The Set-Indexed Bandit Problem

Giacomo Aletti⋆**, Ely Merzbach**⋆⋆

[1] - Dept. of Mathematics and Computer Science

Bar-Ilan University, Ramat Gan 52900, ISRAEL

- Dipartimento di Matematica

Università di Milano, 20133 Milan, ITALY

[2] Dept. of Mathematics and Computer Science

Bar-Ilan University, Ramat Gan 52900, ISRAEL

The date of receipt and acceptance will be inserted by the editor

Abstract We give a proof of the existence of an optimal solution to a set-indexed formulation of the bandit problem.

Key words Set-indexed processes, bandit problem, stochastic control, randomization

Mathematics Subject Classification: Primary: 60g40; Secondary: 93e20

⋆ e-mail: `giacomo.aletti@unimi.it`

⋆⋆ e-mail: `merzbach@macs.biu.ac.il`

Correspondence to: G. Aletti,

Dipartimento di Matematica, Università di Milano,

Via Saldini $n^\circ 50$, 20133 Milano, ITALY

1 Formulation of the problem

The aim of this paper is to present a formulation of the Bandit problem in the framework of the set-indexed processes, and to prove the existence of an optimal solution to this problem. Recall that the classical bandit problem was studied extensively and a main tool for proving the existence of an optimal solution was the Baxter-Chacon topology (see [1]). The multi-armed bandit problem was studied by serveral authors. We refer to [7] and [8] or, more recently, [6] for motivation and description of the state of art.

In this first section, we give the notation and state the main existence result. In order to prove it, we need two kinds of tools: the first one is fuzzy stopping sets and fuzzy optional increasing paths that will be the subject of the next section. The other tool is a kind of set-indexed integration; it will be developed in section 3. The last section is devoted to the proof of the main theorem.

Our notation follows that of [5]: let T be a compact complete separable space. A nonempty class $\mathcal{A}$ of compact, connected subsets of T is called an *indexing collection* if it satisfies the following:

1. $\varnothing, T \in \mathcal{A}$, and $A^\circ \neq A$ if $A \neq \varnothing$ or T.
2. $\mathcal{A}$ is closed under arbitrary intersections and if $A, B \neq \varnothing$, then $A \cap B \neq \varnothing$. If $(A_i)_{i\in\mathbb{N}}$ is increasing, then $\overline{\cup_i A_i} \in \mathcal{A}$.

3. $\sigma(\mathcal{A}) = \mathfrak{B}_T$, where $\mathfrak{B}_T$ is the collection of all Borel sets of T.

4. *Separability from above*

   There exists an increasing sequence of finite subclasses $\mathcal{A}_n = \{A_1^n, \dots, A_{k_n}^n\} \subseteq \mathcal{A}$ closed under intersections with $\varnothing, B_n \in \mathcal{A}_n$, and a sequence of functions $g_n : \mathcal{A} \longrightarrow \mathcal{A}_n(u)$ ($\mathcal{A}'(u)$ is the class of finite union of elements of $\mathcal{A}' \subseteq \mathcal{A}$) such that:

   (a) g_n preserves arbitrary intersections and finite unions (i.e. $g_n(\cap_{A\in\mathcal{A}'} A) = \cap_{A\in\mathcal{A}'} g_n(A)$ for any $\mathcal{A} \subseteq \mathcal{A}'$, and if $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j$, then $\cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j)$),

   (b) for each $A \in \mathcal{A}$, $A \subseteq (g_n(A))^\circ$,

   (c) $g_n(A) \subseteq g_m(A)$ if $n \geq m$,

   (d) for each $A \in \mathcal{A}$, $A = \cap_n g_n(A)$,

   (e) if $A, \in \mathcal{A}$ and $A' \in \mathcal{A}_n$, then $g_n(A) \cap A' \in \mathcal{A}_n$,

   (f) $g_n(\varnothing) = \varnothing \ \forall n$.

   (Note: "$\subsetneq$" indicates strict inclusion and "$\overline{(\cdot)}$" and "$(\cdot)^\circ$" denotes respectively the closure and the interior of a set.)

We define the Hausdorff metric on the non-empty compact subsets of T:

$$d_H(A, B) = \inf\{\varepsilon : B \subseteq A^\varepsilon \text{ and } A \subseteq B^\varepsilon\},$$

where $A^\varepsilon = \{t \in T : d(t, A) \leq \varepsilon\}$, for any $\varepsilon > 0$ and $d(t, A) = \inf\{d(t, s) : s \in \mathcal{A}\}$.

