FRACTIONAL POISSON FIELDS

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Abstract. Using inverse subordinators and Mittag-Leffler functions, we present a new definition of a fractional Poisson process parametrized by points of the Euclidean space $\mathbb{R}_+^2$. Some properties are given and, in particular, we prove a long-range dependence property.

1. Introduction

There are essentially four different approaches to the concept of Fractional Poisson process on the real line: The "integral representation" method follows the integral representation of the Fractional Brownian motion, replacing the Brownian motion by the Poisson process (Wang, Wen [32]). For some other aspects of this approach, see [14],[6].

Another approach that we can call the "Renewal" approach consists of considering the characterization of the Poisson process as a sum of independent non-negative random variables, and instead of the assumption that these random variables have an exponential distribution, we assume that they have the Mittag-Leffler distribution (Mainardi et al. [19],[20], Repin, Saichev [28]). In [3], the renewal approach to the fractional Poisson process is developed and it is proved that its one-dimensional distributions coincide with the solution to fractionalized state probabilities.

A third approach, the "differential" one, uses the differential equations of the Poisson process and replaces them by fractional derivatives (Beghin, Orsingher [3],[4]). Finally, using "inverse subordinator", a kind of fractional Poisson process can be constructed (Meerschaert, Nane, Vellaisamy [23]). Many formulas attributed to [23] have been obtained in [14], [3],[4].

Here, we will follow the fourth method to generalize and define a Fractional Poisson field parametrized by points of the Euclidean space $\mathbb{R}_+^2$, as it has been done for fractional Brownian fields, see, e.g., [10], [18] (see also [11],[12]).

The starting point of our extension will be the set-indexed Poisson process which is a well-known concept, see, e.g., [29] or [24],[25].

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2. Subordinators and inverse subordinators

This section collects some known results from the theory of subordinators and inverse subordinators, see Bingham [5], Veillette and Taqqu [30], [31], [22], among the others.

2.1. **Inverse subordinator.**

Let $L(t),\ t \geq 0$ be a Lévy subordinator with Laplace exponent

$$\phi(s) = \mu s + \int_{(0,\infty)} (1 - e^{-sx})\Pi(dx),\ s \geq 0, \tag{1}$$

where $\mu \geq 0$ is the drift and the Lévy measure Π on $\mathbb{R}_+ \cup \{0\}$ satisfies

$$\int_0^\infty (1 \wedge x)\Pi(dx) < \infty. \tag{2}$$

This means that

$$\mathrm{E}e^{-sL(t)} = e^{-t\phi(s)},\ s \geq 0. \tag{3}$$

Consider the inverse subordinator $Y(t),\ t \geq 0$, which is given by the first-passage time of L

$$Y(t) = \inf \{u \geq 0 : L(u) > t\},\ t \geq 0. \tag{4}$$

The process $Y(t),\ t \geq 0$, is non-decreasing and its sample paths are a.s. continuous if L is strictly increasing. Also Y is, in general, non-Markovian with non-stationary and non-independent increments.

We have

$$\{L(u_i) < t_i, i = 1, \ldots, n\} = \{Y(t_i) > u_i,\ i = 1, \ldots, n\}. \tag{5}$$

Let

$$U(t) = \mathrm{E}Y(t) \tag{6}$$

be the renewal function of $Y(t)$, and let

$$H_u(t) = \mathrm{P}\{Y(u) < t\}. \tag{7}$$

From (1) we have

$$\int_0^\infty e^{-st} dH_u(t) = e^{-t\phi(s)}, \tag{8}$$

and

$$\tilde{U}(s) = \int_0^\infty U(t)e^{-st}dt = \frac{1}{s\phi(s)}; \tag{9}$$

thus, $\tilde{U}$ characterizes the process Y (since ϕ characterizes L).

