On the upper bound of the $L_p$ discrepancy of Halton’s sequence and the Central Limit Theorem for Hammersley’s net

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Abstract

Let $(H_s(n))_{n \geq 1}$ be an $s$–dimensional Halton’s sequence, and let $\mathcal{H}_{s+1,N} = (H_s(n), n/N)_{n=0}^{N-1}$ be the $s+1$–dimensional Hammersley point set. Let $D(x, (H_n)_{n=0}^{N-1})$ be the local discrepancy of $(H_n)_{n=0}^{N-1}$, and let $D_{s,p}((H_n)_{n=0}^{N-1})$ be the $L_p$ discrepancy of $(H_n)_{n=0}^{N-1}$. It is known that $\limsup_{N \to \infty} N(\log N)^{-s/2}D_{s,p}(H_s(N))_{n=0}^{N-1} > 0$. In this paper, we prove that

$$D_{s,p}((H_s(N))_{n=0}^{N-1}) = O(N^{-1} \log^{s/2} N) \text{ for } N \to \infty.$$ 

I.e., we found the smallest possible order of magnitude of $L_p$ discrepancy of Halton’s sequence. Then we prove the Central Limit Theorem for Hammersley net:

$$N^{-1}D(\bar{x}, \mathcal{H}_{s+1,N})/D_{s+1,2}(\mathcal{H}_{s+1,N}) \overset{w}{\to} \mathcal{N}(0,1),$$

where $\bar{x}$ is a uniformly distributed random variable in $[0,1]^{s+1}$. The main tool is the theorem on $p$-adic logarithmic forms.

Key words: Halton’s sequence, ergodic adding machine, central limit theorem

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1. Introduction

Let $\mathcal{P}_N = (\beta_{n,N})_{n=0}^{N-1}$ be an $N$-element point set in the $s$-dimensional unit cube $[0,1]^s$. The local discrepancy function of $\mathcal{P}_N$ is defined as

$$D(x, \mathcal{P}_N) = \sum_{n=0}^{N-1} (1_{B_n}(\beta_{n,N}) - x_1 \cdots x_s), \quad (1)$$
where \( 1_{B_\mathbf{x}}(\mathbf{y}) = 1 \), if \( \mathbf{y} \in B_\mathbf{x} \), and \( 1_{B_\mathbf{x}}(\mathbf{y}) = 0 \), if \( \mathbf{y} \notin B_\mathbf{x} \) with \( B_\mathbf{x} = [0, x_1) \times \cdots \times [0, x_d) \).

We define the \( L_\infty \) and \( L_p \) discrepancy of an \( N \)-point set \( \mathcal{P}_N \) as

\[
D_{s,\infty}(\mathcal{P}_N) = \sup_{0 < x_1, \ldots, x_s \leq 1} |D(x, \mathcal{P}_N)|/N, \quad D_{s, p}(\mathcal{P}_N) = \|D(x, \mathcal{P}_N)\|_{s, p} / N, \quad (2)
\]

\[
\|f(x)\|_{s, p} = \left( E_s(|f(x)|^p) \right)^{1/p}, \quad E_s(f(x)) = \int_{(0,1)^s} f(x) dx.
\]

**Definition 1.** A sequence of point sets \( (\mathcal{P}_N) \) is of low discrepancy (abbreviated l.d.s.) if \( D_{s, \infty}(\mathcal{P}_N) = O(N^{-1}(\log N)^{s}) \) for \( N \to \infty \).

A sequence of point sets \( (\mathcal{P}_N) \) is of low discrepancy (abbreviated l.d.p.s.) if \( D_{s, \infty}(\mathcal{P}_N) = O(N^{-1}(\log N)^{s-1}) \) for \( N \to \infty \).

For examples of such a sequence, see, e.g., [BC], [Ni].

In 1954, Roth proved that there exists a constant \( C_s > 0 \), such that

\[
ND_{s, \infty}(\beta_n^{(s)}_{N=n=0}^{-1}) > C_s(\log N)^{1-s} \quad \text{and} \quad \lim_{N \to \infty} ND_{s, \infty}(\beta_n^{(s)}_{N=n=0}^{-1})(\log N)^{-s/2} > 0
\]

for all \( N \)-point sets \( (\beta_n^{(s)}_{N=n=0}^{-1}) \) and all sequences \( (\beta_n^{(s)})_{n=0} \).

According to the well-known conjecture (see, e.g., [BC, p.283], [Ni, p.32]), these estimates can be improved to

\[
ND_{s, \infty}(\beta_n^{(s)}_{N=n=0}^{-1})(\log N)^{-\tilde{s}+1} > C_{\tilde{s}}' \quad \text{and} \quad \lim_{N \to \infty} ND_{s, \infty}(\beta_n^{(s)}_{N=n=0}^{-1})(\log N)^{-\tilde{s}} D_{s, \infty}(\beta_n^{(s)}_{N=n=0}^{-1}) > 0
\]

(3)

for all \( N \)-point sets \( (\beta_n^{(s)}_{N=n=0}^{-1}) \) and all sequences \( (\beta_n^{(s)})_{n=0} \) with some \( C_{\tilde{s}}' > 0 \).

In 1972, W. Schmidt proved (3) for \( \tilde{s} = 1 \) and \( s = 2 \). In 1989, Beck proved that \( ND_{3, \infty}(\tilde{s}) \geq \tilde{c} \log N(\log \log N)^{1/8-\epsilon} \) for \( s = 3 \) and some \( \tilde{c} > 0 \). In 2008, Bilik, Lacey and Vagharshakyan (see [Bi, p.147]) proved in all dimensions \( s \geq 3 \) that there exists some \( \tilde{c}(s), \eta > 0 \) for which the following estimate holds for all \( N \)-point sets : \( ND_{s, \infty}(\mathcal{P}_N) > \tilde{c}(s)(\log N)^{-\frac{1}{s}+\eta} \). In [Le1]-[Le3], Levin proved that (3) is true for Hammersly’s net, known constructions of \( (t, m, s) \) nets and for Frolov’s net. It is known that

\[
ND_{s, p}(\beta_n^{(s)}_{N=n=0}^{-1}) > C_{s, p}(\log N)^{1-s} \quad \text{and} \quad \lim_{N \to \infty} ND_{s, p}(\beta_n^{(s)}_{N=n=0}^{-1})(\log N)^{-s/2} > 0
\]

for all \( N \)-point sets \( (\beta_n^{(s)}_{N=n=0}^{-1}) \) and all sequences \( (\beta_n^{(s)})_{n=0} \) with some \( C_{s, p} > 0 \) (see Roth for \( p = 2 \), Schmidt for \( p > 1 \) [BeCh], and Proinov [Pr]).

**Definition 2.** A sequence \( (\beta_n)_{n=0} \) is of \( L_p \) low discrepancy (abbreviated l.d.s.) if \( D_{s, p}(\beta_n^{(s)}_{N=n=0}^{-1}) = O(N^{-1}(\log N)^{s/2}) \) for \( N \to \infty \).

A sequence of point sets \( (\beta_n^{(s)}_{N=n=0}^{-1})_{N=1} \) is of \( L_p \) low discrepancy (abbreviated l.d.p.s.) if \( D_{s, p}(\beta_n^{(s)}_{N=n=0}^{-1}) = O(N^{-1}(\log N)^{(s-1)/2}) \) for \( N \to \infty \).
The existence of \( L_p \) l.d.p.s. was proved by Roth for \( p = 2 \) and by Chen for \( p > 1 \) [Ch1], [Ch2]. The first explicit construction of \( L_p \) l.d.p.s. was obtained by Chen and Skriganov for \( p = 2 \) and by Skriganov for \( p > 1 \) (see [ChSk], [Sk]). The next explicit construction of \( L_p \) l.d.p.s. was proposed by Dick and Pillichshammer (see [Di], [DP], [Ma]). The first explicit construction of \( L_p \) l.d.s. was obtained by Dick, Hinrichs, Markhasin and Pillichshammer [DHMP]. All these explicit constructions were obtained by using \((t, m, s)\) nets.

In this paper we obtain a similar result for Halton’s sequence.

Let \( p_1, \ldots, p_s \geq 2 \) be pairwise coprime integers, \( n = \sum_{j=1}^{s} e_{i,j}(n) p_i^{j-1} \), \( e_{i,j}(n) \in \{0, 1, \ldots, p_i - 1\} \), and \( \phi_i(n) = \sum_{j=1}^{s} e_{i,j}(n) p_i^{j-1} \). (4)

Van der Corput proved that \((\phi_1(n))_{n \geq 0}\) is the 1-dimensional l.d.s.

The first example of multidimensional l.d.s. was proposed by Halton

\( H_s(n) = (\phi_1(n), \ldots, \phi_s(n)), \quad n = 0, 1, 2, \ldots \) (5)

The first example of multidimensional l.d.p.s. was obtained by Hammersley

\( H_{s+1,N} = (H_s(n), n/N)_{n=0}^{N-1} \).

In this paper we will prove that Halton’s sequence is of \( L_p \) l.d.s.:

**Theorem 1.** Let \( s \geq 2 \), \( p \geq 1 \). Then

\[ D_{s,p}(H_s(n))_{n=Q}^{Q+N-1} = O(N^{-1} \log^{s/2} N). \] (6)

For the sake of simplicity, we will consider only the case of primes \( p_1, \ldots, p_s \). For \( n < 0 \), we consider (4) in the sense of \( p \)-adic representation. Note that (4) is also true for generalized Halton’s sequences (see e.g. [L2]) and for the \( s \)-dimensional ergodic adding machine [L2].

Similarly to [L4], in Theorem 2, we prove that the local discrepancy of Hammersley’s point set satisfies the Central Limit Theorem (abbreviated CLT) for \( s \geq 3 \). This result is not true for \( s = 2 \) because the normalised expectation \( E_{s+1}(D(\bar{x}, H_{s+1,N})) / \|D(\bar{x}, H_{s+1,N})\|_{s+1,2} \) does not vanish for \( N \to \infty \), where \( \bar{x} = (x, x_{s+1}) = (x_1, \ldots, x_s) \). The simplest way to avoid this problem is to take \( D(\bar{x}, H_{s+1,N}) - E_{s+1}(D(\bar{x}, H_{s+1,N})) \) instead of \( D(\bar{x}, H_{s+1,N}) \). But we prefer a different way. In Theorem 3 we get the asymptotic property of \( L_p \) discrepancy of Hammersley’s point set for \( p > 0 \). Theorem 3 is the corollary of Theorem 1 and Theorem 2. For this reason, we want to prove CLT exactly for the discrepancy function. The normalised expectation of the symmetrized Hammersley set \( H_{s+1,N}^{sym} = (H_s(n), |n|/N)_{-N < n < N} \) vanishes for \( N \to \infty \). So for
Theorem 2. Let $s \geq 2$, $\bar{x}$ be a uniformly distributed random variable in $[0,1]^{s+1}$. Then

$$\frac{D(\bar{x}, \mathcal{H}_{s+1,N})}{\|D(\bar{x}, \mathcal{H}_{s+1,N})\|_{s+1,2}} \xrightarrow{w} \mathcal{N}(0,1) \text{ for } s \geq 3,$$

$$\frac{D(\bar{x}, \mathcal{H}_{3,N}^{sym})}{\|D(\bar{x}, \mathcal{H}_{3,N}^{sym})\|_{3,2}} \xrightarrow{w} \mathcal{N}(0,1).$$

Theorem 3. Let $s \geq 2$ and $p > 0$. Then

$$\frac{D_{s+1,p}(\mathcal{H}_{s+1,N})}{D_{s+1,2}(\mathcal{H}_{s+1,N})} \xrightarrow{N \to \infty} \kappa_p^{1/p}, \quad s \geq 3,$$

$$\frac{D_{3,p}(\mathcal{H}_{3,N}^{sym})}{D_{3,2}(\mathcal{H}_{3,N}^{sym})} \xrightarrow{N \to \infty} \kappa_p^{1/p},$$

where $\kappa_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^{p} e^{-u^2/2} du$, $\kappa_{2r} = \frac{(2r)!}{2^{r}r!}$ for integer $r \geq 1$.

We note that Theorem 2 and Theorem 3 are also true for other symmetrizations. For example, for $\mathcal{H}_{s+1,N}^{sym} = (H_{s}^{sym}(n), n/(2^s N))_{0 \leq n < 2^s N}$, where $H_{s}^{sym}(n) = (m_1 + (-1)^{m_1} \phi_1(m_{s+1}), ..., m_s + (-1)^{m_s} \phi_s(m_{s+1}))$, with

$$n = m_1 + 2m_2 + ... + 2^{s-1}m_s + 2^s m_{s+1}, \quad m_i \in \{0,1\}, \quad i = 1, ..., s, \quad m_{s+1} \geq 0.$$

For the case $s = 1$ see e.g. [LM].

Now we describe the structure of the paper. In §2, we get simple estimates of Fourier’s series of truncated discrepancy function of Halton’s sequence.

In §3, we apply the theorem on $p$-adic logarithmic forms to obtain the first estimates of the $L_p$ discrepancy of Halton’s sequence. This is the main chapter of the paper.

In §4, we finish the proof of Theorem 1 and, using the moment’s method, we prove Theorem 2. Next, using the standard tools of probability theory, we derive Theorem 3 from Theorem 1 and Theorem 2.

The main tools of the proofs is the theorem on $p$-adic logarithmic forms [Yu].

2. Beginning of the proof of Theorems.

By the moment method, Theorem 2 follows from the following statement:

Lemma 1. Let $s \geq 2$, $h \geq 1$. With notations as above

$$\lim_{N \to \infty} \frac{E_{s+1} D^h(\bar{x}, \mathcal{H}_{s+1,N})}{\|D(\bar{x}, \mathcal{H}_{s+1,N})\|_{s+1,2}^h} d\bar{x} = \begin{cases} \frac{h!}{2^{h/2}(h/2)!}, & \text{if } h \text{ is even}, \\ 0, & \text{if } h \text{ is odd}, \end{cases} \quad (7)$$
where \( \hat{\mathcal{H}}_{s+1,N} = \mathcal{H}_{s+1,N} \) for \( s \geq 3 \) and \( \hat{\mathcal{H}}_{s+1,N} = \mathcal{H}_{s+1,N}^{sym} \) for \( s = 2 \).

The proof of the Lemma 1 is given below. We will use notation \( A \ll B \) equal to \( A = O(B) \). Let

\[
\Delta(\overline{x}) = \begin{cases} 
1, & \text{if } \overline{x} \text{ is true}, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\delta_M(a) = \begin{cases} 
1, & \text{if } a \equiv 0 \pmod{M}, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( [y] \) be the integer part of \( y \),

\[
I_M = -[(M - 1)/2], [M/2], \quad I'_M = -[(M - 1)/2], [M/2] \setminus \{0\}.
\]

Note that the integers of the interval \( I_M \) are a complete set of residues \( \mod M, \, M \geq 1 \). By [Ko, Lemma 2, p. 2], we have

\[
\delta_M(a) = \frac{1}{M} \sum_{k \in I_M} e\left( \frac{ak}{M} \right), \quad \text{where } e(x) = \exp(2\pi ix).
\]

By [Ko, Lemma 1, p. 1], we get

\[
\left| \frac{1}{M} \sum_{k=0}^{M-1} e(ka) \right| \leq \min \left( 1, \frac{1}{2M\langle\alpha\rangle} \right), \quad \text{where } \langle\alpha\rangle = \min(\{\alpha\}, 1 - \{\alpha\}).
\]

Let \( \bar{m} = \max(1, |m|) \leq M/2 \). From [Ko, p. 2], we obtain for \( R \leq M \):

\[
\left| \frac{1}{M} \sum_{k=0}^{R-1} e\left( \frac{mk}{M} \right) \right| \leq \min \left( 1, \left| \frac{e(R/M) - 1}{M(e(M/M) - 1)} \right| \right) \leq \frac{\sin(\pi R/M)}{\bar{m}} \leq \frac{1}{\bar{m}}.
\]

Let \( x_i = 0.x_{i1}x_{i2}... = \sum_{j \geq 1} x_{ij}p_i^{-j} \), with \( x_{ij} \in \{0,1,...,p_i-1\} \), \( i = 1, ..., s \). We define the truncation

\[
[x_i]_r = \sum_{1 \leq j \leq r} x_{ij}p_i^{-j} \quad \text{with } \quad r \geq 1.
\]

If \( \mathbf{x} = (x_1, ..., x_s) \in [0,1)^s \), then the truncation \( [\mathbf{x}]_r \) is defined coordinatewise, that is \( [\mathbf{x}]_r = ([x_1]_{r_1}, ..., [x_s]_{r_s}) \), where \( \mathbf{r} = (r_1, ..., r_s) \). By (4), we have

\[
[\phi_i(k)]_r = [x_i]_r \iff k \equiv \sum_{1 \leq j \leq r} x_{ij}p_i^{-j-1} (\text{mod } p_i).
\]

Let \( p_0 = p_1p_2...p_s \),

\[
P_r = p_1^{r_1}p_2^{r_2}...p_s^{r_s}, \quad M_{i,r} \equiv (P_r/p_i)\textsuperscript{r_i} \pmod{p_i}, \quad M_{i,r} \in [0,p_i^{r_i}), \, 1 \leq i \leq s.
\]
Applying (5) and the Chinese Remainder Theorem, we get
\[ [H_s(k)]_r = [x]_r \iff k \equiv \hat{x}_r \pmod{P_r}, \quad (13) \]
\[ \hat{x}_r \equiv \sum_{i=1}^{s} M_{i,r} P_i r_i x_{i,r_i} \pmod{P_r}, \quad \hat{x}_{r_i} = \sum_{1 \leq j \leq r_i} x_{i,j} P_i^{j-1}, \quad \hat{x}_r \in [0, P_r), \quad (14) \]
\[ \hat{x}_{r_i} \in [0, P_i^{r_i}), \quad 1 \leq i \leq s. \]

Let \( n = [\log_2 N] + 1 \). From [Ni, p. 29, 30], we get
\[ \mathcal{D}(Q,N) := D([x], (H(k))_{k \leq Q}^{Q+N-1}) = D(x, (H(k))_{k \leq Q}^{Q+N-1}) + \epsilon s, \quad |\epsilon| \leq 1. \quad (15) \]

Let
\[ \hat{x}_{r,b} \equiv \sum_{i=1}^{s} M_{i,r} P_i r_i \left( \sum_{1 \leq j \leq r_i} x_{i,j} P_i^{j-1} + b_i P_i^{r_i-1} \right) \pmod{P_r}, \quad \hat{x}_{r,b} \in [0, P_r). \quad (16) \]