We require that $\mathcal{A}$ is compact in the Hausdorff metric.

Remark 1. For sake of simplicity, we required that T is compact and $T \in \mathcal{A}$. Using Alexandroff compactification, all our results will still hold supposing only that T is locally compact.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A *set-indexed filtration* is a family $\{\mathcal{F}_A, A \in \mathcal{A}\}$ of sub-σ-algebras of $\mathcal{F}$ which is:

- complete ($\mathcal{F}_\varnothing$ contains all the $\mathbb{P}$-null sets);
- increasing (if $A \subseteq B$, $A, B \in \mathcal{A}$, then $\mathcal{F}_A \subseteq \mathcal{F}_B$);
- outer-continuous ($\mathcal{F}_{\cap_{n=1}^\infty A_n} = \bigcap_{n=1}^{\infty} \mathcal{F}_{A_n}$, where $\{A_n\}$ is a decreasing sequence in $\mathcal{A}$).

Definition 1. A random set $\xi : \Omega \to \mathcal{A}$ is a *stopping set* if $\{A \subseteq \xi\} \in \mathcal{F}_A$, for any $A \in \mathcal{A}$. An *optional increasing path* (o.i.p.) is a family $\boldsymbol{\xi} = \{\xi_t, t \in [0,1]\}$ of stopping sets s.t.

1. $\xi_0 = \varnothing' = \cap_{A \in \mathcal{A}} A$
2. $\xi_1 = T$
3. $\xi_s \subseteq \xi_t$ if $s \leq t$ a.s.
4. $\xi_t \subseteq g_n^2(\xi_s)$ if $s \leq t < s + 2^{-\beta(n)}$, where $\lim_{n\to\infty} \beta(n) = \infty$ a.s.

The set of all o.i.p.'s will be denoted by $\boldsymbol{\Xi}$.

Remark 2. In [5], what we called stopping set is called simple stopping set. We omitted the word simple since we'll use only these kinds of sets. Since the concept of stopping set is related to that of stopping line, this paper is not a direct extension of [4]. The formulation of the bandit problem will be done for the set-indexed theory.

Condition 4 relates to $|Z_u| = u$ in [4]. It states that the speed of o.i.p.'s growing is bounded. It will be necessary in Lemma 9 as $|Z_u| = u$ is in [4].

Following the formulation of the bandit problem given in [4], let $X = \{X_A : A \in \mathcal{A}\}$ be a real-valued set-indexed process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with upper-semicontinuous sample paths such that $E[\sup_{A\in\mathcal{A}} |X_A|] < \infty$ and let $V = \{V_s, s \in [0,1]\}$ be a bounded non-negative right-continuous process with nondecreasing sample paths.

Main Theorem. *For each $\boldsymbol{\xi} \in \Xi$, set $R(\boldsymbol{\xi}) = E(\int_{[0,1]} X_{\xi_t} \mathrm{d}V_t)$. Then there exists an optional increasing path $\boldsymbol{\xi}^* \in \Xi$ s.t. $R(\boldsymbol{\xi}^*) = \sup_{\boldsymbol{\xi}\in\Xi} R(\boldsymbol{\xi})$, i.e., $\boldsymbol{\xi}^*$ is an optimal o.i.p.*

This theorem will be proved in the last section.

2 Fuzzy stopping sets

Definition 2. Let $\mathcal{B}$ be the Borel sets of $[0,1]$. A map $\tau : \Omega \times [0,1] \to \mathcal{A}$ will be called a *fuzzy set* if it is $\mathcal{F} \times \mathcal{B}$-measurable and

1. *nondecreasing* in the second variable: $s \leq t$ implies $\xi(\cdot, s) \subseteq \xi(\cdot, t)$
2. *inner continuous* in the second variable: if $t_n \uparrow t$, $t_n \leq t$, then

$$\overline{\bigcup_n \xi(\cdot, t_n)} = \xi(\cdot, t)\,;$$

and will be called a *fuzzy stopping set* if it is a fuzzy set and $\{(\omega, r)\colon A \subseteq \tau(\omega, r)\} \in \mathcal{F}_A \times \mathcal{B}$, for any $A \in \mathcal{A}$.

Also, any function on Ω will be considered defined on $\Omega \times [0,1]$ in the obvious way. This implies that a stopping set η may be seen as a fuzzy stopping set (where $\xi(\omega, v) = \eta(\omega)$).