Therefore, we get a covariance formula ([30], [31])

$$\operatorname{Cov}(Y(t_1), Y(t_2)) = \int_0^{t_1 \wedge t_2} (U(t_1 - \tau) + U(t_2 - \tau)) dU(\tau) - U(t_1)U(t_2). \tag{10}$$

The most important example is considered in the next section, but there are some others.

2.2. **Inverse stable subordinator.** Let $L_\alpha(t), t \geq 0$, be a $\alpha-$stable subordinator with $\phi(s) = s^\alpha, 0 < \alpha < 1$ (cadlag, continuous in probability, with independent and stationary increments), whose density $g(t,x)$ is such that $L_\alpha(1)$ has pdf

(11)

$$g_\alpha(x) = g(1,x) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(\alpha k+1)}{k!}\frac{1}{x^{\alpha k+1}}\sin(\pi k \alpha) = \frac{1}{x}W_{-\alpha,0}(-x^{-\alpha}),\ x > 0,$$

and with Laplace transform

$$\mathrm{E}e^{-sL_\alpha(t)} = \exp\{-ts^\alpha\},\ s \geq 0.$$

Here we use the Wright' s generalized Bessel function (see, e.g., [9])

$$W_{\gamma,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k)\Gamma(\beta+\gamma k)},\ z \in \mathcal{C},$$

where $\gamma > -1$, and $\beta \in \mathbb{R}^1$, and for $\beta = 0, \gamma = -\alpha \in (-1,0)$

$$W_{-\alpha,0}(z) = \sum_{k=1}^{\infty}\frac{\sin(\pi k\alpha)}{\pi}\frac{z^k\Gamma(1+\alpha k)}{k!} \tag{12}$$

by reciprocity relation for the $\Gamma-$function.

Also

$$g_\alpha(x) \sim \frac{\left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(2-\alpha)}}}{\sqrt{2\pi\alpha(1-\alpha)}}\exp\left\{-(1-\alpha)\left(\frac{x}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}\right\},\ x \to 0,$$

and

$$g_\alpha(x) \sim \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}, x \to \infty.$$

Then

$$Y_\alpha(t) = \inf\{u \geq 0 : L_\alpha(u) > t\}$$

has density

$$f_\alpha(t,x) = \frac{t}{\alpha}x^{-1-\frac{1}{\alpha}}g_\alpha(tx^{-\frac{1}{\alpha}}),\ x \geq 0,\ t > 0. \tag{13}$$

Let

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha > 0,\ z \in \mathbb{C}$$

be the Mittag-Leffler function [8],[9], and recall the following:

i) The Laplace transform of the Mittag-Leffler function is of the form

$$\int_0^\infty e^{-st}E_\alpha(-t^\alpha)dt = \frac{s^{\alpha-1}}{1+s^\alpha},\ 0<\alpha<1,\ t\geq 0.$$

(ii) The Mittag-Leffler function is a solution of the fractional equation with fractional Caputo-Djrbashian derivative D_t^α

$$\mathrm{D}_t^\alpha E_\alpha(at^\alpha) = aE_\alpha(at^\alpha).$$

Proposition.

i)

$$\mathrm{E}e^{-sY_\alpha(t)} = \sum_{n=0}^\infty \frac{(-st^\alpha)^n}{\Gamma(\alpha n+1)} = E_\alpha(-st^\alpha),$$

(ii) *Both processes* $L_\alpha(t), t\geq 0$ *and* $Y_\alpha(t)$ *are self-similar*

$$\frac{L_\alpha(at)}{a^{1/\alpha}} \stackrel{d}{=} L_\alpha(t),\ \frac{Y_\alpha(at)}{a^\alpha} \stackrel{d}{=} Y_\alpha(t),\ a>0.$$

(iii)

$$\frac{\partial\mathrm{E}(Y_\alpha(t_1)\cdots Y_\alpha(t_k))}{\partial t_1\cdots\partial t_k} = \frac{1}{\Gamma^k(\alpha)}\frac{1}{[t_1(t_2-t_1)\cdots(t_k-t_{k-1})]^{1-\alpha}},\ 0<t_1<\cdots<t_k.$$