Using (11), we obtain
\[ \mathcal{D}(Q,N) = \sum_{k=Q}^{Q+N-1} \sum_{r_1,...,r_s=1}^{n} \sum_{b_1=0}^{x_1,r_1-1} \cdots \sum_{b_s=0}^{x_s,r_s-1} \delta_{P_r}(k - \hat{x}_{r,b}) - N[x_1]_{s} \cdots [x_s]_{n} \]
\[ = \sum_{r_1,...,r_s=1}^{n} \mathcal{D}_{Q,N,r}, \quad \mathcal{D}_{Q,N,r} := \sum_{b_1=0}^{x_1,r_1-1} \cdots \sum_{b_s=0}^{x_s,r_s-1} \sum_{k=Q}^{Q+N-1} (\delta_{P_r}(k - \hat{x}_{r,b}) - 1/P_r), \quad (17) \]
\[ |\mathcal{D}_{Q,N,r}| \leq x_1,r_1 \cdots x_s,r_s < p_0. \]

For given \( r \), we define
\[ \hat{T}(r) := \{ i \in [1, s] \mid r_i > V_1 \}, \quad V_1 = [\log_3 n]. \quad (18) \]

Let \( s \in [1, s]. \) We consider subsets \( T_s \subseteq \{1, ..., s\} \) with \( \text{card}(T_s) = s. \) It is easy to see that
\[ \sum_{s=0}^{S} \sum_{T_s \subseteq \{1, ..., s\}, \#T_s = s} \Delta(T_s = \hat{T}(r)) = 1. \]

By (17), we get
\[ \mathcal{D}(Q,N) = \sum_{r_1,...,r_s=1}^{n} \sum_{s=0}^{S} \sum_{T_s \subseteq \{1, ..., s\}} \Delta(T_s = \hat{T}(r)) \mathcal{D}_{Q,N,r} \]
\[ = \sum_{s=0}^{S} \sum_{T_s \subseteq \{1, ..., s\}} \mathcal{D}_{T_s}(Q,N), \quad \text{with} \quad \mathcal{D}_{T_s}(Q,N) = \sum_{r \in \mathbb{U}_{T_s}} \mathcal{D}_{Q,N,r}, \quad (19) \]

where
\[ \mathbb{U}_{T_s} = \{ r = (r_1, ..., r_s) \in [1, n]^s \mid r_i > V_1, \text{ for } i \in T_s, \quad r_i \leq V_1, \text{ for } i \notin T_s \}. \quad (20) \]
From (17) and (18), we obtain

\[ |\mathcal{D}_T(Q, N)| \leq p_0 V_1^s \leq p_0 \log^{3s} n. \]

Using (15) and (19), we derive

\[ |\hat{\mathcal{D}}(Q, N)| \leq \sum_{s=0}^{s-1} \sum_{T_s \subseteq \{1, \ldots, s\}} |\mathcal{D}_T(Q, N)| = \sum_{s=1}^{s-1} \sum_{T_s \subseteq \{1, \ldots, s\}} |\mathcal{D}_T(Q, N)| + O(\log^{3s} n), \]

with \( \hat{\mathcal{D}}(Q, N) := D(x, (H(k))_{k=Q}^{Q+N-1}) - \mathcal{D}_T(Q, N). \) (21)

Lemma 2. With the notations as above, we have

\[ \mathcal{D}_T(Q, N) = \sum_{r \in U(T_s)} \sum_{m \in \mathcal{P}_e} \varphi_{r, Q, N, m} \psi_r(m, x) e\left(\frac{-m}{P_r} \hat{x}_r\right), \]

\[ \varphi_{r, Q, N, m} = \frac{e(m(Q + N)/P_r) - e(mQ/P_r)}{P_r(e(m/P_r) - 1)}, \quad |\varphi_{r, Q, N, m}| \leq \frac{1}{m}, \]

\[ \psi_r(m, x) = \left\{ \begin{array}{ll}
\prod_{i=1}^{s} \psi(i, \{-mM_{i,r}/p_i\}p_i, x_{i,r_i}), & |\psi_r(m, x)| \leq p_0, \\
\psi(i, 0, x_{i,r_i}) = x_{i,r_i}, & \psi(i, m', x_{i,r_i}) = \frac{1 - e(-m'x_{i,r_i}/p_i)}{e(m'/p_i) - 1} \quad \text{for } m' \neq 0. \end{array} \right. \] (22)

Proof. Similarly to [Ni, p. 37-39], we obtain from (9), (8) and (17) that

\[ \hat{\mathcal{D}}_{Q,s} = \sum_{b_1=0}^{x_{1,r_1}-1} \cdots \sum_{b_s=0}^{x_{s,r_s}-1} \sum_{k=1}^{Q+N-1} P_r e\left(\frac{m}{P_r} (k - \hat{x}_{r,b})\right) \]

\[ = \sum_{b_1=0}^{x_{1,r_1}-1} \cdots \sum_{b_s=0}^{x_{s,r_s}-1} \sum_{m \in \mathcal{P}_e} e(m(Q + N)/P_r) - e(mQ/P_r) \frac{P_r(e(m/P_r) - 1)}{P_r} e\left(\frac{-m}{P_r} \hat{x}_{r,b}\right). \]

According to (16), we get

\[ \hat{x}_{r,b} \equiv \hat{x}_r + \sum_{i=1}^{s} M_{i,r} P_r (b_i - x_{i,r_i})/p_i \text{ mod } P_r. \]

Using (22), we get

\[ \hat{\mathcal{D}}_{Q,s} = \sum_{m \in \mathcal{P}_e} e\left(\frac{-m}{P_r} \hat{x}_r\right) \varphi_{r, Q, N, m} \sum_{b_1=0}^{x_{1,r_1}-1} \cdots \sum_{b_s=0}^{x_{s,r_s}-1} e\left(\frac{-m}{P_r} \sum_{i=1}^{s} M_{i,r} (b_i - x_{i,r_i})/p_i\right) \]

\[ = \sum_{m \in \mathcal{P}_e} \varphi_{r, Q, N, m} \psi_r(m, x) e\left(\frac{-m}{P_r} \hat{x}_r\right). \]
By (11) and (22), we obtain

\[ |\varphi_{r,Q,N,m}| \leq \frac{1}{m}, \quad |\psi(i,m,x_{i,r})| \leq p_i, \quad |\psi_r(m,x)| \leq p_0, \quad i = 1, \ldots, s. \]  

(23)

Bearing in mind (19), we get the assertion of Lemma 2. □

Let \( q = \lceil h/2 \rceil \), \( \Xi_h \) be the set of all transposition of the set \( \{1, \ldots, h\} \),

\[ \varpi_{m,2} = \Delta(h = 2q), \quad s = s, \quad \exists r \in \Xi_{2q} : m_{r(2k-1)}/P_{r(2k-1)} = -m_{r(2k)}/P_{r(2k)}, \quad k \in [1,q], \quad \varpi_{r,m,1} = 1 - \varpi_{r,m,2}, \quad \nu \in \{0,1\}, \]  

(24)

\[ \mathcal{D}_{s,h,\nu}(Q,N) = \sum_{r \in U_{2q}} \sum_{j \in [1,h]} \varpi_{r,m,\nu} \prod_{j=1}^{h} \varphi_{r_1,Q,N,m_1} \psi_{r_1}(m_1,x) e(\frac{-m_{r_1}}{P_{r_1}} \hat{x}_{r_1}). \]  

(25)

Hence

\[ \mathcal{D}_{s,h}(Q,N) = \mathcal{D}_{s,h,1}(Q,N) + \mathcal{D}_{s,h,2}(Q,N). \]  

(26)

Let \( \varpi_{m,2} = 1 \) and let \( m_j = \hat{m}_j P_{\alpha_j} \) with \( (\hat{m}_j,p_0) = 1 \). Therefore \( \mu_k := \hat{m}_r(2k) = -\hat{m}_r(2k-1) \) and \( r_{r(2k)} = \alpha_{r(2k)} = r_{r(2k-1)} - \alpha_{r(2k-1)} \).

Bearing in mind that \( |\varphi_{r_1,Q,N,m_1}\psi_r(m_j,x)| \leq p_0/\hat{m}_j \), we get

\[ \mathcal{D}_{s,2q,2}(Q,N) \ll \sum_{r_{s}} \sum_{j_{s}} \sum_{\alpha_{s},\beta_{s} \in [0,p_0,\ldots,\beta_{s}]} \frac{\Delta(r_{s} - \alpha_{s} = r_{s+q} - \alpha_{s+q}, k = 1,\ldots,q)}{P_{s} \cdots P_{s+q} \beta_{s}^{2} \cdots \beta_{s}^{2}} \ll n^{qs}. \]  

(27)

We will get the more precise estimate in §4. Below we will prove that \( \mathcal{D}_{s,h,1}(Q,N) \ll n^{hs/2} \). Let \( E^{(i)}(f(x_i)) = f([x_i])f(x_i)dx_i \).

It is easy to see that

\[ E^{(i)}(f([x_i])) = \frac{1}{p_i^{\frac{1}{l}}} \sum_{x_{i,1}=0}^{p_i-1} \ldots \sum_{x_{i,l}=0}^{p_i-1} f(0,x_{i,1}\ldots x_{i,l}). \]  

(28)

Let

\[ \hat{m}_i = \sum_{j=1}^{h} m_j M_i r_i \hat{r}_i \pmod{p^\hat{r}_i}, \quad \hat{m}_i \in [0,p_i^{\hat{r}_i}], \quad \hat{r}_i := \max r_{i,j}, \quad i \in [1,s]. \]  

(29)
Lemma 3. With the notations as above, we get

\[ |E_s(\mathcal{D}_{n,h,1}(Q,N))| \ll \sum_{r_j \in \mathcal{U}(n)} \sum_{j=1}^h \bar{\omega}_{r,m,1} \prod_{i=1}^s |E(i)(Z_i)|, \]

\[ |E_{s+1}(\mathcal{D}_{n,h,1}([-N x_{s+1}],2[N x_{s+1}])| \ll \sum_{r_j \in \mathcal{U}(n)} \sum_{j=1}^h \bar{\omega}_{r,m,1} |\hat{\gamma}(h)| \]

\[ \times \prod_{i=1}^s |E(i)(Z_i)|, \quad \text{with} \quad \hat{\gamma}(h) := \int_0^1 \prod_{j=1}^h \phi_{r_i,-[N x_{s+1}],2[N x_{s+1}],m_j} dx_{s+1}, \]

\[ \hat{Z}_i = e\left( \sum_{i=1}^s \frac{\hat{m}_i \hat{x}_{r_i}}{p_i} \right) \prod_{j=1}^h \phi_{r_j,\varphi_{r_i},Q,m_j} e\left( -\sum_{j=1}^h m_j \frac{x_{r_i}}{p_i} \right), \quad \hat{m}_i,j = \{-m_j M_{i,j,r}/p_i\} p_i, \]

\[ Z_i = \hat{Z}_i \delta_{r_i} \left( \hat{m}_i + \sum_{j=1}^h m_j M_{i,j,r} \frac{x_{r_i}}{p_i} \right). \]  

Proof. We will prove the first relation. The proof of the second relation is similar. By (2), (23) and (25), we obtain

\[ |E_s(\mathcal{D}_{n,h,1}(Q,N))| \leq \sum_{r_j \in \mathcal{U}(n)} \sum_{j=1}^h \bar{\omega}_{r,m,1} \prod_{j=1}^h |\varphi_{r_j,Q,N,m_j} E_s(Z)| \]

\[ \leq \sum_{r_j \in \mathcal{U}(n)} \sum_{j=1}^h \bar{\omega}_{r,m,1} |E_s(Z)|, \quad Z = \prod_{j=1}^h \phi_{r_j} e\left( -\sum_{j=1}^h m_j \frac{x_{r_i}}{p_i} \right). \]

By (14) and (29), we have

\[ e\left( -\sum_{j=1}^h m_j \frac{x_{r_i}}{p_i} \right) = e\left( -\sum_{i=1}^s \sum_{j=1}^h m_j M_{i,j,\varphi_{r_i}} \frac{x_{r_i}}{p_i} \right) \]

\[ = e\left( -\sum_{i=1}^s \sum_{j=1}^h m_j M_{i,j,\varphi_{r_i}} \frac{x_{r_i}}{p_i} \right) = e\left( \sum_{i=1}^s \frac{\hat{m}_i \hat{x}_{r_i}}{p_i} \right). \]  

Taking into account that \( m_j' \) and \( \hat{m}_i \) are linked by congruences (29), we get from (22) and (30) that \( E_s(Z) = E(1)(Z_1)\cdots E(s)(Z_s) \). Hence, Lemma 3 is proved. \[ \square \]

Let \( \sigma_{i,r} \) be a transposition of the set \( \{1, 2, \ldots, h\} \) satisfies the condition

\[ r_{i,\sigma_{i,r}(j-1)} \leq r_{i,\sigma_{i,r}(j)}, \quad r_{i,\sigma_{i,r}(h)} = \hat{r}_i := \max_j r_{i,j}, \quad r_{i,\sigma_{i,r}(0)} := 0, \]  

(32)
\( r_j = (r_{1,j}, \ldots, r_{s,j}) \) for \( j \in [1, h] \), \( i = [1, s] \). From (29), we derive: \( \hat{m}_i \in [0, p_i^{\hat{\delta}_i}) \),

\[- \hat{m}_i \equiv \sum_{j=1}^{h} m_j M_{i,r_i,p_i^{\hat{\delta}_i}} \equiv \sum_{j=1}^{h} m_{\sigma_i,r(j)} M_{i,r_{\sigma_i,r(j)}} p_i^{\hat{\delta}_i - r_i, \sigma_i,r(j)} \mod p_i^{\hat{\delta}_i}. \tag{33}\]

We define \( m_{i,j} \) (\( i = 1, \ldots, s, j = 1, \ldots, h \)) from the following congruence:

\[ \hat{m}_i \equiv \sum_{j=1}^{h} m'_{i,\sigma_i,r(j)} \hat{p}_i \mod p_i^{\hat{\delta}_i}, \text{ with } m'_{i,\sigma_i,r(j)} \in I_{p_i^{\hat{\delta}_i}} \leq \hat{m}_i. \tag{34}\]

Let \( I_1 = I_{p_i^{\hat{\delta}_i}} = \{0\} \). Let \( \hat{m}_i, \sigma_i,r(j) = m'_{i,\sigma_i,r(j)} + m_{\sigma_i,r(j)} M_{i,r_{\sigma_i,r(j)}}. \) By (33), we get

\[ \sum_{j=1}^{h} m_{i,\sigma_i,r(j)} p_i^{\hat{\delta}_i - r_i, \sigma_i,r(j)} \equiv \sum_{j=1}^{h} (m'_{i,\sigma_i,r(j)} + m_{\sigma_i,r(j)} M_{i,r_{\sigma_i,r(j)}}) p_i^{\hat{\delta}_i - r_i, \sigma_i,r(j)} \equiv 0 \mod p_i^{\hat{\delta}_i}. \tag{35}\]

**Lemma 4.** With notations as above, we have

\[ |E^{(i)}(Z_i)| \leq \prod_{j=1}^{h} \frac{4p_i^{\hat{\delta}_i + 1}}{m_{i,\sigma_i,r(j)}} \delta_{p_i^{\hat{\delta}_i}} \left( \sum_{j=1}^{h} m_{i,\sigma_i,r(j)} p_i^{\hat{\delta}_i - r_i, \sigma_i,r(j)} \right). \]

**Proof.** From (29) and (33) - (35), we have that \( \hat{m}_i, \hat{m}_i \) and \( m'_{i,j} \) are linked by congruence (35). By (30) and (33), we get

\[ Z_i = \hat{Z}_i, \delta_{p_i^{\hat{\delta}_i}} \left( \sum_{j=1}^{h} m_{i,\sigma_i,r(j)} p_i^{\hat{\delta}_i - r_i, \sigma_i,r(j)} \right). \]

Let \( J = \{ j \in [1, h] \mid r_{i,\sigma_i,r(j)} \leq r_{i,\sigma_i,r(j-1)} + h + 1 \} \), \( h' = \text{card}(J) \). According to (14), (23), (28), (30), (34) and (35), we derive \( |\psi(i, m, x)| \leq p_i \).
and

\[
p_{i}^{-h}|E^{(i)}(\hat{Z}_{i})| = \frac{1}{p_{i}^{h}} \left| \sum_{x_{i,k} \in \{0,p_{i}\}, k \in [1,\hat{r}_{i}]} e\left( \frac{\hat{m}_{i} \hat{x}_{i,k}}{p_{i}^{k}} \right) \prod_{j=1}^{k} \hat{\psi}(i, \hat{m}_{i,j}, x_{i,r_{i,j}}) \right| \\
\leq \frac{1}{p_{i}^{\hat{r}_{i}+k'}} \left| \sum_{x_{i,k} \in \{0,p_{i}\}, k \in [1,\hat{r}_{i}]} e\left( \frac{\hat{m}_{i} \sum_{k=1}^{\hat{r}_{i}} x_{i,k}p_{i}^{k-1}}{p_{i}^{\hat{r}_{i}}} \right) \right| \\
= \frac{1}{p_{i}^{\hat{r}_{i}+k'}} \prod_{j \in [1,h] \setminus J} \left| \sum_{k \in [r_{i,j}, r_{i,j}(j)-1]} e\left( \frac{\hat{m}_{i} \sum_{k=r_{i,j}(j)-1+1}^{r_{i,j}(j)-1} x_{i,k}p_{i}^{k-1}}{p_{i}^{\hat{r}_{i}}} \right) \right| \\
= \prod_{j \in [1,h] \setminus J} \left| \frac{p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}}{p_{i}^{\hat{r}_{i}} \hat{m}_{i,j}(j)} \right| \sum_{x=0}^{\hat{r}_{i}-r_{i,j}(j)-1} e\left( \frac{\hat{m}_{i,j}(j)}{p_{i}^{\hat{r}_{i}-r_{i,j}(j)-1}} \right).
\]