Let $\xi : \Omega \times [0,1] \to \mathcal{A}$ be nondecreasing and inner continuous in the second variable. We denote for every $v \in [0,1]$

$$\xi_v : \Omega \to \mathcal{A} \qquad \xi_v(\omega) = \xi(\omega, v)\,.$$

Proposition 1. *Let $\xi : \Omega \times [0,1] \to \mathcal{A}$ be nondecreasing and inner continuous in the second variable. ξ is a fuzzy stopping set iff ξ_v is a stopping set for each $v \in [0,1]$. In particular, taking $\mathcal{F}_A = \mathcal{F}$, ξ is a fuzzy set iff ξ_v is $\mathcal{F}$-measurable for each $v \in [0,1]$.*

Proof. ξ is a fuzzy stopping set iff for any $A \in \mathcal{A}$, $\{A \not\subseteq \xi\} \in \mathcal{F}_A \times \mathcal{B}$ iff (by 1 and 2) $\{A \not\subseteq \xi\} = F_A \times (t_A, 1]$. □

Define the "probability" $P(\omega, \cdot)$ on $\mathcal{A}$ by

$$P(\omega, A) = \lim_{\varepsilon \to 0} \left(\sup\{v \in [0,1]\colon A^\varepsilon \not\subseteq \xi(\omega, v)\} \right) \tag{1}$$

We note that $0 \leq P(\omega, A) \leq 1$ and is:

nondecreasing If $A \subseteq B$, then $P(\omega, A) \leq P(\omega, B)$,

compatible If $P(\omega, A) \leq v$ and $P(\omega, B) \leq v$, then

$$P(\omega, \cap\{C \in \mathcal{A}\colon A, B \subseteq C\}) \leq v\,,$$

outer continuous If $A_n \searrow A$, then $P(\omega, A_n) \to P(\omega, A)$.

Denote by $\mathfrak{B}$ (resp. $\mathfrak{B}'$) the set of the all nondecreasing compatible outer continuous process as $\{P(\cdot, A), A \in A\}$ taking values in $[0,1]$ which are measurable (resp. adapted).

Proposition 2. *Let $\xi : \Omega \times [0,1] \to \mathcal{A}$. ξ is a fuzzy stopping set if and only if $P(\cdot, A) \in \mathfrak{B}'$ and ξ is a fuzzy set if and only if $P(\cdot, A) \in \mathfrak{B}$.*

Proof. Let

$$\xi(\omega, v) = \overline{\cup\{A \in \mathcal{A} \colon P(\omega, A < v)\}}\,. \tag{2}$$

By (1) and (2) we have $A \subseteq \xi(\omega, v) \iff P(\omega, A) \leq v$. $\square$

Corollary 3. *$P \in \mathfrak{B}'$ corresponds to a stopping set iff $P(\cdot, A) \in \{0,1\}$.*

Theorem 4. *The set $\mathfrak{B}$ (resp. $\mathfrak{B}'$) has the following properties:*

1. *$\mathfrak{B}$ (resp. $\mathfrak{B}'$) is convex.*
2. *The set of the extremal elements in $\mathfrak{B}'$ is the set of those which correspond to the set of the generalized stopping set.*
3. *$\mathfrak{B}$ (resp. $\mathfrak{B}'$) is compact in the Baxter-Chacon topology.*

Proof.

1. Obvious.

2. Suppose that there exists A s.t. $\mathbb{P}[\{\omega\colon 0 < P(\omega, A) < 1\}] > 0$. Then we may choose a convenient λ such that if

$$P'(\omega, B) = \frac{P(\omega, B) \vee \lambda}{\lambda}$$

and

$$P''(\omega, B) = \frac{(P(\omega, B) - \lambda) \wedge 0}{1 - \lambda}$$

for any $B \in \mathcal{A}$, then P' and P'' are distinct. We have $P = \lambda P' + (1 - \lambda)P''$. Moreover P' and P'' are nondecreasing, compatible and outer continuous, since P is. By Corollary 3, if P does not correspond to a stopping set, it cannot be one of the extremal elements of $\mathfrak{B}'$.