In particular,

(A)

$$\mathrm{E}Y_\alpha(t) = \frac{t^\alpha}{\Gamma(1+\alpha)};$$

(B)

$$\mathrm{Cov}(Y_\alpha(t),Y_\alpha(s)) =$$

$$= \frac{1}{\Gamma(1+\alpha)\Gamma(\alpha)}\int_0^{\min(t,s)}\left((t-\tau)^\alpha+(s-\tau)^\alpha\right)\tau^{\alpha-1}d\tau - \frac{(st)^\alpha}{\Gamma^2(1+\alpha)}. \tag{14}$$

Comments. 1. Notice that this last property can be interpreted as long-range dependence.

2. There is a (complicated) form of all finite-dimensional distributions of $Y_\alpha(t), t\geq 0$, in the form of Laplace transforms, see [5].

2.3. **Mixture of inverse subordinators.** Different kinds of inverse subordinators can be considered.

Here we deal with a mixture of α-stable subordinators. Following (1)

$$\phi(s) = \int_0^1 q(w)s^\alpha dw = \int_0^\infty (1-e^{-sx})l_q(x)dx,$$

where $q(w)$ is a probability density on $(0,1)$, and the density $l_q(x)$ of the Lévy measure is given by

$$l_q(x) = \int_0^1 \frac{x^{-\alpha-1}}{\Gamma(-\alpha)} q(\alpha) d\alpha.$$

The α-stable subordinator corresponds to the choice

$$q(w) = \delta(w-\alpha);$$

thus we can consider

$$q(w) = C_1\delta(w-\alpha_1) + C_2\delta(w-\alpha_2), C_1 + C_2 = 1,$$

with $\alpha_1 > \alpha_2$. This is a sum of two independent subordinators with

$$\phi(s) = C_1 s^{\alpha_1} + C_2 s^{\alpha_2}, C_1 + C_2 = k.$$

It was proved in [17] that

$$\tilde{U}(t) = \frac{1}{C_1 s^{\alpha_1} + C_2 s^{\alpha_2}}, U(t) = \frac{1}{C_1} t^{\alpha_1 - 1} E_{\alpha_1-\alpha_2,\alpha_1}(-C_2 t^{\alpha_1-\alpha_2}/C_1),$$

where $E_{\alpha,\beta}(z)$ is the two-parametrical Mittag-Leffler function ([8],[9])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ 0 < \alpha < 2,\ \beta > 0,\ z \in \mathbb{C}.$$

Another possibility is to consider the uniform mixture with $q(x) = 1, x \in (0,1)$.

One can also consider the tempered stable inverse subordinator, an inverse subordinator to the Poisson process, compound Poisson process, Gamma and inverse Gaussian Lévy processes, for additional details see Veillette and Taqqu [30], [31].

3. Homogeneous Poisson field

Let $\mathcal{B}$ be the class of Borel subsets of $\mathbb{R}^d$.

It is well known (see, e.g., Stoyan, Kendall, Mecke [29]) that the Poisson field, also called the spatial Poisson process, is an integer-valued random measure $N(A)$, $A \in \mathcal{B}$, which has the following properties:

(P1) for any finite collection $\{A_1, ..., A_n\}$ of disjoint Borel subsets

$$N(A_1), \ldots, N(A_n)$$

are independent;

(P2) the probability distribution function of $N(A)$ depends on the set A only through its size (or Lebesgue measure) $|A|$;

(P3) there exists $\lambda > 0$ such that

$$\mathrm{P}\left(N(A) \geq 1\right) = \lambda\,|A| + o(|A|), |A| \to 0;$$

(P4) the probability of points overlapping is zero

$$\lim_{|A|\to 0} \frac{\mathrm{P}\left(N(A) \geq 1\right)}{\mathrm{P}\left(N(A) = 1\right)} = 1.$$

