estimates obtained in [2].

**REMARK 1.** If \( m = 1 \), that is, if \( \lambda_1 \) is a Pisot number, then the uniform distribution remainder of the sequence \( \{ \theta_1 x^2 \} \) for \( x = 1, \ldots, P \) is \( O (P^{1/3} (\log P)^{1/3}) \).

**REMARK 2.** It is easy to see that the condition on \( f(\lambda) \) to be irreducible involves no loss of generality.

**REMARK 3.** A similar method allows us to obtain a generalization of this result to the case of a recurrent sequence of \( \nu \)-dimensional vectors.

**References**

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