By (8), (10) and (34), we have that

\[
\frac{|E^{(i)}(\hat{Z}_{i})|}{p_{i}^{h}} \leq \hat{Z}_{i,1}\hat{Z}_{i,2}\cdots\hat{Z}_{i,h}, \quad \hat{Z}_{i,j} = 1 \leq \frac{p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}}{\hat{m}_{i,j}(j)} \leq \frac{p_{i}^{-h}}{\hat{m}_{i,j}(j)} \text{ for } j \in J.
\]

\[
\hat{Z}_{i,j} = \min \left( 1, \frac{1}{2p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}} \left\| \frac{\hat{m}_{i,j}}{p_{i}^{\hat{r}_{i}-r_{i,j}(j)-1}} \right\| \right), \text{ for } j \in [1,h] \setminus J. \quad (36)
\]

Let \( j \in [1,h] \setminus J \). If \( |m'_{i,s_{1},r(j)}| \leq 2h \), then we will use the estimate \( \hat{Z}_{i,j} \leq 1 \leq 2p_{i}^{h}/\hat{m}_{i,s_{1},r(j)} \). Consider the case \( |m'_{i,s_{1},r(j)}| > 2h \). Bearing in mind (34), we get

\[
\hat{m}_{i,j} \equiv \sum_{k=1}^{\hat{r}_{i}} p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1} \frac{\hat{m}_{i} \hat{x}_{i,k}}{p_{i}^{k}} \equiv \hat{m}_{i,j} := \sum_{k=1}^{\hat{r}_{i}} p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1} \mod p_{i}^{\hat{r}_{i}-r_{i,j}(j)-1},
\]

\[
\left\| \frac{\hat{m}_{i,j}}{p_{i}^{\hat{r}_{i}-r_{i,j}(j)-1}} \right\| = \left\| \frac{\hat{m}_{i,j}}{p_{i}^{\hat{r}_{i}-r_{i,j}(j)-1}} \right\|.
\]

From (33), (8) and the previous condition, we obtain \( 2h < |m_{i,s_{1},r(j)}'| \leq \frac{1}{2}p_{i}^{r_{i,j}(j)} \times p_{i}^{r_{i,j}(j)-1} \). Taking into account that \( j \notin J \), we get \( r_{i,j}(j) \geq r_{i,j}(j-1) \) + \( h + 2 \). Hence

\[
\left| \sum_{k=1}^{h} \frac{m_{i,j}(k)}{p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}} \right| = \left| \frac{m_{i,s_{1},r(j)}'}{p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}} \right| + \varepsilon h \geq \left| \frac{m_{i,s_{1},r(j)}'}{2p_{i}^{r_{i,j}(j)-r_{i,j}(j)-1}} \right|, \text{ with } |\varepsilon| \leq 1,
\]

\[
\text{with } |\varepsilon| \leq 1.
\]
and

$$\sum_{k=j}^{h} \frac{1}{p_i} \leq \frac{h}{p_i^{h+2}} \leq \frac{1}{4}, \quad \frac{3}{4} \geq \left| \frac{m_{i,s,r}^{r}}{p_i^{h}} \right| \geq \left| \frac{m_{i,s,r}^{r}}{p_i^{h}} \right| + 1/4$$

$$\geq \left| \sum_{k=j}^{h} \frac{m_{i,s,r}^{r}}{p_i^{h}} \right| = \left| \frac{\tilde{m}_{i,j}}{p_i^{h}} \right| = \left| \frac{m_{i,s,r}^{r}}{p_i^{h}} \right| / 2.$$  

Bearing in mind that if $|\alpha| \leq 3/4$ then $4\langle \alpha \rangle \geq |\alpha|$, we get

$$2\left\langle \frac{\tilde{m}_{i,j}}{p_i^{h}} \right\rangle = 2\left\langle \frac{\tilde{m}_{i,j}}{p_i^{h}} \right\rangle \geq \left| \frac{\tilde{m}_{i,j}}{p_i^{h}} \right| / 4 \geq \left| \frac{m_{i,s,r}^{r}}{p_i^{h}} \right| / 4.$$  

Using (36), we obtain

$$p_i^{-h}|E(i)(\hat{Z}_i)| \leq \prod_{j \in [1,h]} \frac{p_i^{h}}{\tilde{m}_{i,s,r}^{r}(j)} \prod_{j \in [1,h]} \frac{4p_i^{h}}{\tilde{m}_{i,s,r}^{r}(j)} \leq \prod_{j \in [1,h]} \frac{4p_i^{h}}{\tilde{m}_{i,s,r}^{r}(j)}.$$  

Hence, Lemma 4 is proved.  

Lemma 5. With the notations as above

$$|E_s(\mathcal{D}_{T_s,h}(Q,N))| \ll n^{-s} + D_{T_s}, \text{ with } D_{T_s} = \sum_{m_{j} \in I_{V_2}} \sum_{r_j \in U_{T_s}} \frac{\Sigma_{r,m}}{m_1 \cdots m_h} \times \prod_{i=1}^{h} \prod_{j=1}^{h} \frac{1}{\tilde{m}_{i,j}} \delta_{p_i^{h}} \left( \sum_{j=1}^{h} \tilde{m}_{i,s,r}^{r}(j) p_i^{h} \right),  \quad V_2 = n^{4h},$$

$$|E_{s+1}(\mathcal{D}_{T_s,h}([N \cdot x_{s+1}],2[N \cdot x_{s+1}]))| \ll n^{-s} + D_{T_s}, \text{ with } D_{T_s} = \sum_{m_{j} \in I_{V_2}} \sum_{r_j \in U_{T_s}} \frac{\Sigma_{r,m}}{m_1 \cdots m_h} \times \sum_{r_j \in U_{T_s}} \frac{\Sigma_{r,m}}{m_1 \cdots m_h} \delta_{p_i^{h}} \left( \sum_{j=1}^{h} \tilde{m}_{i,s,r}^{r}(j) p_i^{h} \right).$$  

(37)

where $\tilde{m}_{i,s,r}^{r}(j) = m_{i,j}^{r} + m_{i,s,r}^{r}(j) M_{i,s,r}^{r}(j).$

Proof. We will prove the first relation. The proof of the second relation is similar. Using Lemma 3 and Lemma 4, we obtain:

$$|E_s(\mathcal{D}_{T_s,h}(Q,N))| \ll \sum_{r_j \in U_{T_s}} \sum_{m_{j} \in I_{V_2}} \frac{\Sigma_{r,m}}{m_1 \cdots m_h} \times \prod_{i=1}^{h} \prod_{j=1}^{h} \frac{1}{\tilde{m}_{i,j}} \delta_{p_i^{h}} \left( \sum_{j=1}^{h} \tilde{m}_{i,s,r}^{r}(j) p_i^{h} \right).$$  

(38)
By \([8]\), we have \(\|\mathcal{P}_T (Q, N)\|_h^b \ll n^{h(s+1)}\). Hence, the part of the right hand of \([38]\), satisfying to the condition \(|m_{i,j}'| > V_2 = n^{4\delta s}\) for some \((i, j) \in [1, s] \times [1, h]\), is equal to \(O(n^{h(s+1) - 4\delta s})\). Therefore

\[
|E_s(\mathcal{P}_T^h (Q, N))| \ll n^{-s} + \sum_{r_0 \in \mathcal{T}_T, j = 1, \ldots, h} \sum_{m_{i,j}' \in \mathcal{I}_{r_0, j}} \frac{\omega_{r,m}}{m_1 \cdots m_h} \times \prod_{j=1}^h \prod_{i=1}^s \frac{1}{m_{i,j}} \delta_{j,j}' \left( \sum_{j=1}^h m_{i,j} \sigma_{i,j}(j)p_i \right) \Delta(|m_{i,j}'| \leq V_2).
\]

Let

\[
j_1(m) = \Delta(\max_j |m_j| < (V_2 - 1)/2), \quad j_2(m) = 1 - j_1(m).
\]

Hence

\[
|E_s(\mathcal{P}_T^{h_1, h_2} (Q, N))| \ll n^{-s} + \bar{\mathbb{D}}_{T, 1} + \bar{\mathbb{D}}_{T, 2}, \quad \text{with} \quad \bar{\mathbb{D}}_{T, \nu} = \sum_{r_0 \in \mathcal{T}_T, j = 1, \ldots, h} \sum_{|m_{i,j}| < V_2} \frac{1}{m_{i,j}} \prod_{j=1}^h \prod_{i=1}^s \frac{\omega_{r,m}}{m_{i,j}} \delta_{j,j}' \left( \sum_{j=1}^h m_{i,j} \sigma_{i,j}(j)p_i \right), \quad \nu = 1, 2.
\]

From \([37]\), we obtain that \(\bar{\mathbb{D}}_{T, 1} \leq \mathbb{D}_{T, 2}\), and the assertion of Lemma 5 follows from the estimate :

\[
\bar{\mathbb{D}}_{T, 2} \ll n^{-s}.
\]  

(39)

Now we consider \(\bar{\mathbb{D}}_{T, 2}\) (the case of \(j_2(m) = 1\)). Let \(|m_{j_0}| \geq (V_2 - 1)/2\) with some \(j_0 \in [1, h]\), and let \(i_0 \in \mathcal{T}_s\). By \([18]\) and \([20]\), we get that \(r_{i_0, j} > V_1 = [\log^3 n]\) \((j = 1, \ldots, h)\). It is easy to verify that \([39]\) follows from the next inequality

\[
W := \sum_{|m_{i,j}| \geq (V_2 - 1)/2} \frac{1}{m_{j_0}} \delta_{i,j_0} \left( \sum_{j=1}^h m_{i,j} \sigma_{i,j}(j)p_{i_0} \right) \ll n^{-2\delta s}.
\]  

(40)

We fix \(m_{i,j}' (i \in [1, s], j \in [1, h])\), \(r_1, \ldots, r_h\) and \(m_j (j \in [1, h] \setminus \{j_1\})\) with
\[ j_1 = \sigma_{i_0,r}(j_0). \]

By (35) we get

\[
\sum_{j=1}^{h} (m'_{i_0,\sigma_{i_0,r}(j)} + m_{\sigma_{i_0,r}(j)} M_{i,r_{\sigma_{i_0,r}(j)}}) \equiv \frac{\hat{r}_{i_0} - r_{i_0,\sigma_{i_0,r}(j)}}{\hat{r}_{i_0}} p_{i_0} \mod p_{i_0},
\]

\[
\equiv - \sum_{j \in [1,h] \setminus \{j_1\}} (m'_{i_0,\sigma_{i_0,r}(j)} + m_{\sigma_{i_0,r}(j)} M_{i,r_{\sigma_{i_0,r}(j)}}) \hat{r}_{i_0} \mod p_{i_0},
\]

\[
m_{\sigma_{i_0,r}(j)} \equiv \left( - \sum_{j \in [1,h] \setminus \{j_1\}} (m'_{i_0,\sigma_{i_0,r}(j)} + m_{\sigma_{i_0,r}(j)} M_{i,r_{\sigma_{i_0,r}(j)}}) \hat{r}_{i_0} \right) \hat{r}_{i_0}^{-1} \mod p_{i_0}.
\]

Hence there exists an integer \( A \) such that

\[
m_{j_0} \equiv A \mod p_{i_0}^{r_{i_0,j_0}}, \quad \text{with} \quad j_0 = \sigma_{i_0,r}(j_1), \quad \text{and} \quad |A| \leq p_{i_0}^{r_{i_0,j_0}/2}, \quad r_{i_0,j_0} \geq \log_2 n.
\]

By (40), we have

\[
W \ll \sum_{k \in \mathbb{Z}} \frac{1}{|A + k p_{i_0}^{r_{i_0,j_0}}|} \Delta(p_{i_0}^n \geq |A + k p_{i_0}^{r_{i_0,j_0}}| \geq (V_2 - 1)/2)
\]

\[
\ll \sum_{k=0}^{p_{i_0}^n} \frac{1}{V_2 + k n^{5h_s}} \ll V_2^{-1} + n^{-5h_s} \log(p_{i_0}^n) \ll n^{-4h_s}, \quad \text{with} \quad V_2 = n^{4h_s}.
\]

According to (40), Lemma 5 is proved. \( \blacksquare \)

3. The main lemmas.

3.1. \textbf{\( p \)-adic logarithmic forms.} Let \( \hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}} \quad (\hat{n} \geq 1) \) be non-zero algebraic numbers and \( K \) be a number field containing \( \hat{\alpha}_1, \ldots, \hat{\alpha}_{\hat{n}} \) with \( d = [K : \mathbb{Q}] \). Denote by \( \mathfrak{d} \) a prime ideal of the ring \( \mathcal{O}_K \) of integers in \( K \), lying above the prime number \( p \), and by \( f_0 \) the residue class degree of \( \mathfrak{d} \). For \( \gamma \in K, \gamma \neq 0 \), write \( \text{ord}_\mathfrak{d}(\gamma) \) for the exponent to which \( \mathfrak{d} \) divides the principal fractional ideal generated by \( \gamma \) in \( K \). Define

\[
h'(\hat{\alpha}_j) = \max(h_0(\hat{\alpha}_j), f_0(\log p)/d) \quad (1 \leq j \leq \hat{n}),
\]

where \( h_0(\gamma) \) denotes the absolute logarithmic Weil height of an algebraic number \( \gamma \), i.e.,

\[
h_0(\gamma) = k^{-1}\left( \log a_0 + \sum_{i=1}^{k} \log \max(1, |\gamma^{(i)}|) \right),
\]

where \( k \) is the degree of the field \( K \).
where the minimal polynomial for $\gamma$ is

$$a_0x^k + a_1x^{k-1} + \cdots + a_k = a_0(x - \gamma^{(1)}) \cdots (x - \gamma^{(k)}), \quad a_0 > 0.$$ 

**Theorem A.** [Yu, Theorem 1] Let $\hat{\Xi} = \hat{\alpha}_1^{b_1} \cdots \hat{\alpha}_n^{b_n} - 1 \neq 0$. Suppose that

$$\text{ord}_B(\hat{\alpha}_1) = 0 \quad (1 \leq j \leq \hat{n}).$$

Then there exists a constant $C$, depending only on $\hat{n}, d$ and $\mathfrak{d}$, such that

$$\text{ord}_B(\hat{\Xi}) < C h'(\hat{\alpha}_1) \cdots h'(\hat{\alpha}_n) \log \hat{B},$$

where

$$\hat{B} = \max(|b_1|, \ldots, |b_n|, 3).$$

We will use Theorem A with $\hat{n} = s, k = d = 1, \mathfrak{d} = p_i, \{\hat{\alpha}_1, \ldots, \hat{\alpha}_{s-1}\} = \{p_1, p_2, \ldots, p_i-1, p_{i+1}, \ldots, p_s\}, \hat{\alpha}_s = k_1/k_2, k_1, k_2 \in \mathbb{Z}$.

**Corollary 1.** Let $0 < |k_j| \leq n^{ths}, j = 1, 2, \text{ord}_p(k_1/k_2) = 0 \ (i \in [1, s])$ and $\hat{B} = n$. Then there exists a constant $C_1 > 0$ such that

$$\text{ord}_{p_i}(\hat{\Xi}) < C_1 \log n \log \hat{B} = C_1 \log^2 n, \quad \text{with} \quad \hat{\Xi} = (k_1/k_2) \prod_{1 \leq j \leq s, j \neq i} p_j^{b_j - 1} \neq 0.$$

### 3.2. The applications of Theorem A.

Let $\Xi_h$ be the set of all transpositions of $\{1, 2, \ldots, h\}$ and let $\sigma_{i,r} \in \Xi_h$ with $r_{i,\sigma_{i,r}(j)} \geq r_{i,\sigma_{i,r}(j)}$ for $j = 1, 2, \ldots, h-1$. For a given nondecreasing sequence $(r_{i,\sigma_{i,r}(j)})_{1 \leq j \leq h}$, we define the following partition of the interval $[0, h]$:

$$b'_{i,r,0} < b'_{i,r,1} < \cdots < b'_{i,r,a_i,r_i} = h.$$ 

More precisely, we define integer variables $a_{i,r_i}, b'_{i,r,k}, b_{i,r,k} \geq 1$ from the following conditions:

$$b'_{i,r,k} = b'_{i,r,k-1} + b_{i,r,k} \quad 1 \leq k \leq a_{i,r_i}, \quad b'_{i,r,0} = 0, \quad b'_{i,r,a_i,r_i} = h, \quad b_{i,r,1} + \cdots + b_{i,r,a_i,r_i} = h,$$

with

$$0 \leq r_{i,\sigma_{i,r}(j+1)} - r_{i,\sigma_{i,r}(j)} \leq V_1 \quad \text{if} \quad j, j+1 \in (b'_{i,r,k-1}, b'_{i,r,k}], \quad k \in [1, a_{i,r_i}],$$

$$r_{i,\sigma_{i,r}(b'_{i,r,k})} - r_{i,\sigma_{i,r}(b'_{i,r,k-1})} > V_1 \quad \text{for} \quad k \in [2, a_{i,r_i}], \quad i = 1, \ldots, s.$$  \hspace{1cm} (41)
From (37), we get

\[
\mathbb{D}_{T_s} = \sum_{m_j \in I_{V_2}'} \sum_{l_i,j = 1, \ldots, s, j = 1, \ldots, h} \prod_{i=1}^s \delta_{p_i} \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right)
\]

Changing the order of the summation, we obtain

\[
\mathbb{D}_{T_s} = \sum_{m_j \in I_{V_2}'} \sum_{l_i,j = 1, \ldots, s, j = 1, \ldots, h} \prod_{i=1}^s \delta_{p_i} \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right)
\]

with

\[
\mathbb{D}_{T_s,a,\lambda,\tau} = \sum_{m_j \in I_{V_2}'} \sum_{l_i,j = 1, \ldots, s, j = 1, \ldots, h} \prod_{i=1}^s \delta_{p_i} \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right)
\]

where

\[
\Phi_{m,\tau} = \sum_{r_j \in U_T} \prod_{i=1}^s \delta_{p_i} \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right)
\]

By (41), we get

\[
\Phi_{m,\tau} \leq \sum_{r_j \in U_T} \prod_{i=1}^s \delta_{p_i} \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right) \left( \sum_{j=1}^h \Phi_{m,\tau} \right)
\]

where

\[
\Lambda_{i,k} = \Lambda_{i,k-1} + \lambda_{i,k}, \quad 1 \leq k \leq a_i, \quad \Lambda_{i,0} = 0, \quad \Lambda_{i,a_i} = h, \quad \sum_{k=1}^{a_i} \lambda_{i,k} = h.
\]