3. Let $\{P_n, n \in \mathbb{N}\}$ be a sequence of elements in $\mathfrak{B}'$. Let $A \in \mathcal{A}$ be fixed. If we define $j_n : \Omega \to [0, 1]$, $j_n(\omega) = P_n(\omega, A)$, then j_n is a bounded $\mathcal{F}_A$-time. Then there exists a subsequence j_{n_k} s.t. j_{n_k} converges in the Baxter-Chacon topology (see [1]) to a $\mathcal{F}_A$-measurable random time. Then, by a diagonalization method, we may find a subsequence P_{n_k} s.t. P_{n_k} converges for any set $A \in \mathcal{A}_n$ (for each n). We have only to note that we can define the limit P by outer continuity. This limit will be clearly a nondecreasing compatible outer continuous process with values in $[0, 1]$.

$\square$

The conditions that make a family of stopping sets be an optional increasing path may be transported on $\prod_{[0,1]} \mathfrak{B}'$ to fuzzy optional increasing paths in the following way:

Definition 3. A *fuzzy optional increasing path* is a family $\boldsymbol{P} = \{P^t, t \in [0,1]\} \in \prod_{[0,1]} \mathfrak{B}'$ s.t.

1. $P^t \in \mathfrak{B}'\ \forall t \in [0,1]$
2. $P^0(\omega, A) = 1$, for any $A \in \mathcal{A}$
3. $P^1(\omega, A) = 0$, for any $A \in \mathcal{A}, A \neq T$
4. $P^s(\omega, \cdot) \geq P^t(\omega, \cdot)$ if $s \leq t$
5. $P^t(\omega, g_n^2(A)) \geq P^s(\omega, A)$ if $s \leq t < s + 2^{-\beta(n)}$.

We denote by $\mathfrak{B}'$ the set of the all fuzzy optional increasing paths.

Remark 3. We only underline that, if $s \leq t < s + 2^{-\beta(n)}$, we have

$$P^t(\omega, A) \leq P^s(\omega, A) \leq P^t(\omega, g_n^2(A)) \,. \tag{3}$$

Lemma 5. $\{P^t, t \in [0,1]\} \in \mathfrak{B}'$ *corresponds to an optional increasing path iff* $P^t(\cdot, A) \in \{0,1\}$ *for any* $t \in [0,1]$.

Proof. It is a consequence of Corollary 3. □

Proposition 6. *The set* $\mathfrak{B}'$ *has the following properties:*

1. $\mathfrak{B}'$ *is convex*
2. *The set of the extremal elements in* $\mathfrak{B}'$ *is the set of those which correspond to the set of the optional increasing path*

3. $\mathfrak{B}'$ *is compact for the product topology*

Proof.

1. It is a consequence of Theorem 4 and the fact that a convex combination is a linear function.
2. If $\mathbf{P} = \{P^t, t \in [0,1]\} \in \mathfrak{B}'$, we can define two elements ${}^1\mathbf{P}$ and ${}^2\mathbf{P}$ by the formulas

$$ {}^1P^t(\omega, A) = \min(2P^t(\omega, A), 1) $$

$$ {}^2P^t(\omega, A) = \max(2P^t(\omega, A) - 1, 0)\,. $$

It is easy to check that ${}^1\mathbf{P}, {}^2\mathbf{P} \in \mathfrak{B}'$ and $2\mathbf{P} = {}^1\mathbf{P} + {}^1\mathbf{P}$ and so $\mathbf{P}$ can be an extremal element in $\mathfrak{B}'$ only if $\mathbf{P} = {}^1\mathbf{P} = {}^2\mathbf{P}$. Then $P^t(\omega, A) \in \{0,1\}$ and the proof is finished via Lemma 5.

3. It is sufficient to note that $\mathfrak{B}'$ is a closed subset of $\prod_{[0,1]} \mathfrak{B}'$ and thus is compact (see [4]).

$\square$

3 Integration on $\mathcal{A}$

The set-indexed bandit problem is expressed here as the maximization of a function ϕ_X defined on the set of the optional increasing paths by

$$ (\xi_t)_{t\in[0,1]} \mapsto E\Big[\int_{\mathbb{R}_+} X_{\xi_t}\, \mathrm{d}V_t\Big]\,, $$

where $\{X_A, A \in \mathcal{A}\}$ is a (sufficient regular) integrable stochastic process and $\{V_t, t \in [0,1]\}$ is a bounded process with nondecreasing simple paths. We note that

$$E\Big[\int_{\mathbb{R}_+} X_{\xi_t}\,\mathrm{d}V_t\Big] = E\Big[\int_{\mathbb{R}_+}\Big(\int_{[0,1]} X(\xi_t(\cdot,v))\,\mathrm{d}v\Big)\mathrm{d}V_t\Big].$$

since the fuzzy set obtained by a stopping set is constant.