If these axioms are satisfied, we have for any finite $n = 1, 2, \dots,$ and for any disjoint bounded Borel sets $A_1, ..., A_n$

$$\mathrm{P}\left(N(A_1) = k_1, ..., N(A_n) = k_n\right) \tag{15}$$

$$= \frac{\lambda^{k_1+...+k_n}}{k_1! \cdot .. \cdot k_n!} (|A_1|)^k \cdots (|A_n|)^{k_n} \exp\left\{-\sum_{j=1}^{n} \lambda |A_j|\right\}, \ k_j = 0, 1, 2, ...,$$

while

$$\mathrm{E}N(A) = \lambda |A|, \ \mathrm{Cov}(N(A_1), N(A_2)) = \lambda |A_1 \cap A_2|.$$

Also the emptiness probabilities

$$\mathrm{P}\left(N(A) = 0\right) = e^{-\lambda|A|}.$$

Particular emptiness probabilities are given by the so-called contact distribution function

$$H_A(r) = 1 - \mathrm{P}\left(N(rA) = 0\right), \ r \geq 0.$$

Here A is a compact "test set" with $|A| > 0$ containing the origin $0 \in A$, and $rA = \{rx, x \in A\}$ denotes the dilation of A. Obviously $H_A(r)$ is a distribution function.

4. Fractional Poisson fields

For the sake of simplicity, we restrict our discussion to the two-dimensional case $d = 2$.

Following ideas of Meershaert, Nane and Vellaisamy [23] (see also Mainardi, Gorenflo, Scalas [19], Beghin and Orsingher [3], [4]), we define the fractional Poisson field as follows:

$$N_{FP}(t_1, t_2) = N(Y^{(1)}(t_1), Y^{(2)}(t_2)), \ (t_1, t_2) \in \mathbb{R}_+^2, \tag{16}$$

where $Y^{(1)}(t), t \geq 0$ and $Y^{(2)}(t), t \geq 0$ are two independent inverse subordinators, and

$$N(t_1, t_2) = N([0, t_1] \times [0, t_2]), \ (t_1, t_2) \in \mathbb{R}_+^2.$$

is a restriction of homogeneous Poisson field on $\mathbb{R}_+^2$.We also assume that the Poisson field $N(t_1, t_2)$ is independent of the inverse subordinators $Y^{(1)}(t)$ and $Y^{(2)}(t)$.

More precisely, if $Y_{\alpha_1}^{(1)}(t), t \geq 0$ and $Y_{\alpha_2}^{(2)}(t), t \geq 0$ are two independent inverse stable subordinators with indices $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$, which are independent of the Poisson field $N(t_1, t_2)$, then we denote the fractional Poisson process

$$N_{\alpha_1, \alpha_2}(t_1, t_2) = N(Y_{\alpha_1}^{(1)}(t), Y_{\alpha_2}^{(2)}(t_2)), \ (t_1, t_2) \in \mathbb{R}_+^2. \tag{17}$$

For the dimension $d = 1$, this construction is known as the fractional Poisson process (see Meershaert, Nane and Vellaisamy [23]). It is denoted by

$$N_\alpha(t) = N(Y_\alpha(t)), \ t \geq 0, \ \alpha \in (0, 1),$$

where $N(t) = N([0,t]), t \geq 0$, is the classical homogeneous Poisson process with parameter $\lambda > 0$, which is independent of the inverse stable subordinator $Y_\alpha(t).t \geq, \alpha \in (0,1)$.