Therefore

\[
\Phi_{m,\tau} \leq V_1^{h(s-s)} \prod_{i \in T} n^{a_i} V_1^{h} = n^{a_1 + \ldots + a_s} V_1^{hs}.
\]
In the following we fix \( i_0 \in T_s \) and \( \tau_{i_0} \in \Xi_h \). Let

\[
\rho_{i,j} = r_{i,\tau_{i_0}(j)}, \quad \rho_j = r_{\tau_{i_0}(j)}, \quad \hat{\tau}_i = \tau_{i_0}^{-1}(\tau_i), \quad i = 1, \ldots, s; \quad j = 1, \ldots, h. \tag{46}
\]

By \((24)\), we get \( \varpi_{\rho,m,1} = \varpi_{r,m,1} \) with \( \rho = (\rho_1, \ldots, \rho_h), \ r = (r_1, \ldots, r_h) \).
Hence, we have proved the following lemma:

**Lemma 6.** Let \( i_0 \in T_s \). Then

\[
\Phi_{p,m,r} \leq \sum_{\rho_j \in U_T, \ j \in [1,h]} \chi_{\rho,j} \ 1_{i_0,\rho}, \quad 1_{i_0,\rho} = \delta_{\rho_{i_0}} \left( \sum_{j=1}^{h} \tilde{m}_{i_0,j} \rho_{i_0}^{-\rho_{i_0,j}} \right). \tag{47}
\]

where

\[
\chi_{\rho} = \prod_{i=1}^{a_i} \chi_{i,\rho}, \quad \chi_{i,k,j} = \prod_{k=1}^{a_i} \chi_{i,k,j}, \quad \chi_{i,k,j} = \prod_{j \in \Lambda_{i,k-1}^2, \Lambda_{i,k}} \chi_{i,k,j,0} \varpi_{\rho,m,1},
\]

\[
\chi_{i,k,j,0} = \Delta \left( 0 \leq \rho_i, \hat{\tau}_i(j+1) - \rho_i, \hat{\tau}_i(j) \leq V_1 \right. \quad \text{for} \ j, j + 1 \in (\Lambda_{i,k-1}^2, \Lambda_{i,k}), \ 
\]

\[
\rho_{i,j}(\Lambda_{i,k-1}^2) - \rho_i, \hat{\tau}_i(\Lambda_{i,k-1}^2) > V_1 \right). \tag{48}
\]

By \((32), (44)\) and \((46)\), we get:

\[
\rho_{i_0,0} = 0, \quad \rho_{i_0,h} = \hat{\rho}_{i_0} = \max_j \rho_{i_0,j}, \quad \rho_{i_0,j} \geq \rho_{i_0,j-1} (j > 1), \quad \Lambda_{i_0,0} = 0, \quad \Lambda_{i_0,a_{i_0}+1} = \Lambda_{i_0,\max_{j} \rho_{i_0,j}} = h. \tag{49}
\]

**Lemma 7.** Let \( i_0 \in T_s, \ j_{k,0} = \Lambda_{i_{0,k-1}^2} + 1, \ j_{k,1} = \Lambda_{i_{0,k}^2} + 1, \ k \in [1,a_{i_0}] \). Then

\[
1_{i_0,\rho} \leq \prod_{k=1}^{a_{i_0}} 1_{i_0,k,\rho}, \quad \text{where} \ 1_{i_0,k,\rho} := \delta_{\rho_{i_0},j_{k,1}^2-\rho_{i_0,j_{k,0}^2}},
\]

\[
L_{k,1} = \sum_{j=0}^{j_{k,1}^{-1}} \left( m_{i_0,j} + m_j M_{i_0,\rho_j} \right) \rho_{i_0,j_{k,1}^2-\rho_{i_0,j_{k,0}^2}}, \quad L_{k,2} = \sum_{j=0}^{j_{k,1}^{-1}} \left( m_{i_0,j} + m_j M_{i_0,\rho_j} \right) \rho_{i_0}^{-\rho_{i_0,j}}. \tag{50}
\]

**Proof.** Let

\[
\hat{1}_{i_0,\rho} := \delta_{\rho_{i_0}} \left( L_{k,1} p_{i_0}^{-\rho_{i_0,j_{k,1}^2-1}} + L_{k,2} \right) = \delta_{\rho_{i_0}} \left( L_{k,1} p_{i_0}^{-\rho_{i_0,j_{k,0}^2-1}} \right) = 1. \tag{51}
\]

Then

\[
L_{k,2} \equiv 0 \mod p_{i_0}^{-\rho_{i_0,j_{k,1}^2-1}}, \quad L_{k,1} + L_{k,2} p_{i_0}^{-\rho_{i_0,j_{k,0}^2-1}} \equiv 0 \mod p_{i_0}^{-\rho_{i_0,j_{k,1}^2-1}},
\]

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Hence
\[ \hat{l}_{i,k,\rho} \leq \delta_{\rho_{i,j},k,1} \hat{\rho}_{i,j,0}^{-1} (L_{i,k,1} + L_{i,k,2} \hat{p}_{i,\rho}) = \hat{l}_{i,k,\rho} \hat{l}_{i,k+1,\rho}. \]  

(52)

According to (49) - (51), we get
\[ \hat{j}_{\alpha_{i,0},1} = \Lambda_{\alpha_{i,0},a_{i,0}} + 1 = h + 1, \quad L_{\alpha_{i,0},2} = 0, \quad \text{and} \quad \hat{l}_{i,\alpha_{i,0},a_{i,0},1} = \delta_{\rho_{i,\rho}} (L_{\alpha_{i,0},2}) = \delta_{\rho_{i,\rho}} (0) = 1. \]

By (35), (49) - (51) and (47), we have \( j_{0,1} = j_{1,0} = \Lambda_{\alpha_{i,0},a_{i,0}}, \rho_{i,0,0} = 0 \) and
\[ \sum_{j=1}^{h} \sum_{j=1}^{h} (m_{i,j} + m_{i,j} M_{i,\rho}) \hat{p}_{i,j} = L_{0,2}, \quad \hat{l}_{i,\rho} = \hat{l}_{i,1,\rho}. \]

(53)

Using (52) - (53), we obtain
\[ \hat{l}_{i,\rho} = \hat{l}_{i,1,\rho} \hat{l}_{i,2,\rho} \cdots \hat{l}_{i,a_{i,0},\rho} \hat{l}_{i,a_{i,0},1} = \hat{l}_{i,1,\rho} \hat{l}_{i,2,\rho} \cdots \hat{l}_{i,a_{i,0},\rho}. \]

Therefore, Lemma 7 is proved. ■

**Lemma 8.** Let \( i_0 \in T_s, k \in [1, a_{i_0}], \mu_k \in [\Lambda_{i_0,k-1} + 1, \Lambda_{i_0,k}], \)
\[ H_{i_0,k} := \sum_{\rho \leq U_{T_s} \cap [\Lambda_{i_0,k-1} + 1, \Lambda_{i_0,k}]} \chi_{i_0,k,\rho} \hat{l}_{i_0,k,\rho}, \quad \check{H}_{i_0,k,\rho} = \sum_{\rho < \mu_k \in [1, n]} \chi_{i_0,k,\rho} \hat{l}_{i_0,k,\rho}. \]

(54)

Then
\[ H_{i_0,k} \leq \sum_{j \in [\Lambda_{i_0,k-1} + 1, \Lambda_{i_0,k}]} \sum_{\rho \leq U_{T_s} \cap [\Lambda_{i_0,k-1} + 1, \Lambda_{i_0,k}]} \check{H}_{i_0,k,\rho}, \quad \text{with} \quad \check{H}_{i_0,k,\rho} \leq 1. \]

(55)

**Proof.** By (18), we get the first assertion in (55). Now we examine the second assertion of statement (55). Suppose that \( \hat{l}_{i_0,k,\rho} = 1. \) After multiplying by \( \check{p}_{i_0} \)
\[ \sum_{j=1}^{h} (m_{i,j} + m_{i,j} M_{i,\rho}) \hat{p}_{i,j} = 0 \mod \hat{p}_{i,\rho}, \quad j_0 = \Lambda_{i_0,k-1} + 1. \]

We fix \( r_j \) for \( j \in \{1, \ldots, h\} \setminus \{\mu_k\} \) and we have for some \( \alpha_1 \) that
\[ m'_{i_0,\rho} + m_{\mu_k} M_{i_0,\rho} \equiv \alpha_1 \mod \hat{p}_{i_0}. \]

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Hence
\[ m_{\mu_k} M_{i_0, \rho_{\mu_k}} \equiv \alpha_2 \mod p_i^{\rho_{i_0, \mu_k} - \rho_{i, j_k, 0}^{-1}}, \quad \text{with } \alpha_2 = \alpha_1 - m_{i_0, \mu_k}. \]

By (37), (33), we get \( 0 < |m_{\mu_k}| \leq V_2 = n^4 s. \) Let \( \beta = \text{ord}_{p_i} (m_{\mu_k}), m_{\mu_k} = \hat{m}_{\mu_k} p_i^\beta, (\hat{m}_{\mu_k}, p_i) = 1. \) Hence \( \beta \ll \log n. \) According to (18) and (54), we have \( \rho_{i_0, \mu_k} - \rho_{i, j_k, 0}^{-1} > V_1 = [\log^3 n] \) and \( \rho_{i_0, \mu_k} - \rho_{i, j_k, 0}^{-1} - \beta > V_1 - \beta \geq [V_1/2]. \) Using (12) and taking into account that \( \rho_{i_0, \mu_k} \geq V_1, \) we obtain
\[ \prod_{1 \leq i \leq \rho_{i_0, \mu_k}} p_i^{\rho_{i_0, \mu_k} - \rho_{i, \mu_k}} \equiv M_{i_0, \rho_{\mu_k}} \equiv \alpha_2 / \hat{m}_{\mu_k} \mod p_i^{V_1/2}, \quad V_1 = [\log^3 n]. \]

Suppose that \( (\rho_{1, \mu_k}, ..., \rho_{i_0-1, \mu_k}, \rho_{i_0+1, \mu_k}, ..., \rho_{s, \mu_k}) \) and \( (\rho_{1, \mu_k}^\prime, ..., \rho_{i_0-1, \mu_k}^\prime, \rho_{i_0+1, \mu_k}^\prime, ..., \rho_{s, \mu_k}^\prime) \) are two different solutions of this congruence. Then
\[ \prod_{1 \leq i \leq \rho_{i_0, \mu_k}} p_i^{\rho_{i, \mu_k} - \rho_{i, \mu_k}^\prime} = \prod_{1 \leq i \leq \rho_{i_0, \mu_k}} p_i^{\rho_{i, \mu_k}^\prime - \rho_{i, \mu_k}} \left( \prod_{1 \leq i \leq \rho_{i_0, \mu_k}} p_i^{\rho_{i, \mu_k} - \rho_{i, \mu_k}^\prime} - 1 \right) \equiv 0 \mod p_i^{V_1/2}. \]

Here \( V_1 = [\log^3 n] \gg C_1 \log^2 n. \) Therefore, we can apply Corollary 1. We get that this congruence is equality, having only one solution \( \rho_{i, \mu_k}^\prime = \rho_{i, \mu_k}^\prime \) (\( i = 1, ..., i - 1, i + 1, ..., s. \)) By (54), we get \( \beta_{i_0, \mu_k, \rho} \leq 1. \)

Hence, Lemma 8 is proved. \( \square \)

**Lemma 9.** Let \( i_0 \in T_s, \ j_{k, 0} = \Lambda_{i_0, k-1} + 1, \ j_{k, 1} = \Lambda_{i_0, k} + 1, \ k \in [1, a_{i_0}]. \) Then
\[ H_{i_0, k} \ll \tilde{H}_{i_0, k} + O(n (\Lambda_{i_0, k-1})(s-1)V_1^\alpha), \]
where \( \tilde{H}_{i_0, k} = \sum_{\beta \in \text{U}_{\Lambda_{i_0, k+1}} \Lambda_{i_0, k}} \chi_{i_0, k, \rho} \tilde{\nu}_{i_0, k, \rho} \]
\[ \tilde{\nu}_{i_0, k, \rho} := \delta_{\mu_k} v_{i_0} (L_{k, 1}), \quad \alpha = h(s-g) + \lambda_{i_0, k}, \quad v_k = \rho_{i_0, j_{k, 1}} - \rho_{i_0, j_{k, 0}}, \quad V_1 = [\log^3 n]. \]

**Proof.** Let \( \mu_k = j_{k, 1} - 1. \) By Lemma 8, we obtain
\[ H_{i_0, k} \leq \sum_{\beta \in \text{U}_{\Lambda_{i_0, k+1}} \Lambda_{i_0, k} \setminus \{\mu_k\}} \tilde{\vartheta}_{i_0, k, \mu_k, \rho}^*, \text{ with } \tilde{\vartheta}_{i_0, k, \mu_k, \rho}^* = \sum_{\nu_{i_0, \mu_k} \leq n} \vartheta_{i_0, k, \mu_k, \rho}, \quad (57) \]

where \( \vartheta_{i_0, k, \mu_k, \rho} \in [0, 1]. \) Let
\[ \tilde{\vartheta}_{i_0, k, \mu_k, \rho}^* = \sum_{\nu_{i_0, \mu_k} \leq n} \tilde{\vartheta}_{i_0, k, \mu_k, \rho}, \text{ with } \tilde{\vartheta}_{i_0, k, \mu_k, \rho} = \sum_{\rho_{i, \mu_k} \in [1, n]} \chi_{i_0, k, \mu_k, \rho} \tilde{\nu}_{i_0, k, \rho}, \quad (58) \]
\[
\beta = \text{card}\{V_1 \leq \rho_{i_0,\mu_k} \leq n \mid \delta_{i_0,\mu_k,\rho} < \delta_{i_0,\mu_k,\rho} \}.
\]

By (20) and (44) we get, that (56) may be derived from the inequality
\[
\beta \leq (h + 1)V_1.
\]

We fix \(r_j\) for \(j \in \{1, \ldots, h\} \setminus \{\mu_k\}\). Suppose that \(1_{i_0,k,\rho} = 1\). By (44) and (56), we have \(\rho_{i_0,jk,0} - \rho_{i_0,jk,0-1} = v_k + \rho_{i_0,jk,0} - \rho_{i_0,jk,0-1} \geq v_k + V_1\). From (50), we obtain
\[
\varrho := L_{k,1} + L_{k,2}p_{i_0}^{\rho_{i_0,jk,0} - \rho_{i_0,jk,0-1}} \equiv 0 \mod p_{i_0}, \quad \varrho \equiv 0 \mod p_{i_0}^{v_k + V_1}. \tag{60}
\]

If \(L_{k,2} = 0\) then
\[
L_{k,1} \equiv 0 \mod p_{i_0}^{v_k + V_1}, \quad \bar{1}_{i_0,k,\rho} = \delta_{p_{i_0}}(L_{k,1}) = 1, \quad \delta_{i_0,\mu_k,\rho} \leq \delta_{i_0,\mu_k,\rho}.
\]

Hence \(\beta = 0\) and (59) follows. Now let \(L_{k,2} \neq 0\), \(\text{ord}_{p_{i_0}}(L_{k,2}) = \xi, \quad L_{k,2} = L_{k,3}p_{i_0}^{\xi}, \quad (L_{k,3}, p_{i_0}) = 1\). By (60), (56) and (48) we obtain
\[
L_{k,1} + L_{k,3}p_{i_0}^{\xi - \rho_{i_0} + \rho_{i,\mu_k}} \equiv 0 \mod p_{i_0}^{v_k + V_1}, \quad v_k \in [0, \lambda_{i_0,k}V_1]. \tag{61}
\]

It is easy to see that this congruence is false for \(\rho_{i,\mu_k} < -\xi + \hat{\rho}_{i_0}\). Therefore \(\delta_{i_0,\mu_k,\rho} = 0, \quad \delta_{i_0,\mu_k,\rho} = 0\) and \(\delta_{i_0,\mu_k,\rho} \leq \delta_{i_0,\mu_k,\rho}\) for \(\rho_{i,\mu_k} < -\xi + \hat{\rho}_{i_0}\). For \(\rho_{i,\mu_k} \geq -\xi + \hat{\rho}_{i_0} + (\lambda_{i_0,k} + 1)V_1 \geq -\xi + \hat{\rho}_{i_0} + v_k + V_1\), we derive from (61) that \(\bar{1}_{i_0,k,\rho} = \delta_{p_{i_0}}(L_{k,1}) = 1\) and \(\delta_{i_0,\mu_k,\rho} \leq \delta_{i_0,\mu_k,\rho}\). Hence, \(\beta \leq (\lambda_{i_0,k} + 1)V_1\) and (59) follows. Thus, Lemma 9 is proved. \(\blacksquare\)

**Lemma 10.** Let \(i_0 \in T_s\). Then
\[
\Phi_{m,\tau} \leq \prod_{k=1}^{a_{i_0}} \hat{H}_{i_0,k}, \quad \hat{H}_{i_0,k} = \max_{r_1, \ldots, r_8} H_{i_0,k}, \quad H_{i_0,k} = \sum_{r_j \in U_{i_0}} \chi_{i_0,k} \bar{1}_{i_0,k,r} \quad (\text{see} \ (54)).
\]

**Proof.** By Lemma 6 and Lemma 7, we get that this inequality is true for \(a_{i_0} = 1\). Let \(a_{i_0} \geq 2, \ k_1 \in [0, a_{i_0} - 1]\) and let
\[
\hat{\Phi}_{m,\tau,k_1} = \max_{\rho_1, \ldots, \rho_h} \Phi_{m,\tau,k_1}, \quad \text{with} \quad \Phi_{m,\tau,k_1} = \sum_{r_j \in U_{i_0}} \chi_{i_0,k,\rho} \bar{1}_{i_0,k,r} \quad (\text{see} \ (54)).
\]

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Hence \( \Phi_{m,\tau,a_{i_0}-1} = H_{i_0,a_{i_0}} \). According to (49), we have \( \Lambda_{i_0,0} = 0 \), \( \Lambda_{i_0,a_{i_0}} = \hbar \). By Lemma 6 and Lemma 7, we obtain \( \Phi_{m,\tau} \leq \Phi_{m,\tau,0} \). We see that

\[
\Phi_{m,\tau,k_1} = \sum_{\rho_j \in U_{\tau_2}} \prod_{k=k_1+2}^{a_{i_0}} \chi_{i_0,k,\rho} \sum_{j \in [\Lambda_{i_0,k_1+1}] \cup [\Lambda_{i_0,k_1+1}+1,n]} \chi_{i_0,k_1+\rho,1} \sum_{\rho_j \in U_{\tau_2}} \chi_{i_0,k,\rho} \prod_{k=k_1+2}^{a_{i_0}} \chi_{i_0,k,\rho} \prod_{j \in [\Lambda_{i_0,k_1+1}] \cup [\Lambda_{i_0,k_1+1}+1,n]} \chi_{i_0,k,\rho}
\]

and \( \Phi_{m,\tau,k_1} \leq \hat{H}_{i_0,k_1+1} \Phi_{m,\tau,k_1+1} \) for \( k_1 \in [0, a_1 - 2] \). Therefore

\[
\Phi_{m,\tau} \leq \Phi_{m,\tau,0} \leq \hat{H}_{i_0,1} \times \cdots \times \hat{H}_{i_0,a_{i_0}-1} \Phi_{m,\tau,a_{i_0}-1} = \hat{H}_{i_0,1} \times \cdots \times \hat{H}_{i_0,a_{i_0}}.
\]

Hence, Lemma 10 is proved.