We are going to give a definition of integration on $\mathcal{A}$ s.t.

$$\int_{\mathcal{A}} f(A)P(\cdot,\mathrm{d}A) = \int_{[0,1]} f(\xi(\cdot,v))\,\mathrm{d}v\,,$$

so that $\phi_X : \boldsymbol{\Xi} \to \mathbb{R}$ will be extended to an affine function $\Phi_X : \mathfrak{B}' \to \mathbb{R}$ by setting:

$$(P^t)_{t\in[0,1]} \mapsto E\Big[\int_{\mathbb{R}_+}\Big(\int_{\mathcal{A}} X(A)\,P^t(\cdot,\mathrm{d}A)\Big)\mathrm{d}V_t\Big].$$

Let f be any bounded Borel function on $\mathcal{A}$ and ξ be a fuzzy set. We denote

$$f^+(\xi)(\omega) = \int_{[0,1]} \max(f[\xi(\omega,v)],0)\,\mathrm{d}v\,,$$
$$f^-(\xi)(\omega) = \int_{[0,1]} \max(-f[\xi(\omega,v)],0)\,\mathrm{d}v\,.$$

Definition 4. We say that a function $f : \mathcal{A} \to \mathbb{R}$ is *integrable* (and we denote it by $f \in \mathfrak{L}^1_{\mathfrak{B}}[\mathcal{A}]$) if, for any fuzzy set ξ, $f^+(\xi)(\omega) < +\infty$ and $f^-(\xi)(\omega) < +\infty$, and we define

$$f(\xi)(\omega) := f^+(\xi)(\omega) - f^-(\xi)(\omega)\,.$$

Let I be an interval of $\mathbb{R}_+$ of the form $[0,s]$. It is natural to denote:

$$\chi_I(t) = \begin{cases} 0 & \text{if } I \subseteq [0,t] \\ 1 & \text{otherwise} \end{cases} \qquad \mathbf{1}_I(t) = \begin{cases} 1 & \text{if } [0,t] \subseteq I \\ 0 & \text{otherwise} \end{cases}$$

so that, for any $A, A^* \in \mathcal{A}$ we may define

$$\chi_A(A^*) = \begin{cases} 0 & \text{if } A \subseteq A^* \\ 1 & \text{otherwise} \end{cases} \qquad \mathbf{1}_A(A^*) = \begin{cases} 1 & \text{if } A^* \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

Let $A \in \mathcal{A}$. We have

$$\chi_A(\xi(\omega,v)) = \begin{cases} 1 & \text{if } P(\omega,A) \leq v \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{1}_{\xi(\omega,v)}(A) = \begin{cases} 1 & \text{if } P(\omega,A) \leq v \\ 0 & \text{otherwise} \end{cases}$$

so that $P(\omega,B) = \int_{[0,1]} \chi_B(\xi(\omega,v))\, \mathrm{d}v = \int_{[0,1]} \mathbf{1}_{\xi(\omega,v)}(B)\, \mathrm{d}v$.

Let $f : \mathcal{A} \to \mathbb{R}_+ \cup \{+\infty\}$ be a nonnegative extended real function on $\mathcal{A}$. Let $\beth_f : \mathbb{R} \to \overline{\mathcal{A}(u)}$ (where $\overline{\mathcal{A}(u)}$ is the class of limits of finite union of sets in $\mathcal{A}$, as in [5]),

$$\beth_f(t) := \overline{\cup\{A \in \mathcal{A} \colon f(A) \leq t\}}\,.$$

We have $\beth_f(s) \subseteq \beth_f(t)$ when $s \leq t$.

Definition 5. We say that a bounded nonnegative real function f : $\mathcal{A} \to \mathbb{R}$ is *regular* iff $f(A) = \lim_n f_n(A)$, for any $A \in \mathcal{A}$, where

$$f_n(A) = \sum_{i=0}^{n2^n-1} i2^{-n} \Big[\chi_{\beth_f([i+1]2^{-n})}(A) - \chi_{\beth_f(i2^{-n})}(A) \Big]\,.$$

We denote by $\mathcal{L}[\mathcal{A}]$ the monotone class generated by the linear combinations of increasing functions.

Proposition 7. $C(\mathcal{A}) \cap \mathfrak{L}^1_{\mathfrak{B}}[\mathcal{A}] \subseteq \mathcal{L}[\mathcal{A}]$.