One can compute the following expression for the one-dimensional distribution:

$$\begin{aligned}
&\mathrm{P}\left(N_\alpha(t) = k\right) = \frac{(t\lambda)^k}{k!}\int_0^\infty e^{-\lambda x} x^{k-1-\frac{1}{\alpha}} g_\alpha(tx^{-\frac{1}{\alpha}})dx \\
&= \frac{(\lambda t^\alpha)^k}{k!}\sum_{j=1}^\infty \frac{(k+j)!}{j!}\frac{(-\lambda t^\alpha)^j}{\Gamma(\alpha(j+k)+1)} = \frac{(\lambda t^\alpha)^k}{k!}E_\alpha^{(k)}(-\lambda t^\alpha), \\
&k = 0,1,2...,t \geq 0,\ 0 < \alpha < 1
\end{aligned}$$

where $E_\alpha(z)$ is the Mittag-Leffler function evaluated at $z = -\lambda t^\alpha$, and $E_\alpha^{(k)}(z)$ is the $k-$th derivative of $E_\alpha(z)$ evaluated at $z = -\lambda t^\alpha$.

Note that (see [13] and [17])

$$\mathrm{E}N_\alpha(t) = \int_0^\infty \mathrm{E}N(u) f_\alpha(t,u)du = \lambda t^\alpha/\Gamma(1+\alpha), \tag{18}$$

where $f_\alpha(t,u)$ is given by (13), and using the method of the bivariate integration by parts proposed in [15], [16], Leonenko, Meerschaert and Sikorskii ([17]) showed that

$$\begin{aligned}
\mathrm{Cov}(N_\alpha(t), N_\alpha(s)) &= \int_0^\infty\int_0^\infty \mathrm{Var}N(1)\min(u,v)H_{t,s}(du,dv) \\
&+ (\mathrm{E}N(1))^2\mathrm{Cov}(Y_\alpha(t), Y_\alpha(s)) \\
&= \frac{\lambda(\min(t,s))^\alpha}{\Gamma(1+\alpha)} + \lambda^2\mathrm{Cov}(Y_\alpha(t), Y_\alpha(s)),
\end{aligned} \tag{19}$$

where $\mathrm{Cov}(Y_\alpha(t), Y_\alpha(s))$ is given in (14), and

$$H_{t,s}(u,v) = \mathrm{P}(Y_\alpha(t) < u, Y_\alpha(s) < v).$$

In this case (see again [23]), $N_\alpha(t), t \geq 0$, is a renewal process with Mittag-Leffler waiting times between the events

$$\begin{aligned}
N_\alpha(t) &= \max\{n : T_1 + ... + T_n \leq t\} = \sum_{j=1}^\infty \mathrm{I}\{T_1 + ... + T_j \leq t\} \\
&= \sum_{j=1}^\infty \mathrm{I}\{U_j \leq G_\alpha(t)\},\ t \geq 0,
\end{aligned} \tag{20}$$

where $\{T_j\}, j = 1,2,...$ are iid random variables with the strictly monotone Mittag-Leffler distribution function

$$F_\alpha(t) = \mathrm{P}\left(T_j \leq t\right) = 1 - E_\alpha(-\lambda t^\alpha),\ t \geq 0,\ 0 < \alpha < 1,\ j = 1,2,.....,$$

and

$$G_\alpha(t) = \mathrm{P}\left(T_1 + ... + T_j \leq t\right) = \int_0^t h^{(j)}(x)dx.$$

Here we denote an indicator as $\mathrm{I}\{\cdot\}$ and $U_j, j = 1, 2, ...$ are an iid uniformly distributed on $[0, 1]$ random variables. $h^{(j)}(x)$ is the pdf of $j - th$ convolution of the Mittag-Leffler distributions which is known as the generalized Erlang distribution and it is of the form

$$h^{(j)}(x) = \alpha\lambda^j \frac{x^{j\alpha-1}}{(j-1)!} E_\alpha^{(j)}(-\lambda x^\alpha),\ \alpha \in (0,1),\ x > 0.$$