**Lemma 11.** With notation as above

\[
H_k \ll n^{(\lambda_{i_0,k-1})+(s-1)+1} V_1^{h(s-\xi)+\lambda_{i_0,k}} \quad \text{and} \quad \Phi_{m,\tau} \ll n^{h(s-1)-(s-2)a_{i_0} \log^{3h^{2+}} n}.
\]

**Proof.** Taking into account (43), (48) and that \( \partial_{i_0,k,\mu,\rho} \leq 1 \), we get from Lemma 8 the estimate for \( H_k \). Using Lemma 10 and (44), we get the estimate for \( \Phi_{m,\tau} \):

\[
\Phi_{m,\tau} \ll \prod_{k=1}^{a_{i_0}} n^{\lambda_{i_0,k} (s-1) + 1} V_1^{h(s-\xi)+\lambda_{i_0,k}} \ll n^v V_1^{h^{2+} n}, \quad \text{with}
\]

\[
v = \sum_{k=1}^{a_{i_0}} (\lambda_{i_0,k} (s-1) - s + 2) = h(s-1) - (s-2)a_{i_0}, \quad V_1 = [\log^{3} n]. \quad (62)
\]

Hence, Lemma 11 is proved.

**Lemma 12.** Let \( s \geq 3 \), and let \( s > s \) or \( s = s \) and \( \exists i_0 \) with \( a_{i_0} \neq \hbar/2 \). Then

\[
\Phi_{m,\tau} \ll n^{h s/2-1/3}. \quad (63)
\]

**Proof.** Let \( \exists i_0 \in T_s \) with \( a_{i_0} \geq (h+1)/2 \). By Lemma 11, we get

\[
\Phi_{m,\tau} \ll n^v \log^{3h^{2+} n}, \quad v = h(s-1) - (s-2)a_{i_0} \leq h(s-1) - (s-2)(h+1)/2 \quad (64)
\]

\[= h s/2 - (s - 2)/2 =: v_1.\]
If \( s \geq 3 \), then \( v_1 \leq h/s/2 - 1/2 \). If \( s = 2 \), then \( v_1 = h/s/2 \leq h/s/2 - 1/2 \). If \( s = 1 \), then \( v_1 = h/s/2 + 1/2 \leq h/s/2 - 1/2 \). Hence (63) is true.

Suppose that \( a_{i_0} \leq h/2 \) for all \( \exists i_0 \in T_s \) and \( s < s, s \geq 3 \). From (45), we get

\[
\Phi_{m, \tau} \ll n^{hs/2} \log^{3hs} n \ll n^{hs/2-1/3}.
\]

Now suppose that \( s = s, s \geq 3 \) and \( \exists i_0 \in T_s \) with \( a_{i_0} \neq h/2 \). We have considered the case \( \exists i_0 \in T_s \) with \( a_{i_0} \geq (h + 1)/2 \) in (64). Let \( a_i \leq h/2 \) for all \( i \in [1, s] \) and \( a_i < h/2 \) for some \( i_0 \in [1, s] \). By (45), we obtain

\[
\Phi_{m, \tau} \ll n^{hs/2-1/2} \log^{3hs} n \ll n^{hs/2-1/3}.
\]

Hence, Lemma 12 is proved. ■

4. Completion of the proofs of the Theorems.

4.1. Proof of Theorem 1. Bearing in mind the monotony of the \( L_p \) norm, we get that it is enough to consider only the case of \( p = h = 2q \). Now we show that the assertion of Theorem 1 is a simple consequence of Lemma 13. Indeed. From (12) and Lemma 13, we derive \( D_{T_s}(Q, N) \ll n^{q^s} \). By Lemma 5, we have \( E_s(D_{T_s, 2q, 1}(Q, N)) \ll n^{q^s} \). According to (27), we obtain \( E_s(D_{T_s, 2q, 2}(Q, N)) \ll n^{q^s} \). Using (26), we get

\[
E_s(D_{T_s, 2q}(Q, N)) \ll n^{q^s}. \tag{65}
\]

Hence \( \| D_{T_s}(Q, N) \|_{s, 2q} \ll n^{q^s/2} \). Applying (21) and Minkowski’s inequality, we derive:

\[
\| D(x, (H(k))^{Q-N-1}) \|_{s, 2q} \ll \sum_{s=1}^{s} \sum_{T_s \subseteq [1, \ldots, s]} \| D_{T_s}(Q, N) \|_{s, 2q} + \log^s n \ll n^{s/2}. \tag{66}
\]

Therefore, Theorem 1 is proved. ■

Lemma 13. We have for \( s \geq 3 \) or for \( s = s = 2 \) that

\[
D_{T_s, a, \lambda, \tau}(Q, N) \ll n^{hs/2-1/4}, \quad \text{and} \quad D_{T_s, a, \lambda, \tau}(Q, N) \ll n^{hs/2} \text{ for } s = 2, s = 1. \tag{67}
\]

We will prove Lemma 13 separately for the cases \( s = 1, s = 2, s \geq 3 \), \( \min \lambda_{i,j} = 1, \max \lambda_{i,j} \geq 3 \) and \( \min \lambda_{i,j} = \max \lambda_{i,j} = 2 \).

4.1.1. Case \( s = 1 \).
Lemma 14. Let $s = 1$. Then (67) is true.

Proof. Using (43) and Lemma 12 with $s \geq 3$, $s = 1$, we get (67).

Now let $s = 2$ and $s = 1$. Consider the case $T_s = \{1\}$. The case $T_s = \{2\}$ is similar. Suppose that $a_1 < \delta_j$. By (43), we have $\Phi_{m,\eta} \ll n^{h-1}\log^{3h} n$. Applying (43), we obtain (67).

Let $a_1 = \delta_j$. Hence $\lambda_{1,k} = 1$ for all $k \in [1, \delta_j]$. By Lemma 9, we get $H_{1,k} \ll \tilde{H}_{1,k} + O(\log^{3h} n)$. From Lemma 10 and Lemma 11, we obtain $\Phi_{m,\eta} \ll \prod_{k=1}^h \tilde{H}_{1,k} + O(n^{h-3/4})$. Consider $\tilde{H}_{1,k}$. Using Lemma 7 and Lemma 9, we derive

$$m_{1,k}' + m_k M_{1,\rho_k} \equiv 0 \mod p_1^{V_1}, \quad k = 1, \ldots, \delta_j,$$

(68)

with $\max(|m_{1,k}'|, |m_k|) \leq V_2 = n^{4h}, \ m_k \neq 0$. Let $\text{ord}_{p_1}(m_k) = \beta_k, \ m_k = m_k p_1^{\beta_k}$, $(p_1, m_k) = 1$. Hence $\beta_k \leq 4h \log_2 n \leq 1/2V_1 = [\log_2 n]/2$. According to (12), $M_{1,\rho} \equiv (p_\rho/p_1^{\rho_1})^{-1} \mod p_1^{\rho_1}$. Therefore

$$m_{1,k}' = m_{1,k}' p_1^{\beta_k}, \quad (p_1, m_{1,k}') = 1$$

$$m_{1,k}' p_2^{\rho_2,k} - 1 \equiv 0 \mod p_1^{[V_1/2]}, \quad k = 1, \ldots, \delta_j.$$

Applying Corollary 1, we get that this congruence is equivalence. Hence

$$-m_{1,k}' p_2^{\rho_2,k} = m_k, \quad k = 1, \ldots, \delta_j.$$

(69)

Using (43), we obtain

$$\mathbb{D}_{T_s,a,\lambda,\tau} \ll \sum_{|m_{1,k}'|, |m_k| \leq V_2} \sum_{j \in [1, \delta_j]} \prod_{i=1, j \in [1, \delta_j]}^h 1 \left( m_{1,i,j}' p_2^{\rho_2,k} \right) \ll \sum_{\rho_i,j \in [1, \delta_j]} \prod_{i=1, j \in [1, \delta_j]}^h 1 \left( p_2^{\rho_2,k} \right) \ll n^h.$$

Hence, Lemma 14 is proved. ■

4.1.2. Case $s \geq 2$ and there exists $(i, k)$ with $\lambda_{i,k} = 1$.

Lemma 15. Let $s \geq 2$, and let $\lambda_{i_0,k_0} = 1$ for some $i_0 \in T_s$ and $k_0 \in [1, a_{i_0}]$. Then

$$\mathbb{D}_{T_s,a,\lambda,\tau} \ll n^{hs/2-1/4}.$$

Proof. Let $j_{k_0} = \Lambda_{i_0,k_0-1} + 1 = \Lambda_{i_0,k_0}$. By Lemma 7 and Lemma 9, we get

$$H_{i_0,k_0} = \tilde{H}_{i_0,k_0} + O(\log^{3h} n), \quad \tilde{H}_{i_0,k_0} = \sum_{\rho_{j_{k_0}} \in T_s} \delta_{p_{i_0}} \left( m_{j_{k_0}} M_{i_0,\rho_{j_{k_0}}} + m_{i_0, j_{k_0}}' \right).$$
Suppose that $\tilde{H}_{i_0,k_0} \geq 1$. Then
\[ m_{j_0} M_{i_0,j_0} + m'_{i_0,j_0} \equiv 0 \mod p_{i_0}^{V_1}. \]
Similarly to (68) and (69), we derive
\[ -m'_{i_0,j_0} \prod_{j \in [1,n], j \neq i_0} p_{i,j_0}^{\mu_i} = m_{j_0}. \]

Bearing in mind that $s \geq 2$, we obtain from (18) and (20) that $\max_{i \neq i_0} \rho_{i,j_0} \geq V_1 = [\log_2 n]$. Taking into account that $\max(|m'_{i,j_0}|, |m_{j_0}|) \leq V_2 = n^{\Delta s}$, we have a contradiction. Thus $H_{i_0,k_0} = O(\log^3 n)$.

Repeating the proof of Lemma 11 and Lemma 12, we get
\[ \Phi_{m,\tau} \ll n^{-1} \prod_{k=1}^{\alpha_{i_0}} n^{(\lambda_{i_0,k}^{-1})(s-1)+1} \log^3 n \ll n^{h(s-1)-(s-2)\alpha_{i_0}-1} \log^{3h^2 s} n. \]

If $s = 2$, then $\Phi_{m,\tau} \ll n^{h-3/4}$. Let $s \geq 3$, and let $s > s$ or $s = s$ and $3i_0$ with $a_{i_0} \neq h/2$. By Lemma 12, we get $\Phi_{m,\tau} \ll n^{h s/2-1/4}$. Now let $s = s \geq 3$ and $a_{i_0} = h/2$. Then
\[ \Phi_{m,\tau} \ll n^{h(s-1)/2+h/2-1} \log^{2h^2 s} n \ll n^{h s/2-1} \log^{3h^2 s} n \ll n^{h s/2-1/2}. \]
By (43), we get $\mathbb{D}_{T_s,a,k,\tau} \ll n^{h s/2-1/4}$. Therefore, Lemma 15 is proved.}

**4.1.3. Case $s \geq 2$, $\lambda_{i,k} \geq 2$ for all $i,k$ and $\lambda_{i_0,k_0} \geq 3$ for some $i_0,k_0$.**

**Lemma 16.** Let $s \geq 2$, $\lambda_{i,k} \geq 2$ for all $i,k$, and let there exist $i_0, i_1 \in \mathbb{T}_s$, $i_0 \neq i_1$, $k_1 \neq k_2 \leq a_{i_0}$, $l \in [1, a_{i_1}]$, $\mu_{k_1} \in [\Lambda_{i_0,k_1-1} + 1, \Lambda_{i_0,k_1}]$, $\mu_{k_2} \in [\Lambda_{i_0,k_2-1} + 1, \Lambda_{i_0,k_2}]$, $j_1, j_2 \in [\Lambda_{i_1,l-1} + 1, \Lambda_{i_1,l}]$, with $\hat{t}_{i_1}(j_1) = \mu_{k_1}$, $\hat{t}_{i_1}(j_2) = \mu_{k_2}$. Then
\[ \Phi_{m,\tau} \ll n^{h s/2-3/4}, \quad \mathbb{D}_{T_s,a,k,\tau} \ll n^{h s/2-1/2}. \] (70)

**Proof.** Applying (48), Lemma 6 and Lemma 7, we obtain
\[ \Phi_{m,\tau} \leq \sum_{\rho_j \in \mathbb{T}_{i_1}} \chi_{i_1,\rho} \prod_{k=1}^{\alpha_{i_0}} \chi_{i_0,k,\rho} \frac{1}{i_0,k,\rho} \leq A \max_{\rho_1,\ldots,\rho_n} B_{\rho} C, \] (71)
where
\[ A = \prod_{k=1}^{k_1-1} \max_{\rho_1,\ldots,\rho_n} \sum_{\rho_j \in \mathbb{T}_{i_1}} \chi_{i_0,k,\rho} \frac{1}{i_0,k,\rho}, \quad C = \prod_{k=k_1+1}^{\alpha_{i_0}} \max_{\rho_1,\ldots,\rho_n} \sum_{\rho_j \in \mathbb{T}_{i_1}} 1 \times \chi_{i_0,k,\rho} \frac{1}{i_0,k,\rho}, \quad B_{\rho} = \sum_{\rho_j \in \mathbb{T}_{i_1}} \chi_{i_1,l,\rho} \chi_{i_0,k,\rho} \frac{1}{i_0,k,\rho}. \]

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Consider $A$ and $C$. Using Lemma 10 and Lemma 11, we get

$$A \leq \prod_{k=1}^{k_1-1} H_{1,k} \ll \prod_{k=1}^{k_1-1} n(\lambda_{1,k-1})^{(z-1)+1} \log^{3h_s} n,$$

$$C \leq \prod_{k=k_1+1}^{a_{k_0}} H_{1,k} \ll \prod_{k=k_1+1}^{a_{k_0}} n(\lambda_{1,k-1})^{(z-1)+1} \log^{3h_s} n. \quad (72)$$

Consider $B_\rho$. By (48), we get that if $\chi_{i_0,k_1,\rho} = 1$, then

$$z_\rho = 1, \quad \text{where} \quad z_\rho = \prod_{\rho \in \Lambda_{i_0,k_1}} \Delta(0 \leq \rho_{i_0,j} - \rho_{i_0,j-1} \leq V_1). \quad (73)$$

Bearing in mind that $\lambda_{i_0,k_1} \geq 2$, we have that there exists $\mu_0 \in [\Lambda_{i_0,k_1-1} + 1, \Lambda_{i_0,k_1}] \setminus \{\mu_k\}$. It is easy to see that

$$B_\rho = \sum_{\rho_0 \in U_{k_0}} B_{\rho,\mu_0}, \quad B_{\rho,\mu_0} = \sum_{\rho \in U_{k_0}} z_\rho \chi_{i_1,\rho} \chi_{i_0,k_1,\rho} \chi_{i_0,k_1,\rho} \chi_{i_0,k_1,\rho}. \quad (74)$$

By (48) and conditions of Lemma 16, we obtain

$$\chi_{i_1,\rho} \leq \prod_{j \in [\Lambda_{i_1,k_{1-1}} + 1, \Lambda_{i_1,k_{1}}]} \Delta(0 \leq \rho_{i_1,j} - \rho_{i_1,j-1} \leq hV_1).$$

Using Lemma 8 and (73), we derive

$$B_{\rho,\mu_0} \leq \Delta(0 \leq \rho_{i_1,j} - \rho_{i_1,j-1} \leq hV_1) \sum_{V_1 \leq \rho_{i_0,j} \leq n} \sum_{\rho \in [1,n]} \chi_{i_0,k_1,\rho} \chi_{i_1,\rho} \leq \Delta(0 \leq \rho_{i_0,j} - \rho_{i_0,j-1} \leq V_1).$$

Hence

$$B_\rho \leq \sum_{V_1 \leq \rho_{i_0,j} \leq n} \sum_{j \in [\Lambda_{i_0,k_{1-1}} + 1, \Lambda_{i_0,k_1}]} \Delta(0 \leq \rho_{i_0,j} - \rho_{i_0,j-1} \leq V_1) \leq \sum_{V_1 \leq \rho_{i_0,j} \leq n} B_1 B_2 B_3,$$

where

$$B_1 = \sum_{\rho \in [1,n]} \chi_{i_1,\rho} \ll V_1^{\lambda_{0,k_{1-1}}} \ll \log^{2\lambda_{0,k_{1-1}}} n,$$

$$B_2 = \sum_{j \in [\Lambda_{i_0,k_{1-1}} + 1, \Lambda_{i_0,k_1}]} \chi_{i_0,k_1,\rho} \ll n^{\lambda_{0,k_{1-1}}} V_1, \quad B_3 = \#\{1 \leq \rho_{i,j} \leq n \mid \rho_{j} \in U_{k_0}, \rho_{i} \in U_{k_0}, 1 \leq \rho_{i,j} \leq \rho_{i,j-1}, \rho_{i,j} \leq V_1 \} \ll n^{(\lambda_{0,k_{1-1}})(s-2)} V_1^{(s-s)} \lambda_{0,k_{1}}.$$
Therefore
\[ B_\rho \ll n^{1+(\lambda_0,k_0-2)+q(\lambda_0,k_0-1)(s-2)}V_{1}^{h_0} n \ll n^{(\lambda_0,k_0-1)(s-1)} \log^{3h_0} n. \]

By (71), (72), (74) and (62) we get
\[ \Phi_{m,\tau} \ll n^{-1} \prod_{k=1}^{a_0} n^{(\lambda_0,k-1)(s-1)+1} \log^{3h_0} n \ll n^{h(s-1)-(s-2)a_0-1+\frac{1}{100}}. \tag{75} \]