Proof. It is sufficient to prove that, given a set $B \in \overline{\mathcal{A}(u)}$, the function $\mathbf{1}_B(A)$ belongs to $\mathcal{L}[\mathcal{A}]$. For any $n \in \mathbb{N}$, we define

$$\begin{aligned} f_n^B(A) &= \min_{\substack{A^* \in \mathcal{A}_n(u) \\ A^* \not\subseteq g_n(B)}} \chi_{A^*}(g_n(A)) \\ &= 1 - \max_{\substack{A^* \in \mathcal{A}_n(u) \\ A^* \not\subseteq g_n(B)}} [1 - \chi_{A^*}(g_n(A))] \\ &= 1 - {f'}_n^B(A)\,. \end{aligned}$$

We note that, if $A \subseteq B$, then $A^* \not\subseteq g_n(A)$, for any $A^* \not\subseteq g_n(B)$, and hence $\chi_{A^*}(g_n(A)) = 1$ and $f_n(A) = 1$, for any $n \in \mathbb{N}$.

If $A \not\subseteq B$, there will be an $n \in \mathbb{N}$ s.t. $g_n(A) \not\subseteq g_n(B)$. Then $f_n(A) = 0$ and so $\mathbf{1}_B = \inf f_n^B = 1 - \sup {f'}_n^B$.

Since $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$, then $\{f_n, n \in \mathbb{N}\}$ is a monotone sequence of functions on $\mathcal{A}$ and then we have only to prove that ${f'}_n^B \in \mathcal{L}[\mathcal{A}]$, for each $n \in \mathbb{N}$. It is sufficient to note that $[1 - \chi_{A^*}(g_n(A))]$ is an increasing function (and hence ${f'}_n^B$). $\square$

Proposition 8. *The space $\mathcal{L}[\mathcal{A}]$ is the space of the Borel integrable functions on $\mathcal{A}$.*

Proof. The proof follows by Proposition 7 and the fact that

$$\chi_B(A) = \sup_{\substack{f\in C(\overline{\mathcal{A}(u)})\\ 0\leq f\leq 1\\ f(A)=0 \text{ when } B\subseteq A}} f(A)\,,$$

since $\mathfrak{L}^1_{\mathfrak{B}}[\mathcal{A}]$ is a monotone class. □

– Let f be a bounded positive regular function on $\overline{\mathcal{A}(u)}$ and ξ be a fuzzy set. We define

$$\int_{\mathcal{A}} f(A)P(\omega,\mathrm{d}A) := \lim_n \sum_{i=0}^{n2^n-1} i2^{-n}\left[P(\omega,\beth_f([i+1]2^{-n})) - P(\omega,\beth_f(i2^{-n}))\right].$$

We have

$$\begin{aligned} f(\xi)(\omega) &= \int_{[0,1]} \left(\lim_n f_n(\xi(\omega,v))\right)\mathrm{d}v \\ &= \lim_n \sum_{i=0}^{n2^n-1} i2^{-n} \int_{[0,1]} \Big[\chi_{\beth_f([i+1]2^{-n})}(\xi(\omega,v)) \\ &\qquad\qquad - \chi_{\beth_f(i2^{-n})}(\xi(\omega,v))\Big]\,\mathrm{d}v \\ &= \lim_n \sum_{i=0}^{n2^n-1} i2^{-n}\left[P(\omega,\beth_f([i+1]2^{-n})) - P(\omega,\beth_f(i2^{-n}))\right] \\ &= \int_{\mathcal{A}} f(A)P(\omega,\mathrm{d}A)\,. \end{aligned}$$

– It is not difficult to show that the extension (via linear combinations and monotone class, as usual) of the integral operator to $\mathcal{L}[\mathcal{A}]$ gives coherent definitions and all those definitions are well-posed. We only underline that, for any $f \in \mathcal{L}[\mathcal{A}]$, $f(\xi)(\omega) = \int_{\mathcal{A}} f(A)P(\omega, \mathrm{d}A)$ and $\int_{\mathcal{A}} f(A)P(\omega, \mathrm{d}A)$ is a linear function on $\mathfrak{B}'$.

– If ξ doesn't depend on the second variable for some ω ($\xi(\omega_0, v) = A$, for any $v \in [0,1]$), we have

$$f(\xi)(\omega_0) = \int_{[0,1]} f(\xi(\omega_0, v))\,\mathrm{d}v = f(A)\,.$$

Then $\Phi_X : \mathfrak{B}' \to \mathbb{R}$, where

$$\Phi_X((P^t)_{t\in[0,1]}) = E\Big[\int_{\mathbb{R}_+}\Big(\int_{\mathcal{A}} X(A)\,P^t(\cdot,\mathrm{d}A)\Big)\mathrm{d}V_t\Big]\,,$$

is an affine function on $\mathfrak{B}'$ that extends $\phi_X : \Xi \to \mathbb{R}$.