Note that $S(t) = (t/L_\alpha(1))^\alpha$ is a distribution, and

$$\mathrm{P}\left(T_j > t\right) = \mathrm{E}e^{-\lambda S(t)},\ t \geq 0.$$

If $h_1(t, x)$ and $h_2(t, x)$ are densities (if they exist) of $Y^{(1)}(t)$, $t \geq 0$ and $Y^{(2)}(t)$, $t \geq 0$, resp., then (assuming that they are independent), we have the following:

$$\mathrm{P}\left(N_{FP}(t_1,t_2) = k\right) = \int_0^\infty \int_0^\infty \mathrm{P}(N(x_1,x_2) = k) h_1(t_1,x_1) h_2(t_2,x_2) dx_1 dx_2,$$
$$k = 0, 1, 2 \ldots .$$

In particular, for an inverse stable subordinator, we obtain

(21)
$$\mathrm{P}\left(N_{\alpha_1,\alpha_2}(t_1,t_2) = k\right) = \int_0^\infty \int_0^\infty \frac{e^{-\lambda x_1 x_2}(\lambda x_1 x_2)^k}{k!} f_{\alpha_1}(t_1,x_1) f_{\alpha_2}(t_2,x_2) dx_1 dx_2,$$
$$k = 0, 1, 2...,$$

where $f_a(t, x)$ is given by (13). In other words,

(22)
$$\mathrm{P}\left(N_{\alpha_1,\alpha_2}(t_1,t_2) = k\right)$$
$$= \frac{t_1 t_2 \lambda^k}{\alpha_1\alpha_2 k!} \int_0^\infty \int_0^\infty e^{-\lambda x_1 x_2} x_1^{k-1-\frac{1}{\alpha_1}} x_2^{k-1-\frac{1}{\alpha_2}} g_{\alpha_1}(t_1 x_1^{-\frac{1}{\alpha_1}}) g_{\alpha_2}(t_2 x_2^{-\frac{1}{\alpha_2}}) dx_1 dx_2,$$
$$= \frac{\lambda^k}{k! t_1 t_2} \int_0^\infty \int_0^\infty e^{-\lambda x_1 x_2} x_1^{k+\frac{1}{\alpha_1}} x_2^{k+\frac{1}{\alpha_2}} W_{-\alpha_1,0}(-\frac{x_1}{t_1^{\alpha_1}}) W_{-\alpha_2,0}(-\frac{x_2}{t_2^{\alpha_2}}) dx_1 dx_2$$
$$(t_1,t_2) \in \mathbb{R}_+^2,\ k = 0, 1, 2...,$$

where the Wright' s generalized Bessel function is defined by (12), and

(23)
$$g_{\alpha_i}(t_i x_i^{-\frac{1}{\alpha_i}}) = \frac{1}{\pi} \sum_{j_i=1}^\infty (-1)^{j_i+1} \frac{\Gamma(\alpha_i j_i + 1)}{j_i!} \frac{x_i^{j_i+1/\alpha_i}}{t_i^{\alpha_i j_j+1}} \sin(\pi j_i \alpha_i),\ i = 1, 2.$$

Using the formula

$$\sin(\pi x) = \frac{\pi}{\Gamma(x)\Gamma(1-x)},$$

one can rewrite (23) in the form

$$g_{\alpha_i}(t_i x_i^{-\frac{1}{\alpha_i}}) = \sum_{j_i=1}^\infty (-1)^{j_i+1} \frac{\Gamma(\alpha_i j_i + 1)}{j_i!} \frac{x_i^{j_j+1/\alpha_i}}{t_i^{\alpha_i j_i+1}} \frac{1}{\Gamma(\pi j_i\alpha_i)\Gamma(1-\pi j_i\alpha_i)},\ i = 1, 2,$$

of the Mittag-Leffler type function or the H-function of Fox, see, e.g., [9].