Let \( s = 2 \), then \( \Phi_{m,\tau} \ll n^{h-1+\frac{1}{100}} < n^{h(s-2)/3}. \)

Let \( s \geq 3. \) By (45), Lemma 11 and Lemma 12, we need to check only the case \( s = s, h = 2g \) and \( a_0 = q. \) We see that \( \Phi_{m,\tau} \ll n^{q(2(s-1)+2)-1+\frac{1}{100}} \ll n^{q-3/4}. \)

Now, by (43), we get the assertion of Lemma 16. \( \blacksquare \)

**Lemma 17.** Let \( s \geq 2, \lambda_{i,k} \geq 2 \) for all \( i, k, \lambda_{i,l} \geq 3 \) for some \( i_1 \in T_s \) and \( l \in [1, a_i]. \) Then
\[ \mathbb{D}_{T_s,a,\lambda,\tau} \ll n^{h(s-2)/4}. \tag{76} \]

**Proof.** Let \( i_0 \in T_s, i_0 \neq i_1, j \in [\Lambda_{i_0,l-1} + 1, \Lambda_{i_0,l}] \) and let \( \hat{\tau}_i (j) = \mu_k \in [\Lambda_{i_0,k-1} + 1, \Lambda_{i_0,k}] \) for some \( k = k(i_0, i_1, j). \) Suppose that \( k(i_0, i_1, j_1) \neq k(i_0, i_1, j_2) \) for some \( j_1, j_2 \in [\Lambda_{i_1,l-1} + 1, \Lambda_{i_1,l}]. \) Then (70) follows from Lemma 16. Now let \( k_0 := k(i_0, i_1, \lambda_{i_1,l}) = k(i_0, i_1, j) \) for all \( j \in [\Lambda_{i_1,l-1} + 1, \Lambda_{i_1,l}]. \) Bearing in mind that \( \lambda_{i_1,l} \geq 3, \) we have that there exists \( j_1 < j_2 < j_3 \) with \( j_\nu \in [\Lambda_{i_1,l-1} + 1, \Lambda_{i_1,l}], \nu = 1, 2, 3. \)

By (48), we get that if \( \chi_{i_1,l,\rho} = 1, \) then
\[ z_\rho = 1, \quad \text{where} \quad z_\rho = \prod_{j \in [\Lambda_{i_1,l-1}+2, \Lambda_{i_1,l}]} \Delta(0 \leq \rho_{i_1,j}(j) - \rho_{i_1,j}(j-1) \leq V_1). \]

Similarly to the proof of Lemma 16, we get \( \Phi_{m,\tau} \leq A \max_{\rho} B_{i_1,l,\rho} C \) (see (70) - (74)), where
\[ A = \prod_{k=1}^{l-1} n^{(\lambda_{i_1,k-1})(s-1)+1} \log^{3h_0} n, \quad C = \prod_{k=l+1}^{a_1} n^{(\lambda_{i_1,k-1})(s-1)+1} \log^{3h_0} n, \]
\[ B_{i_1,l,\rho} = \sum_{\rho_{i_1,j} \in \mathbb{U}_{T_s}, j \in [\Lambda_{i_1,l-1}+1, \Lambda_{i_1,l}]} \chi_{i_0,k_0,\rho} \chi_{i_1,l,\rho} 1_{i_1,l,\rho}. \]

It is easy to see that
\[ B_{i_1,l,\rho} \leq \sum_{\rho_{i_1,j} \in \mathbb{U}_{T_s}, j \in [\Lambda_{i_1,l-1}+1, \Lambda_{i_1,l}] \setminus \{j_3\}} B_{\rho_{i_1,j}} \quad B_{\rho_{i,j}} = \sum_{\rho_{i_1,j} \in \mathbb{U}_{T_s}} z_\rho \chi_{i_0,k_0,\rho} \chi_{i_1,l,\rho} 1_{i_1,l,\rho}. \tag{77} \]
By (48), we have
\[ \chi_{i_0, k_0, \rho} \leq \prod_{j \in [\Lambda_{i_0, k_0 - 1} + 1, \Lambda_{i_0, k_0}]} \chi_{i_0, k_0, j, \rho} \leq \Delta(\rho_{i_0, \tau_1(j_1)} - \rho_{i_0, \tau_1(j_2)}) \leq hV_1. \]

Using Lemma 8, we derive
\[
B_{p_{i_0,j_1},i_0,j_2} \leq \Delta(\rho_{i_0,\tau_1(j_1)} - \rho_{i_0,\tau_1(j_2)}) \leq hV_1 \sum_{V_1 \leq \rho_{i_0,\tau_1(j_3)} \leq n} \sum_{\rho_{i_0,\tau_1(j_3)} \in [1, n]} \prod_{i \in [1, s]} \chi_{i_1, l, j, i, \rho} \sum_{\rho_{i_1, \tau_1(j_3)} \leq j \neq i_1} \chi_{i_1, l, j, \rho} \leq hV_1 \sum_{V_1 \leq \rho_{i_0,\tau_1(j_3)} \leq n} \prod_{j \in [\Lambda_{i_1, l + 1}, \Lambda_{i_1, l} \leq n]} \Delta(0 \leq \rho_{i_1, \tau_1(j)} - \rho_{i_1, \tau_1(j - 1)} \leq V_1). \]

Hence
\[
B_{i_1, l, \rho} \leq \sum_{V_1 \leq \rho_{i_1, \tau_1(j_3)} \leq n} \rho_{i_1, \tau_1(j_3)} \in U_{T_{k_0}} \prod_{j \in [\Lambda_{i_1, l + 1}, \Lambda_{i_1, l} \leq n]} \Delta(0 \leq \rho_{i_1, \tau_1(j)} - \rho_{i_1, \tau_1(j - 1)} \leq V_1) \leq \sum_{V_1 \leq \rho_{i_1, \tau_1(j_3)} \leq n} B_1 B_2 B_3, \]

where
\[
B_1 = \sum_{\rho_{i_1, \tau_1(j)} \leq \rho_{i_1, \tau_1(j_3)} \leq hV_1} V_1^\lambda_{i_1, j - 1} \ll \log^{3\lambda_{i_1, j}} n, \quad B_2 = \sum_{1 \leq \rho_{i_0, \tau_1(j)} \leq hV_1} \rho_{i_0, \tau_1(j)} \ll n^{\lambda_{i_1, j - 1}} V_1, \quad B_3 = \#\{1 \leq i, j \leq n | \rho_j \in U_{T_{k_0}}, 1 \leq i \leq s, i \neq i_0, i_1, j \in [\Lambda_{i_1, l + 1}, \Lambda_{i_1, l} \leq n] \ll n^{(\lambda_{i_1, l - 1})/2} V_1^{(s - 2)} \lambda_{i_1, j}. \}
\]

Therefore
\[
B_{i_1, l, \rho} \ll n^{1 + (\lambda_{i_1, l - 2})/2} V_1 h^s \ll n^{(\lambda_{i_1, l - 1})(s - 1)} \log^{3h} n. \]

By (70), (72), (74) and (62) we get
\[
\Phi_{m, \tau} \ll n^{-1} \prod_{k=1}^{a_{i_1} \lambda_{i_1, k - 1} (s - 1) + 1} \log^{3h} n \ll n^{h(s - 1) - (s - 2) a_{i_1} - 1 + \frac{1}{s - 2}}. \]

Which is the same estimate as (75). Similarly to the end of the proof of Lemma 16, we get the assertion of Lemma 17.
4.1.4. Case $s \geq 2$ and $\lambda_{i,k} = 2$ for all $i,k$.

**Lemma 18.** Let $s \geq 2$, and let $\lambda_{i,k} = 2$ for all $i,k$. Then

$$D_{T_{s,a,\lambda},\tau} \ll n^{h_{s}/2 - 1/4}. \quad (78)$$

**Proof.** If $s > s \geq 2$, then $s \geq 3$ and (78) follows from Lemma 12. Now let $s = s \geq 2$. Taking into account that $\lambda_{i,k} = 2$ for all $i,k$, we get $\Lambda_{i,k} = 2k$ for all $i,k$. Hence $h = 2q$.

From (43), we have $\varpi_{\rho, m, 1} = 1$. Using (24) and (16), we get that there exists $k_1 \in [1, q]$ such that

$$m_{2k_1 - 1} / P_{2k_1 - 1} \neq -m_{2k_1} / P_{2k_1}.$$ \quad (79)

Let $i_0, i_1, \in T_s = [1, s]$, $l \in [1, q]$, $j_1 = \Lambda_{i_1, l} - 1 = 2l - 1$, $j_2 = \Lambda_{i_2, l} = 2l$ and let $\tilde{\tau}_{i_1}(j_1) = \mu_{k_1} \in \Lambda_{i_0, k_1 - 1, \Lambda_{i_0, k_1}}$, $\tilde{\tau}_{i_2}(j_2) = \mu_{k_2} \in [\Lambda_{i_0, k_2 - 1, \Lambda_{i_0, k_2} = [2k_2 - 1, 2k_2]}$

for some $k_2 \in [1, q]$. If $k_1 \neq k_2$, then (78) follows from Lemma 16. Now let $k_1 = k_2$ for all $i_1 \in [1, s] \setminus \{i_0\}$. Hence

$$\{\tilde{\tau}_{i_1}(2l - 1), \tilde{\tau}_{i_2}(2l)\} = \{2k_1 - 1, 2k_1\} \quad \forall i_1 \in [1, s] \setminus \{i_0\}.$$ 

Therefore, applying (44) and (46), we obtain

$$|\rho_{i_1, 2k_1} - \rho_{i_2, 2k_1 - 1}| = |\rho_{i_1, \tilde{\tau}_{i_1}(2l)} - \rho_{i_2, \tilde{\tau}_{i_2}(2l-1)}| \leq V_1 = \log^3 n, \quad l = l(i_1), \forall i_1 \in [1, s].$$

Hence

$$\# \{\rho_{2k_1} - \rho_{2k_1 - 1}\} \leq (2V_1 + 1)^s \ll \log^3 n. \quad (80)$$

According to Lemma 10 and Lemma 11, (78) follows from the estimate $H_{i_0, k_1} \ll n^{s - 3/4}$. By Lemma 9, it is enough to prove that $H_{i_0, k_1} \ll n^{s - 3/4}$. We examine Lemma 9 with $\Lambda_{i_0, k_1} = 2k_1, j_{k_1, 1} = 2k_1 - 1, j_{k_1, 1} = 2k_1 + 1$.

Suppose that $i_{i_0, k_1, \rho} = 1$. Applying Lemma 7 and Lemma 9, we obtain

$$\sum_{j=2k_1 - 1}^{2k_1} (m_{i_0, \rho} + M_{i_0,\rho}) p_{i_0, 2k_1 - 1} \equiv 0 \mod p_{i_0}^{v_{k_1} + V_1}, \; v_{k_1} = \rho_{i_0, 2k_1} - \rho_{i_0, 2k_1-1} \leq V_1. \quad (81)$$

Consider the case $v_{k_1} \geq V_1/4$. By (81), we get $m_{i_0, 2k_1} + M_{i_0,\rho_2k_1} \equiv 0 \mod p_{i_0}^{V_1/4}$, with max$(|m_{i_0, 2k_1}|, |m_{2k_1}|) \leq V_1 < n^{h_{s}}, m_{2k_1} \neq 0$. Let ord$_{p_{i_0}}(m_{2k_1}) = \beta_{k_1}, m_{2k_1} = m_{2k_1} \beta_{k_1}, (p_{i_0}, m_{2k_1}) = 1$. Hence $\beta_{k_1} \leq 4L \log n < V_1/8$. We get from Corollary 1 that the congruence

$$(-m_{i_0, 2k_1} / p_i^{\beta_{k_1}}) m_{2k_1}^{-1} \prod_{1 \leq l \leq 8, i \neq i_0} p_i^{\rho_{i, 2k_1} - 1} \equiv 0 \mod p_i^{V_1/8}$$

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is equality. But this is impossible because $s = s \geq 2$ and $\rho_{i, j} \geq V_1 = \lfloor \log_2 n \rfloor$ for all $j$ (see (18) and (20)).

Consider the case $v_k < V_1/4$ and $g_k := m_{i_0, k_1} + m_{i_0, k_1} - P_{i_0}^{v_k} = 0$. By (81) and (12), we get
\[
m_{2k_1} \prod_{1 \leq s, t \neq i_0} p_i^{\rho_{i, 2k_1} - \rho_{i, k_1}} = -p_{i_0}^{v_k} m_{2k_1-1} \mod p_{i_0}^{v_k} + V_1
\]
with $0 < |m_{2k_1-1}, m_{2k_1}| \leq V_3 = n^{\text{th}}$. Let $\text{ord}_{p_{i_0}}(m_j) = \beta_j$, $m_j = m_j p_{i_0}^{\beta_j}$, $(p_{i_0}, m_j) = 1$, $j = 2k_1 - 1, 2k_1$. By Corollary 1, we get that the congruence
\[
\prod_{1 \leq s, t \neq i_0} p_i^{\rho_{i, 2k_1} - \rho_{i, k_1}} \equiv -p_{i_0}^{v_k + \beta_{2k_1-1} - \beta_{2k_1}} m_{2k_1-1} / m_{2k_1} \mod p_{i_0}^{v_k + V_1 - \beta_{2k_1}}
\]
is equality (here $V_1 - \beta_{2k_1} \geq V_1/2$, $\beta_{2k_1}, \beta_{2k_1-1} \ll \log n$ and $v_{k_1} + \beta_{2k_1-1} - \beta_{2k_1} = 0$). Hence $m_{2k_1-1} / P_{2k_1} = -m_{2k_1}/P_{2k_1}$. But according to (78), it is impossible.

Consider the case $v_k < V_1/4$ and $g_k = m_{i_0, 2k_1} + m_{i_0, 2k_1-1} - P_{i_0}^{v_k} \neq 0$. Let
\[
\xi_k = m_{2k_1} M_{i_0, 2k_1} M_{i_0, 2k_1-1}^{-1} + P_{i_0}^{v_k} m_{2k_1-1} - 1.
\]
By (81), we obtain
\[
g_k + \xi_k M_{i_0, 2k_1} M_{i_0, 2k_1-1}^{-1} \equiv 0 \mod p_{i_0}^{v_k + V_1}. \tag{82}
\]
We fix $m_{2k_1-1}, m_{2k_1}, m_{i_0, 2k_1-1} - m_{i_0, 2k_1}$ and $\rho_{i, 2k_1} - \rho_{i, 2k_1-1}$ for $i = 1, \ldots, s$.

Let $\beta_1 = \text{ord}_{p_{i_0}}(g_k)$ and let $g_k = \hat{g}_k p_{i_0}^{\beta_1}$, $(p_{i_0}, \hat{g}_k) = 1$. Let $\beta_2 = \text{ord}_{p_{i_0}}(\xi_k)$ and let $\hat{\xi}_k = \hat{\xi}_k p_{i_0}^{\beta_2}$, $(p_{i_0}, \hat{\xi}_k) = 1$. We see that $\beta_1 \leq v_{k_1} + V_1/2$. By (82), we get $\beta_1 = \beta_2$ and
\[
\prod_{1 \leq s, t \neq i_0} p_i^{\rho_{i, 2k_1-1}} \equiv -\hat{\xi}_k / \hat{g}_k \mod p_{i_0}^{V_1/2}. \tag{83}
\]
By Corollary 1, we get that, for fixed $\rho_{i_0, 2k_1-1}$, the number of solutions of this congruence is no more than one. Therefore, the number of vectors $(\rho_{1, 2k_1-1}, \ldots, \rho_{s, 2k_1-1})$ satisfying (83) is less than $n + 1$. According to (80), there are only $O(\log^3 n)$ opportunities to choose $\xi_k$. Applying Lemma 9, we obtain $H_{i_0, k_1} \ll n \log^3 n \ll n^{s-3/4}$, $s \geq 2$. Thus Lemma 18 is proved. ■

4.2. Proof of Theorem 2. The assertion of Theorem 2 follows essentially from Lemma 21. To prove Lemma 21, we need firstly to compute the main value of the product of functions $\varphi_{r, 0, N, m}$ (see (22)):
\[
\hat{\gamma}_{r, m}^{(q)} = \int_0^1 \prod_{j=1}^q |\varphi_{r, 0, [N x_{s+1}], m}|^2 dx_{s+1}, \quad \varphi_{r, 0, N, m} = \frac{e(mN/P_r) - 1}{P_r (e(m/P_r) - 1)}. \tag{84}
\]
Taking into account (11), (22) and that \(|\sin(x)| \leq |x|\), we obtain

\[
|\gamma_{r,m}^{(q)}| \leq \prod_{j=1}^{q} \min(1, \pi N|m_j|/P_{r_j})/\tilde{m}_j^2.
\]  

(85)

Lemma 19. Let \(0 < |m_j| \leq n^{4s}\), \(j = 1, \ldots, q\). Then

\[
q := \sum_{r_j \in U_0 \atop j \in [1, q]} \left| \gamma_{r,m}^{(q)} - \theta_{r,m}^{(q)} \right| \ll n^{q-1} \log^3 n \frac{2}{m_1^2 \cdots m_q^2}, \quad \theta_{r,m}^{(q)} := \prod_{j=1}^{q} \frac{2}{P_{r_j}(1 - e(m_j/P_{r_j}))^2}, \ s \geq 3,
\]  

(86)

\[
\sum_{r_j \in U_0 \atop j \in [1, q]} \left| \gamma_{r,m}^{(2q)} - \theta_{r,m}^{(2q)} \right| \ll n^{q-1} \log^3 n \frac{2}{m_1^2 \cdots m_q^2}, \quad \gamma_{r,m}^{(2q)} = \int_0^1 \prod_{j=1}^{q} |\varphi_{r_j, [N x_3], 2[N x_3], m_j}|^2 dx_3, \ s = 2,
\]  

\[
\sum_{r_j \in U_0 \atop j \in [1, 2q]} \left| \gamma_{r,m}^{(3q)} - \theta_{r,m}^{(3q)} \right| \ll n^{q-1} \log^3 n \frac{2}{m_1^2 \cdots m_q^2}, \quad \gamma_{r,m}^{(3q)} = \int_0^1 \prod_{j=1}^{3q} |\varphi_{r_j, [N x_3], 2[N x_3], m_j}|^2 dx_3, \ s = 2
\]  

(87)

(see (50)), where \(U_s = \{ r \in [1, n] | P_r \leq 2^{n+\log^3 n} \}\), \(U_0 = U_2 \cap \{ r_1, r_2 \leq V_1 \}\).