4 Proof of the Main Theorem

Let $\mathcal{M}$ denote the deterministic elements of $\mathfrak{B}'$, i.e., $\boldsymbol{\alpha} = \{\alpha^t, t \in [0,1]\} \in \mathcal{M}$ provided each $\alpha^t : \mathcal{A} \to [0,1]$ is a nondecreasing outer continuous function such that $\alpha^t(T) = 1$ and

1. $\alpha^s(\cdot) \geq \alpha^t(\cdot)$ if $s \leq t$
2. $\alpha^t(g_n^2(A)) \geq \alpha^s(A)$ if $s \leq t < s + 2^{-\beta(n)}$.

As in (3), we may affirm that if $s \leq t < s + 2^{-\beta(n)}$,

$$\alpha^t(A) \leq \alpha^s(A) \leq \alpha^t(g_n^2(A))\,. \tag{4}$$

Let x be an integrable function on $\mathcal{A}$. Let $\boldsymbol{\alpha}(x) : [0,1] \to \mathbb{R}$ be defined by

$$\boldsymbol{\alpha}(x)(t) = \begin{cases} \int_T x(A)\, \alpha^t(\mathrm{d}A) & \text{when } t \in [0,1) \\ x(T) & \text{when } t = 1 \end{cases}$$

Lemma 9. *If $x : \mathcal{A} \to \mathbb{R}$ is a continuous function on $\mathcal{A}$, then $\{\boldsymbol{\alpha}(x), \boldsymbol{\alpha} \in \mathcal{M}\}$ is an equicontinuous set of functions on $[0,1]$, i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - \bar{t}| < \delta$ then*

$$|\boldsymbol{\alpha}(x)(t) - \boldsymbol{\alpha}(x)(\bar{t})| < \varepsilon\,, \tag{5}$$

for any $\boldsymbol{\alpha} \in \mathcal{M}$.

Proof. Since a continuous function on a compact set is uniformly continuous, then (5) has to be checked for all fixed $t \in [0,1]$. The case $t = 1$ follows immediately from the continuity of x at 1, so we assume $t \in [0,1)$. Now

$$\begin{aligned} |\boldsymbol{\alpha}(x)(t) - \boldsymbol{\alpha}(x)(\bar{t})| &= \Big| \int_{\mathcal{A}} x(A)\big[\alpha^t(\mathrm{d}A) - \alpha^{\bar{t}}(\mathrm{d}A)\big] \Big| \\ &= \Big| \int_{[0,1]} \big[x(\xi^t(v)) - x(\xi^{\bar{t}}(v))\big]\, \mathrm{d}v \Big| \end{aligned} \tag{6}$$

We must prove that for any $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ s.t.

$$|t - \bar{t}| < 2^{-k}, \boldsymbol{\alpha} \in \mathcal{M} \quad \Rightarrow \quad \Big| \int_T x(A)\big[\alpha^t(\mathrm{d}A) - \alpha^{\bar{t}}(\mathrm{d}A)\big] \Big| < 2^{-n}\,.$$

Since x is uniformly continuous, then $d(A,B) < \delta$ implies $|x(A) - x(B)| < \varepsilon$. Then the statement follows by the assumption in Remark 1, by (4), (6) and (2). $\square$

The proof of the following result will follow that of Nualart in [4]. We'll write it for completeness.

Lemma 10. *Suppose $E[\sup_{A\in\mathcal{A}}|X_A|]<\infty$. Then Φ_X is continuous on $\mathfrak{B}'$.*

Proof. Let V^k denote the increasing process defined by

$$V_t^k=\sum_{j=1}^{2^k}\Delta V_j^k\, I_{\{j2^{-k}\le t\}}\qquad\text{where}\qquad \Delta V_j^k=V_{j2^{-k}}-V_{(j-1)2^{-k}}\,.$$

Since $E[\sup_{A\in\mathcal{A}}|X_A|]<\infty$, it is not difficult to see using Lemma 9 that

$$\sup_{\mathbf{P}\in\Xi_A}\Big|\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t-\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t^k\Big|\underset{k\to\infty}{\longrightarrow}0\qquad\mathbb{P}\text{-a.s.}$$