It is known (see, e.g., [1, Theorems 1.1. and 1.2]) that a single point random field M can be regarded as a random set of points and could be characterized either by all systems of finite-dimensional distributions as

$$\mathrm{P}\left(M(B_1)=k_1,...,M(B_m)=k_m\right)$$

for all integers $m>0$ and all compacts $B_1,..,B_m$, or the so-called capacity functional

$$T_M(K)=1-P(M(K)=0),$$

for any compact K. By results of ([1], Theorems 1.1. and 1.2)), the distribution of the field N_{α_1,α_2} is completely characterized by its capacity functional

(24)

$$T_{N_{\alpha_1,\alpha_2}}([0,t_1]\times[0,t_2])=1-\frac{t_1t_2}{\alpha_1\alpha_2}\int_0^\infty\int_0^\infty e^{-\lambda x_1x_2}\frac{g_{\alpha_1}(t_1x_1^{-\frac{1}{\alpha_1}})g_{\alpha_2}(t_2x_2^{-\frac{1}{\alpha_2}})}{x_1^{1+\frac{1}{\alpha_1}}x_2^{1+\frac{1}{\alpha_2}}}dx_1dx_2.$$

Using (18) and (19) (see [13],[15], [16],[17] for details), we obtain

$$\begin{aligned}\mathrm{E}N_{\alpha_1,\alpha_2}(t_1,t_2)&=\int_0^\infty\int_0^\infty \mathrm{E}N(u_1,u_2)f_{\alpha_1}(t_1,u_1)f_{\alpha_2}(t_2,u_2)du_1du_2\\ &=\lambda t_1^{\alpha_1}t_2^{\alpha_2}/[\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)]\end{aligned}\tag{25}$$

and

$$\begin{aligned}&\mathrm{Cov}(N_{\alpha_1,\alpha_2}(t_1,t_2),N_{\alpha_1,\alpha_2}(s_1,s_2))\\ &\quad=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{Cov}\{N(u_1,v_1)N(u_2,v_2)\}\cdot H_{t_1,s_1}(du_1,dv_1)H_{t_2,s_2}(du_2,dv_2)\\ &\qquad+\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{E}N(u_1,v_1)N(u_2,v_2)\cdot H_{t_1,s_1}(du_1,dv_1)H_{t_2,s_2}(du_2,dv_2)\\ &\qquad-\lambda^2\mathrm{E}Y_{\alpha_1}^{(1)}(t_1)\mathrm{E}Y_{\alpha_1}^{(1)}(s_1)\mathrm{E}Y_{\alpha_2}^{(2)}(t_2)\mathrm{E}Y_{\alpha_2}^{(2)}(s_2).\end{aligned}$$

We have

(26)

$$\begin{aligned}&\mathrm{Cov}(N_{\alpha_1,\alpha_2}(t_1,t_2),N_{\alpha_1,\alpha_2}(s_1,s_2))\\ &=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{Var}N(1,1)\{\min(u_1,v_1)\min(u_2,v_2)\}\cdot H_{t_1,s_1}(du_1,dv_1)H_{t_2,s_2}(du_2,dv_2)\\ &\quad+\lambda^2(\mathrm{E}Y_{\alpha_1}^{(1)}(t_1)Y_{\alpha_1}^{(1)}(s_1)\mathrm{E}Y_{\alpha_2}^{(2)}(t_2))Y_{\alpha_2}^{(2)}(s_2)-\mathrm{E}Y_{\alpha_1}^{(1)}(t_1)\mathrm{E}Y_{\alpha_1}^{(1)}(s_1)\mathrm{E}Y_{\alpha_2}^{(2)}(t_2)\mathrm{E}Y_{\alpha_2}^{(2)}(s_2))\\ &=\lambda(\min(t_1,s_1))^{\alpha_1}(\min(t_2,s_2))^{\alpha_2}/[\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)]\\ &\quad+\lambda^2[\frac{1}{\Gamma(1+\alpha_1)\Gamma(\alpha_1)\Gamma(1+\alpha_2)\Gamma(\alpha_2)}\int_0^{\min(t_1,s_1)}\int_0^{\min(t_2,s_2)}((t_1-\tau_1)^{\alpha_1}+(s_1-\tau_1)^{\alpha_1})\\ &\quad\times((t_2-\tau_2)^{\alpha_2}+(s_2-\tau_2)^{\alpha_2})\,\tau_1^{\alpha_1-1}\tau_2^{\alpha_2-1}d\tau_1d\tau_2-\frac{(s_1t_1)^{\alpha_1}(s_2t_2)^{\alpha_2}}{\Gamma^2(1+\alpha_1)\Gamma^2(1+\alpha_2)}],\end{aligned}$$