Proof. We will prove the first statement. The proof of the second and third statements are similar. By (84) and (86), we obtain

\[
\gamma_{r,m}^{(q)} = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{q} |\varphi_{r, 0, k, m_j}|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{q} \frac{2 - 2 \cos(2\pi m_j k/P_{r_j})}{P_{r_j}(1 - e(m_j/P_{r_j}))^2} \gamma_{r,m}^{(q)} + \epsilon g_{r,m},
\]

\[
g_{r,m} := \sum_{J \subseteq \{1, \ldots, q\}, J \neq \emptyset} \left| \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j \in J} \cos(2\pi k m_j/P_{r_j}) \right|, \ |\epsilon| \leq 1.
\]  

(88)

It is easy to see that

\[
g_{r,m} \leq \sum_{J \subseteq \{1, \ldots, q\}, J \neq \emptyset} \sum_{\nu_1, \ldots, \nu_J \in \{-1, 1\}} \left| X_r \right|, \quad X_r = \frac{1}{N} \sum_{k=0}^{N-1} e\left( k \sum_{j \in J} \nu_j m_j/P_{r_j} \right).
\]  

(89)

Using (10), we have

\[
\left| X_r \right| \leq \min\left(1, \frac{1}{2N \left\langle Y'_r \right\rangle} \right), \quad \text{with} \quad Y'_r = \sum_{j \in J} \nu_j m_j/P_{r_j}.
\]

Let \(J = \text{card}(J), J = (j_1, \ldots, j_3), J = j_3\). By (11) and (86) - (89), we get

\[
q \leq \sum_{J \subseteq \{1, \ldots, q\}, J \neq \emptyset} \sum_{\nu_1, \ldots, \nu_J \in \{-1, 1\}} \sum_{r_j \in [1, n] \atop j \in [1, q], P_{r_j} \leq 2^{n+\log^3 n}} \frac{\delta_J}{\tilde{m}_1^2 \cdots \tilde{m}_q^2}, \quad \delta_J = \sum_{r_j \in U_s} \left| X_r \right|.
\]  

(90)
We fix $m_j, r_j, j \in J \setminus J$. Then $Y_r' = f' + \nu_J m_J/P_{r_J}$ for some $f'$. Let $f \equiv f'$ mod 1 and $f \in (-1/2, 1/2]$. Taking into account that the function $\langle z \rangle$ has period one, we get $\langle Y_r \rangle = \langle Y_r' \rangle$ with $Y_r = f + \nu_J m_J/P_{r_J}$. We see that $\bar{g}_J = \bar{g}_{J,1} + \cdots + \bar{g}_{J,5}$ with

$$\bar{g}_{J,i} = \sum_{r_J \in \{1, n\}^s, P_{r_J} \leq 2^{n+\log_3 n}} \min \left(1, \frac{1}{2N\langle Y_r \rangle} \right) \Delta(b_r = i), \quad m_J \neq 0, \quad (91)$$

$$b_r = \begin{cases} 1, & \text{if } \langle Y_r \rangle \geq n^s/N, \\ 2, & \text{if } |m_J|/P_{r_J} < 4n^s/N, \\ 3, & \text{if } |m_J|/P_{r_J} \geq 1/4, \\ 4, & \text{if } \langle Y_r \rangle < n^s/N, \quad 1/4 > |m_J|/P_{r_J} \geq 4n^s/N, \quad |f| > 2n^s/N, \\ 5, & \text{if } \langle Y_r \rangle < n^s/N, \quad 1/4 > |m_J|/P_{r_J} \geq 4n^s/N, \quad |f| \leq 2n^s/N. \end{cases} \quad (92)$$

Consider the case $b_r = 1$. By (91) and (92), we obtain

$$\bar{g}_{J,1} \leq \sum_{r_J \in \{1, n\}^s} \frac{1}{N} \frac{N}{n^s} \leq 1.$$ 

Consider the case $b_r = 2$. By (91) and (92), we derive

$$0.25N/n^s \leq 0.25N|m_J|/n^s \leq P_{r_J} \leq 2^{n+\log_3 n} \quad (1 \leq r_{i,J} \leq n, \ i = 1, \ldots, s),$$

$$n - 3 - s\log_2 n \leq \sum_{i=1}^s r_{i,J} \log_2 p_i \leq n + \log_2 n, \quad n = \lfloor \log_2 N \rfloor + 1.$$

It is easy to verify that the number of solutions of this inequality is equal to $O(n^{s-1}\log_2^3 n)$ and $\bar{g}_{J,2} = O(n^{s-1}\log_2^3 n)$.

Consider the case $b_r = 3$. By (91) and (92), we get

$$P_{r_J} \leq 4|m_J| \leq 4n^{4s}.$$ 

We see that the number of solutions of this inequality is equal to $O(\log_2^8 n)$ and $\bar{g}_{J,3} = O(\log_2^8 n)$.

Consider the case $b_r = 4$. We have $|Y_r| \geq |m_J/P_{r_J}| + |f| \leq 3/4$ and $\langle Y_r \rangle < n^s/N, \ n^s/N \to 0$ for $N \to \infty$. Hence $|Y_r| = \langle Y_r \rangle$. Taking into account that $\nu_J m_J/P_{r_J} = -f + Y_r$, we derive $\nu_J m_J/P_{r_J} \in [-f - n^s/N, -f + n^s/N]$. Therefore $|m_J|/P_{r_J} \in [|f|/2, 2|f|]$. Thus

$$\log_2 |m_J| = \sum_{i=1}^s r_{i,J} \log_2 p_i \in \lfloor \log_2 |f| - 1 \rfloor, \log_2 |f| + 1 \right]. \quad (93)$$
Bearing in mind that $|\log_2 |f|| \leq 2\log_2 N \leq 2n$, we get that the number of solutions of (93) is equal to $O(n^{s-1})$. By (91), we have $\tilde{\mathcal{A}}_{s-1} = O(n^{s-1})$.

Consider the case $b_r = 5$. Taking into account that $|f| \leq 2n^s/N$, $n^s/N \rightarrow 0$ and $|m_j|/P_{r,j} \leq 1/4$, we get that $|Y_r| \leq 3/8$. Hence $|Y_r| = \langle Y_r \rangle < n^s/N$. Therefore $4n^s/N \leq |m_j|/P_{r,j} \leq |Y_r| + |f| < 3n^s/N$. We have a contradiction.

Hence $\tilde{\mathcal{A}}_r = \tilde{\mathcal{A}}_{r,1} + \cdots + \tilde{\mathcal{A}}_{r,5} = O(n^{s-1}\log^3 n)$. By (56) and (90), we get $\varrho < n^{s-1}\log n / (\tilde{m}_{s,1}^2 \cdots \tilde{m}_{s,q}^2)$. Therefore, Lemma 19 is proved. ■

**Corollary 2.** Let $s = 1$, $s = 2$, $h = 2q$. Then

$$\varrho := E_{s+1}(\mathcal{D}_{T_1,h,1}([-N x_{s+1}], 2[N x_{s+1}])) = O(n^{hs/2-1/2}). \quad (94)$$

**Proof.** From (20), we have that if $r \in \mathcal{U}_{T_1}$, then $P_r \leq 2^{n+\log_2^2 n}$. Using (31), (37) and Lemma 19, we obtain

$$\varrho \ll n^{-s} + \mathcal{D}_{T_1}, \quad \mathcal{D}_{T_1} \leq \sum_{r \in \mathcal{U}_{T_1}} \sum_{j=1, \ldots, 2q} \sum_{i=1}^{s} \frac{1}{m_{i,j}} \ll n^{qs-1}\log^3 n \prod_{j=1}^{2q} \prod_{i=1}^{s} \frac{1}{m_{i,j}} \ll n^{qs-1}(\log n)^{2q(s+1)+3}.$$  

Hence, Corollary 2 is proved. ■

**Lemma 20.** With notations as above

$$E_{s+1}(\mathcal{D}_{T_s,h,1}(0, [N x_{s+1}])) \ll n^{hs/2-1/5}, \quad s \geq 3, \quad (95)$$

$$\|\mathcal{D}_{T_s}(0, [N x_{s+1}])\|_{s+1,2q} \ll n^{s/2-1/(10q)}, \quad s \geq 3, \quad s > s, \quad (96)$$

$$E_{s+1}(\mathcal{D}_{T_s,h,1}([-N x_{s+1}], 2[N x_{s+1}])) \ll n^{hs/2-1/5}, \quad s = s = 2, \quad (97)$$

$$\|\mathcal{D}_{T_s}([-N x_{s+1}], 2[N x_{s+1}])\|_{s+1,2q} \ll n^{s/2-1/(10q)}, \quad s = 2, \quad s = 1. \quad (98)$$

**Proof.** Using Lemma 5, Lemma 13 and (42), we get (95). By (24) and (26), we have $\mathcal{D}_{T_s}(0, [N x_{s+1}]) = \mathcal{D}_{T_s,2q,1}(0, [N x_{s+1}])$ for $s > s$. Applying (42) and Lemma 13, we obtain (96). Consider the case $s = 2$. According to (24) and (26), we get

$$\mathcal{D}_{T_s}([-N x_{s+1}], 2[N x_{s+1}]) = \mathcal{D}_{T_s,2q,1}([-N x_{s+1}], 2[N x_{s+1}]) \quad \text{for} \quad s > s = 1, \quad (98).$$

Using Corollary 2, we derive (98). Now (97) follows from (42), Lemma 5 and Lemma 13. Hence, Lemma 20 is proved. ■

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The next lemma is the main lemma in this section. Its proof is based on Lemma 19, rearrangements of domains of summations and multiple changes of orders of summations.

**Lemma 21.** With notations as above

$$s_{2q} = E_{s+1}(T_{s, 2q}, 0, [N x_{s+1}]) = \frac{(2q)!}{2^{q}} \left( E_{s+1}(T_{s, 2}, 0, [N x_{s+1}]) \right)^q + O(n^{gs-3/4}), \quad (99)$$

$$E_{s+1}(T_{s, 2q}, [-N x_{s+1}, 2[N x_{s+1}])] = \frac{(2q)!}{2^{q}} \left( E_{s+1}(T_{s, 2}, [-N x_{s+1}, 2[N x_{s+1}])] \right)^q + O(n^{gs-3/4}), \quad s = 2. \quad (100)$$

**Proof.** We will prove the first statement. The proof of the second statement is similar. Using (20), (22), (24) and (25), we obtain

$$s_{2q} = \sum_{r, \iota \in \mathcal{V}, r \in [1, s]} \sum_{i, j \in [1, 2q]} \varpi_{r, m, 2} \psi_{r, m} \beta_{r, m}, \quad \gamma_{r, m} = \int_0^1 \prod_{j=1}^{2q} \varphi_{r, 0, [N x_{s+1}], m_j} dx_{s+1},$$

$$\varphi_{r, 0, N, m} = \frac{e(mN/P_r) - 1}{P_r(e(m/P_r) - 1)}, \quad \beta_{r, m} = E_a \left( \prod_{j=1}^{2q} \psi_{r, 0} (m, x) e(\varphi_{r, m}) \right), \quad (101)$$

where

$$\varpi_{r, m, 2} = \Delta \left( h = 2q, \sigma = s, \exists \tau \in \mathbb{R} : m_{\tau(2k-1)}/P_{r(2k-1)} = -m_{\tau(2k)}/P_{r(2k)}, \right)$$

$$k \in [1, q], \quad \varphi_{r, m} = \sum_{j=1}^{2q} \frac{-m_j r_j}{P_r}, \quad |\varphi_{r, 0, N, m}| \leq \frac{1}{m},$$

$$\psi_r(m, x) = \prod_{i=1}^{s} \psi(i, \{ -m M_{r_i} / p_i \} p_i, x_{i, r_i}), \quad \psi(i, 0, x_{i, r_i}) = x_{i, r_i},$$

$$\psi(i, m', x_{i, r_i}) = 1 - e(-m' x_{i, r_i} / p_i) / e(m' / p_i - 1) \quad \text{for } m' \neq 0, \quad |\psi(i, m', x_{i, r_i})| \leq p_i. \quad (102)$$

By (26), we have

$$I_M^* = \left[ -(M - 1)/2, [M/2] \right] \setminus \{0\}. \quad \text{Let } m_j = \hat{m}_j P_{\alpha_j} \text{ with } (\hat{m}_j, p_0) = 1, \quad P_{\alpha_j} = p_1^{\alpha_1} \cdots p_{s+j}^{\alpha_{s+j}}. \text{ Then}$$

$$m \in I_{P_r}^* \iff \hat{m}_j \in I_{P_r}^{**} := \{ k \in \left[ -(P_r - 1) P_{\alpha}^{-1}/2, [P_r P_{\alpha}^{-1}/2] \right], \quad k \neq 0, \quad (k, p_0) = 1 \}. \quad (103)$$
Let \( A_{m,m,a} = (m_j = \tilde{m}_j P_{\alpha_j}, \ (\tilde{m}_j, p_0) = 1, \ j \in [1,2q]) \). It is easy to see
\[
\sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], m_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \Delta(A_{m,m,a}) = 1.
\]
Changing the order of the summation, we get from (101) and (103):
\[
\sum_{r_{i,j} \in [V_1,n], i \in [1,s], m_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \Delta(A_{m,m,a}) \gamma_{r,m}^* \beta_{r,m}^* 
= \sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], m_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \Delta(A_{m,m,a}) \gamma_{r,m}^* \beta_{r,m}^* 
= \sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], m_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \alpha_i, j \in [V_1,n], i \in [1,s], m_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \Delta(A_{m,m,a}) \gamma_{r,m}^* \beta_{r,m}^* 
\]
with
\[
\tilde{\omega}_{r,m,p_a} := \sum_{\tilde{m}_j \in I_{r_{j,x_0}}^*, j \in [1,2q]} \Delta(A_{m,m,a}) \gamma_{r,m}^* \beta_{r,m}^* 
\]
\[
\psi_{r,m} = 0 \quad \text{and} \quad |\gamma_{r,m}^* \beta_{r,m}^*| \leq p_2^2 |\gamma_{r,m}^*| \leq 2q \prod_{j=1}^{2q} p_0 \min(1, 2\pi N|m_j|/P_{r_j})/\tilde{m}_j 
= \prod_{j=1}^{2q} p_0^{-a_1,j} \cdots p_s^{-a_s,j} \min(1, 2\pi N|m_j|/P_{r_j-a_j})/\tilde{m}_j \leq \prod_{j=1}^{2q} p_0^{-1}\tilde{m}_j. \quad (105)
\]
Let \( A_1 = (\exists \tau \in \Xi_{2q}, \exists k \in [1,q] \ | \ |m_{r_k(2k-1)}| = |\tilde{m}_{r_k(2k)}| > n) \). Taking into account that \( \sum_{|m|>n} |m|^{-2} \leq 4/n \), and that \( \tilde{m}_{r_k(2k-1)} = \tilde{m}_{r_k(2k)} \), we have
\[
\sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], j \in [1,2q]} \tilde{\omega}_{r,m,p_a} \gamma_{r,m,p_a}^* \beta_{r,m,p_a}^* \Delta(A_1) 
\ll n^{-1} \sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], j \in [1,2q]} \tilde{\omega}_{r,m,p_a} \gamma_{r,m,p_a}^* \beta_{r,m,p_a}^* \Delta(A_1) 
\ll n^{-1} \sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], j \in [1,2q]} (P_{\alpha_1} \cdots P_{\alpha_{2q}})^{-1} \Delta(r_j = r_{j+q-\alpha_j+q} \alpha_j, \ j = 1, \ldots, q) 
\ll n^{-1} \sum_{\alpha_i, j \in [0,p_0sn], i \in [1,s], j \in [1,2q]} (P_{\alpha_1} \cdots P_{\alpha_{2q}})^{-1} n^{q_s} < n^{q_s-1}. \quad (106)
\]
Bearing in mind (105), we get for \( j \in [2,3] \), similarly to (106), that
\[
\sum_{\alpha_i, j \in [0,p_02sn], i \in [1,s], j \in [1,2q]} \tilde{\omega}_{r,m,p_a,2} \gamma_{r,m,p_a}^* \beta_{r,m,p_a}^* \Delta(A_j) \ll n^{q_s-3/4},
\]
Therefore, (107) is true for $j = 4$. Hence

$$\tilde{s}_q = \tilde{s}_q + O(n^{98-3/4}), \quad \tilde{s}_q = \sum_{\alpha_{i,j} \in [0, \log_2 n]} \sum_{r_{i,j} \in [V_1, n]} \sum_{\tilde{m}_{i,j} \in I_n} \tilde{\omega}_{r,m_P \alpha} \gamma_{r,m_P \alpha}^{(2q)}$$

$$\times \beta_{r,m_P \alpha}^{(2q)} \prod_{j=3}^4 (1 - \Delta(A_j)), \quad \text{with } I_n := \{k \in [-n, n] \mid k \neq 0, (k, p_0) = 1\}. \quad (108)$$

Taking into account that $\max \alpha_{i,j} \leq \log_2 n$ and $\min r_{i,j} \geq [\log_2 n]$, we get from (102) and (104) that

$$\tilde{\omega}_{r,m_P \alpha} = \Delta(\exists \tau \in \Xi_{2q} \mid \tilde{m}_{r(2k-1)} = -\tilde{m}_{r(2k)}) \quad \text{with } \alpha_{r(2k-1)} - \alpha_{r(2k)} = \alpha_{r(2k)}, \quad k \in [1, q).$$

Let

$$\tilde{\Xi}_q = \{(\sigma_1, \sigma_2), \sigma_i : \{1, \ldots, q\} \to \{1, \ldots, 2q\}, \ i = 1, 2 \mid \{\sigma_1(1), \sigma_2(1), \ldots, \sigma_1(q), \sigma_2(q)\} = \{1, \ldots, 2q\}, \ \sigma_1(k) < \sigma_2(k) \quad \forall k\}$$

and let $\ell_{\sigma_1, \sigma_2, r, \alpha} = \Delta(r_{\sigma_1(k)} - \alpha_{\sigma_1(k)} = r_{\sigma_2(k)} - \alpha_{\sigma_2(k)})$.