Fix $\varepsilon>0$. Again since $E[\sup_{A\in\mathcal{A}}|X_A|]<\infty$, it follows that we may choose $k\in\mathbb{N}$ s.t.

$$E\Big[\sup_{\mathbf{P}\in\Xi_A}\Big|\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t-\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t^k\Big|\Big]<\frac{\varepsilon}{3}\,. \tag{7}$$

Fix $\mathbf{P}\in\mathfrak{B}'$, and define an open set $\mathcal{O}$ of $\mathfrak{B}'$ by

$$\mathcal{O}=\Big\{\tilde{\mathbf{P}}\in\mathfrak{B}'\colon\Big|E\big[\big(\tilde{\mathbf{P}}(X)(j2^{-k})-\mathbf{P}(X)(j2^{-k})\big)\Delta V_j^k\big]\Big|<\frac{\varepsilon}{3}2^{-k}\,,\quad j=1,\dots,2^k\Big\}.$$

Using (7), we see that for $\tilde{\mathbf{P}}\in\mathcal{O}$,

$$\begin{aligned}&\Big|E\Big[\int_{[0,1]}\tilde{\mathbf{P}}(X)(t)\,\mathrm{d}V_t\Big]-E\Big[\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t\Big]\Big|\\&\quad<2\frac{\varepsilon}{3}+\Big|E\Big[\int_{[0,1]}\tilde{\mathbf{P}}(X)(t)\,\mathrm{d}V_t^k\Big]-E\Big[\int_{[0,1]}\mathbf{P}(X)(t)\,\mathrm{d}V_t^k\Big]\Big|<\varepsilon\end{aligned}$$

by the definitions of $\mathcal{O}$ and V^k. This completes the proof. □

Proof of the Main Theorem. This proof is now similar to that of [4] and [10]. By the proof of Proposition 7.1 of [3], there is a nonincreasing sequence $(X^n)_{n\in\mathbb{N}}$ (where $E[\sup_{A\in\mathcal{A}}|X^n_A|]<\infty$) s.t.

$$X_A(\omega) = \lim_{n\to\infty}\downarrow X^n_A(\omega)\,, \qquad \forall A\in\mathcal{A}, \forall\omega\in\Omega\,.$$

Since $\mathrm{d}V_t$ is a nonnegative measure, we obtain

$$\lim_{n\to\infty}\downarrow \Phi_{X^n}(\mathbf{P}) = \Phi_X(\mathbf{P})\,, \qquad \forall\mathbf{P}\in\mathfrak{B}'\,.$$

By Lemma 10, this shows that Φ is upper semicontinuous on $\mathfrak{B}'$. Hence Φ attains his maximum on $\mathfrak{B}'$ and since Φ is affine, this maximum is attained at an extremal element of $\mathfrak{B}'$ (see [2], II.58, Proposition 1). Proposition 6 completes the proof. $\square$

References

1. J. R. Baxter and R. V. Chacon. Compactness of stopping times. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 40(3):169–181, 1977.
2. N. Bourbaki. *Espaces vectoriels topologiques. Chapitres 1 à 5.* Masson, Paris, new edition, 1981. Éléments de mathématique. [Elements of mathematics].
3. R. C. Dalang. Optimal stopping of two-parameter processes on nonstandard probability spaces. *Trans. Amer. Math. Soc.*, 313(2):697–719, 1989.
4. R. C. Dalang. Randomization in the two-armed bandit problem. *Ann. Probab.*, 18(1):218–225, 1990.
5. B. G. Ivanoff and E. Merzbach. *Set-Indexed Martingales.* Chapman & Hall/CRC, Boca Raton, FL, 1999.

6. H. Kaspi and A. Mandelbaum. Multi-armed bandits in discrete and continuous time. *Ann. Probab.*, 8(4):1270–1290, 1998.

7. A. Mandelbaum. Continuous multi-armed bandits and multiparameter processes. *Ann. Probab.*, 15(4):1527–1556, 1987.

8. A. Mandelbaum. Navigating and stopping multiparameter bandit processes. In *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, pages 339–372. Springer, New York, 1988.

9. A. Mandelbaum and R. J. Vanderbei. Optimal stopping and supermartingales over partially ordered sets. *Z. Wahrsch. Verw. Gebiete*, 57(2):253–264, 1981.

10. G. Mazziotto and A. Millet. Stochastic control of two-parameter processes application: the two-armed bandit problem. *Stochastics*, 22(3-4):251–288, 1987.