It is easy to verify that

$$\text{card}(\Xi_q) = \binom{2q}{2} \binom{2q-2}{2} \cdots \binom{2}{2} = \frac{(2q)!}{2^q}, \quad (109)$$

and that

$$\sum_{(\sigma_1, \sigma_2) \in \Xi_q} \ell_{\sigma_1, \sigma_2, r, \alpha} \tilde{\omega}_{r,m_P \alpha} (1 - \Delta(A_4)) = q! \tilde{\omega}_{r,m_P \alpha} (1 - \Delta(A_4)).$$

Therefore

$$\tilde{s}_q = \frac{1}{q!} \sum_{(\sigma_1, \sigma_2) \in \Xi_q} \sum_{\alpha_{i,j} \in [0, \log_2 n]} \sum_{r_{i,j} \in [V_1, n]} \sum_{\tilde{m}_{i,j} \in I_n} \ell_{\sigma_1, \sigma_2, r, \alpha} \tilde{\omega}_{r,m_P \alpha} \gamma_{r,m_P \alpha}^{(2q)}$$

$$\times \beta_{r,m_P \alpha}^{(2q)} \prod_{j=3}^4 (1 - \Delta(A_j)).$$
Therefore
\[
\dot{\mathcal{R}}(\rho, \mu, \sigma) = \Delta(\rho_k = \mathbf{r}_{\sigma_1(k)} - \alpha_{\sigma_1(k)} - \alpha_{\sigma_2(k)}),
\]
\[
\mu_k = \dot{m}_{\sigma_1(k)} = -\dot{m}_{\sigma_2(k)}, \quad k \in [1, q].
\]

It is easy to verify
\[
\sum_{\rho_i, j \in [V_1/2, n]} \sum_{j \in [1, q]} \sum_{r_{i,j} \in [V_1, n]} \sum_{j \in [1, q]} \sum_{\rho_i, j \in [V_1/2, n]} \sum_{j \in [1, q]} \sum_{\mu_j \in I_n^\prime} 1
\]

\[
\times \ell_{\sigma_1, \sigma_2, \sigma, \alpha} \dot{\mathcal{R}}(\rho, \mu, \sigma) \dot{\mathcal{R}}_{\mu, \nu, \sigma} \gamma_{\mu, \nu, \sigma} \gamma_{\mu, \nu, \sigma} \prod_{j=3}^{(2q)} (1 - \Delta(A_j)).
\]

Changing the order of the summation, we get
\[
\hat{S}_2 = \frac{1}{q!} \sum_{(\sigma_1, \sigma_2) \in \mathcal{E}_q} \sum_{i \in [1, s]} \sum_{j \in [1, q]} \sum_{\rho_i, j \in [V_1/2, n]} \sum_{j \in [1, q]} \sum_{\mu_j \in I_n^\prime} Z_0,
\]
where
\[
Z_0 = \sum_{r_{i,j} \in [V_1, n]} \sum_{j \in [1, q]} \ell_{\sigma_1, \sigma_2, \sigma, \alpha} \dot{\mathcal{R}}(\rho, \mu, \sigma) \dot{\mathcal{R}}_{\mu, \nu, \sigma} \gamma_{\mu, \nu, \sigma} \gamma_{\mu, \nu, \sigma} \prod_{j=3}^{(2q)} (1 - \Delta(A_j)).
\]

By (101), (110), (84) and (88), we have
\[
\gamma_{\mu, \nu, \sigma}^{(2q)} = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{2q} \frac{e(\dot{m}_j k / P_{\rho_j - \alpha}) - 1}{P_{\rho_j} (e(\dot{m}_j k / P_{\rho_j - \alpha}) - 1)}
\]
\[
= \frac{1}{NP_{\alpha_1} \cdots P_{\alpha_2q}} \sum_{k=0}^{N-1} \prod_{j=1}^{q} \left| \frac{e(\mu_j k / P_{\rho}) - 1}{P_{\rho} (e(\mu_j k / P_{\rho}) - 1)} \right|^2 = \dot{Z}_0 / (P_{\alpha_1} \cdots P_{\alpha_2q}).
\]

From (101), (102) and (110), we have
\[
\beta_{\rho, \mu, \alpha}^{(2q)} := \beta_{\mu, \nu, \sigma}^{(2q)} = E_x \left( \prod_{j=1}^{q} \psi_{\rho_j + \alpha_{\sigma_1(j)}} (\mu_j P_{\alpha_{\sigma_1(j)}}), x \right) \psi_{\rho_j + \alpha_{\sigma_2(j)}} (\mu_j P_{\alpha_{\sigma_2(j)}}), x \right).
\]
Bearing in mind that $|\rho_{i,j_1} + \alpha_{i,j_1} - \rho_{i,j_2} - \rho_{i,j_2}| \geq V_1$ for $j_1 \neq j_2$, $i = 1,...,s$ we get from (102) that expectation and multiplication can be interchanged:

$$\tilde{\beta}_{(q,\sigma)}(\theta_{1,\mu},\alpha) = \prod_{j=1}^{q} \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha),$$

$$\tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) = E_{\Xi}(\psi_{\rho_j + \alpha_{i,j}(j)}(\mu_j P_{\alpha_{i,j}(j)}; x) \psi_{\rho_j + \alpha_{i,j}(j)}(-\mu_j P_{\alpha_{i,j}(j)}; x)), \ V_1 = [\log^3 n].$$

(112)

Consider $Z_0$ (see (107) - (111)). Changing the order of the summation, we obtain

$$Z_0 = \tilde{\omega}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha)/(P_{\alpha_{1}} \cdots P_{\alpha_{2q}}),$$

with

$$\tilde{\omega}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) = \Delta \left( \min_{i,j} \rho_{i,j} + \alpha_{i,j} \geq V_1, \ \max \rho_{i,j} + \alpha_{i,j} \leq n, \ \max_j P_{\rho_j} \leq 2^{n+\log^3 n}, \right)
\min \rho_{i,j_1} + \alpha_{i,j_1} - \rho_{i,j_2} - \alpha_{i,j_2} \geq V_1).$$

Applying (111), (112) and Lemma 19, we obtain

$$\hat{\xi}_{2q} = \sum_{(\sigma_1,\sigma_2) \in \Xi_q} \sum_{\alpha_{i,j} \in [0,\log^3 n]} \sum_{\rho_{i,j} \in [V_1/2, n]} \sum_{\mu_j \in \mathbb{T}_{n}/j \in [1,q]} \tilde{\omega}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) P_{\alpha_{1}} \cdots P_{\alpha_{2q}} = \hat{\xi}_{2q} + O(n^{q^3/4}),$$

(113)

It is easy to verify that the part of this sum, satisfying the condition $\min_{i,j} \rho_{i,j} + \alpha_{i,j} < V_1$ is equal to $O(n^{q^3/4})$. Similarly for the cases $\max_{i,j} \rho_{i,j} + \alpha_{i,j} > n,$ $\min_{i,j_1,j_2,j_1 \neq j_2} |\rho_{i,j_1} + \alpha_{i,j_1} - \rho_{i,j_2} - \alpha_{i,j_2}| < V_1$. Hence

$$\sum_{(\sigma_1,\sigma_2) \in \Xi_q} \sum_{\alpha_{i,j} \in [0,\log^3 n]} \sum_{\rho_{i,j} \in [V_1/2, n]} \sum_{\mu_j \in \mathbb{T}_{n}/j \in [1,q]} \frac{2P_{\alpha_{i,j}(j)} P_{\alpha_{i,j}(j)} \tilde{\omega}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \rho_j - 1}{|P_{\rho_j}(1 - e(\mu_j/P_{\rho_j}))|^2} \ll n^{3/4} q.$$ 

Therefore $\hat{\xi}_{2q} = \hat{\xi}_{2q} + O(n^{3/4})$ with

$$\hat{\xi}_{2q} = \sum_{(\sigma_1,\sigma_2) \in \Xi_q} \sum_{\alpha_{i,j} \in [0,\log^3 n]} \sum_{\rho_{i,j} \in [V_1/2, n]} \sum_{\mu_j \in \mathbb{T}_{n}/j \in [1,q]} \frac{2P_{\alpha_{i,j}(j)} P_{\alpha_{i,j}(j)} \tilde{\omega}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha) \rho_j - 1}{|P_{\rho_j}(1 - e(\mu_j/P_{\rho_j}))|^2} \tilde{\beta}_{(q,\sigma,j)}(\theta_{1,\mu},\alpha).$$

(114)
Changing the order of the summation, we obtain
\[
\hat{s}_2 q = \frac{1}{q!} \sum_{(\sigma_1, \sigma_2) \in \mathcal{E}_q} \sum_{\rho_{i,j} \in [V_{1/2}, n], \rho_{i,j} \leq 2^{n+\log_2 n}} \sum_{\mu_{j} \in [1, q]} \prod_{j=1}^{q} \left| P_{\rho_j} \left( 1 - e(\mu_j/P_{\rho_j}) \right) \right|^2 G_{\sigma, \rho_j, \mu_j}.
\]

By (112), we have
\[
G_{\sigma, \rho_j, \mu_j} = \sum_{\alpha_{i, \sigma_1(j), \alpha_{i, \sigma_2(j)} \in [0, \log_2 n]} \sum_{i \in [1, s]} P_{\alpha^{-1}_{\sigma_1(j)}} P_{\alpha^{-1}_{\sigma_2(j)}} (q, \sigma, j). (115)
\]

Therefore, \(G_{\sigma, \rho_j, \mu_j}\) does not depend on \((\sigma_1, \sigma_2)\). We fix some \(\sigma_0 = (\sigma_{0,1}, \sigma_{0,2}) \in \mathcal{E}_q\). From (109) and (115), we obtain
\[
\hat{s}_2 q = \frac{(2q)!}{q! 2^q} \sum_{\rho_{i,j} \in [V_{1/2}, n], \rho_{i,j} \leq 2^{n+\log_2 n}} \sum_{i \in [1, s]} \prod_{j=1}^{q} \left| P_{\rho_j} \left( 1 - e(\mu_j/P_{\rho_j}) \right) \right|^2 G_{\sigma_0, \rho_j, \mu_j}.
\]

According to (108), (113) and (114), we get
\[
\hat{s}_2 q = \frac{(2q)!}{q! 2^q} \prod_{j=1}^{q} \left( \sum_{\rho_{i,j} \in [V_{1/2}, n], i \in [1, s]} \sum_{i \in [1, s]} \prod_{j=1}^{q} \left| P_{\rho_j} \left( 1 - e(\mu_j/P_{\rho_j}) \right) \right|^2 \right) + O(n^{q_{s - 4}}).
\]

Using this statement for \(q = 1\), we obtain \(\hat{s}_2 q = \frac{(2q)!}{q! 2^q} \hat{s}_2 + O(n^{q_{s - 3/4}})\). Therefore, Lemma 21 is proved. □

### 4.2.1. End of the proof of Theorem 2.

**Lemma 22.** Let \(\mathbf{x} = (x_1, ..., x_s, x_{s+1})\) and \(\mathbf{x} = (x_1, ..., x_s)\). Then
\[
g := D(\tilde{x}, \mathcal{H}_{s+1, N}) = D(\mathbf{x}, (H_s(k))[N_{x_{s+1}} - 1]) + \epsilon_1, \quad (116)
\]
\[
D(\tilde{x}, \mathcal{H}_{s+1, N}) = D(\mathbf{x}, (H_s(k))[N_{x_{s+1}} - 1]) + 4\epsilon_2, \quad |\epsilon_i| \leq 1, i = 1, 2. (117)
\]

**Proof.** We will prove the first statement. The proof of the second statement is similar. From (1) and (5), we have
\[
g = \text{card}\{0 \leq k < N \mid \phi_i(k) < x_i, \ i = 1, ..., s, \ k/N < x_{s+1} \} = x_1 \cdots x_s x_{s+1} N.
\]

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Lemma 23. With notations as above

\[ \varrho := E_{s+1}(D^h(\bar{x}, \mathcal{H}_{s+1,N})) = E_{s+1}(\mathcal{D}_{T_s,h,2}(0, [N x_{s+1}])) + O(n^{hs/2-1/10}), \quad s \geq 3, \]
\[ E_{s+1}(D^h(\bar{x}, \mathcal{H}^y_{s+1,N})) = E_{s+1}(\mathcal{D}_{T_s,h,2}([-N x_{s+1}], 2[N x_{s+1}] - 1)) \tag{118} \]
+ \( O(n^{hs/2-1/10}) \) for \( s = 2 \).

Proof. Consider the first estimate. Let \( \bar{\mathcal{D}}([N x_{s+1}]) = D(\bar{x}, \mathcal{H}_{s+1,N}) - \mathcal{D}_{T_s}(0, [N x_{s+1}]) \) and \( \mathcal{D}(0, [N x_{s+1}]) = D(\bar{x}, (H(k))_{k=0}^{[N x_{s+1}]-1} - \mathcal{D}_{T_s}(0, [N x_{s+1}]) \). According to Lemma 22, we have \( \| \bar{\mathcal{D}}([N x_{s+1}]) \| \leq \| \mathcal{D}([N x_{s+1}]) \| + 1 \). Applying (21), Minkowski’s inequality and Lemma 20, we obtain for \( s \geq 3 \) and \( \nu \geq 1 : \)

\[ \| \bar{\mathcal{D}}([N x_{s+1}]) \|_{s+1,2\nu} \leq \sum_{s=1}^{s-1} \sum_{T_s \in \{1, \ldots, s\}} \| \mathcal{D}_{T_s}(0, [N x_{s+1}]) \|_{s+1,2\nu} + O(\log n) \ll n^{\frac{s-1}{2} - \frac{1}{2s}}. \tag{119} \]

From (119), we derive

\[ E_{s+1}(\mathcal{D}^{2\nu}_{T_s}(0, [N x_{s+1}])) \leq \sup_{0 \leq x_{s+1} \leq 1} E_s(\mathcal{D}^{2\nu}_{T_s}(0, [N x_{s+1}])) \ll n^{\nu s}, \quad \nu = 1, 2, \ldots. \tag{120} \]

By (119), we have

\[ \varrho = E_{s+1}(\mathcal{D}_{T_s}(0, [N x_{s+1}]) + \bar{\mathcal{D}}([N x_{s+1}])) = E_{s+1}(\mathcal{D}^h_{T_s}(0, [N x_{s+1}])) \]
\[ + \epsilon_1 2^h \sum_{1 \leq N \leq h} E_{s+1}(\mathcal{D}_{T_s}(0, [N x_{s+1}]))^{\frac{h}{1-h}} \| \bar{\mathcal{D}}([N x_{s+1}]) \|^{\nu} . \]

Using (26), (119), (120), Cauchy-Schwarz’s inequality and Lemma 20, we get

\[ \varrho = E_{s+1}(\mathcal{D}^h_{T_s}(0, [N x_{s+1}])) + \epsilon_2 2^h \sum_{1 \leq N \leq h} \left( E_{s+1}(\mathcal{D}_{T_s}(0, [N x_{s+1}]))^{2h-2\nu} \right) \]
\[ \times E_{s+1}(\| \mathcal{D}^{2\nu}_{T_s}(0, [N x_{s+1}]) \|^{2\nu})^{1/2} = E_{s+1}(\mathcal{D}^h_{T_s}(0, [N x_{s+1}])) \]
\[ + O\left( \sum_{1 \leq N \leq h} (E_s(\mathcal{D}_{T_s}(0, [N x_{s+1}]))^{2h-2\nu} n^{\nu s-1/5})^{1/2} \right) = E_{s+1}(\mathcal{D}^h_{T_s}(0, [N x_{s+1}])) \]
\[ + O(\sum_{1 \leq N \leq h} n^{hs/2-1/10}) = E_{s+1}(\mathcal{D}_{T_s,h,2}(0, [N x_{s+1}])) + O(n^{hs/2-1/10}), \quad |\epsilon_i| \leq 1, \quad i = 1, 2. \]
Hence, the first estimate is proved. The proof of the second estimate is similar. We need only use (97) and (98) instead of (95) and (96). Therefore, Lemma 23 is proved.

Consider Lemma 1 for odd $\tilde{h}$. Let $s \geq 3$. From (24) and (26), we get

$$D_{s, \tilde{h}}(0, \lfloor Nx_{s+1} \rfloor) = 0.$$ 

According to Lemma 23, we have $E_{s+1}(D^h(\tilde{x}, H_{s+1,N})) = O(n^{h_{s/2-1/10}})$. By (2) and Roth’s inequality (3), we obtain

$$\lim_{n \to \infty} n^{-s/2} D(\tilde{x}, H_{s+1,N})_{s+1,2} > 0.$$

Therefore, (7) is proved for $s \geq 3$ and odd $\tilde{h}$. The proof for the case $s = 2$ follows from (118).

Consider Lemma 1 with even $\tilde{h} = 2q$. Let $s \geq 3$. Using Lemma 21 and Lemma 23, we get

$$E_{s+1}(D_{2q}(\tilde{x}, H_{s+1,N})) = (2q)! (E_{s+1}(D^q(\tilde{x}, H_{s+1,N})))^q + O(n^{q_{s-1/10}}).$$

Applying (121), we obtain (7) for $s \geq 3$ and even $\tilde{h}$. The proof for the case $s = 2$ follows from (100) and (118). Hence, Theorem 2 is proved.

**Proof of Theorem 3.** We need the following simple variant of the Continuous Mapping Theorem (see [Du, Theorem 3.2.4., p.101]).

**Theorem B.** Let $g$ be a continuous function. If $X_N \xrightarrow{w} X$, then $g(X_N) \xrightarrow{w} g(X)$.

By [Bil, p.31], a simple condition of uniform integrability of a sequence of functions $X_n$ is that $\sup_n E|X_n|^{1+\epsilon} < \infty$. According to [Bil, Theorem 3.5, p.31], we have

**Theorem C.** If $X_N$ are uniformly integrable and $X_N \xrightarrow{w} X$, then $X$ is integrable and $E(X_N) \to E(X)$.

We will consider the case $s \geq 3$. The proof for the case $s \geq 2$ is similar. Let $Y_N := D(x, (H_{s+1,N}))/D_{s+1,2}(H_{s+1,N})$. By Theorem 2, $Y_N \xrightarrow{w} N(0, 1) =: Y$. We take the continuous function $g(x) = |x|^p$. Using Theorem B, we get $g(Y_N) \xrightarrow{w} g(Y)$. Bearing in mind Theorem 1, we get that the functions $g(Y_N)$ ($N = 1, 2, \ldots$) are uniformly integrable.

Now using Theorem C, we get the assertion of Theorem 3.
References


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