$(t_1,t_2),(s_1,s_2)\in\mathbb{R}_+^2$,

in particular
(27)
$\mathrm{Var}N_{\alpha_1,\alpha_2}(t_1,t_2) = \lambda {t_1}^{\alpha_1}{t_2}^{\alpha_2}C_1(\alpha_1,\alpha_2)+\lambda^2{t_1}^{2\alpha_1}{t_2}^{2\alpha_2}C_2(\alpha_1,\alpha_2),\ (t_1,t_2),\in\mathbb{R}^2_+,$

where

$$C_1(\alpha_1,\alpha_2)=\frac{1}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)};C_2(\alpha_1,\alpha_2)=\frac{1}{\alpha_1\alpha_2\Gamma(2\alpha_1)\Gamma(2\alpha_2)}-\frac{1}{(\alpha_1\alpha_2)^2\Gamma^2(\alpha_1)\Gamma^2(\alpha_2)}$$

We can summarize our results in the following

Theorem. *Let* $N_{\alpha_1,\alpha_2}(t_1,t_2),(t_1,t_2)\in\mathbb{R}^2_+$, *be a fractional Poisson process defined by (17). Then*

i) $\mathrm{P}\left(N_{\alpha_1,\alpha_2}(t_1,t_2)=k\right),\ k=0,1,2...$*is given by (22);*
ii) $\mathrm{E}N_{\alpha_1,\alpha_2}(t_1,t_2),\mathrm{Var}N_{\alpha_1,\alpha_2}(t_1,t_2)$ *and* $\mathrm{Cov}(N_{\alpha_1,\alpha_2}(t_1,t_2),N_{\alpha_1,\alpha_2}(s_1,s_2))$ *are given by (25),(27),(26) respectively;*
iii) *the following long-range dependence property holds:*

(28)
$$\frac{N_{\alpha_1,\alpha_2}(t_1,t_2)}{t_1^{2\alpha_1}t_2^{2\alpha_2}}-\frac{\lambda}{t_1^{\alpha_1}t_2^{\alpha_2}[\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)]}\to 0,\ \alpha_1\in(0,1),\ \alpha_2\in(0,1),$$

in probability as $t_1,t_2\to\infty$.

Remark. For $d=1$ the definition of Hurst index for renewal processes is discussed in [7]. Following the ideas of this paper the Hurst index of the fractional Poisson random field in $d=2$ can be defined as follows:

$$H=\inf\left\{\beta:\lim\sup_{T\to\infty}\frac{\mathrm{Var}N_{\alpha_1,\alpha_2}(T,T)}{T^\beta}<\infty\right\},$$

thus $H=2(\alpha_1+\alpha_2)$ and lies in the interval $0<H<4$.

5. Alternative forms of the fractional Poisson-like random field

We present some ideas for different versions of fractional Poisson-like random fields.

Similar to (20) one can define a point random field

$$N_F^{(0)}(t_1,t_2)=\sum_{j=1}^{\infty}\mathrm{I}\left\{U_j\in[0,G_{\alpha_1}(t_1)]\times[0,G_{\alpha_2}(t_2)]\right\},\ (t_1,t_2)\in\mathbb{R}^2_+, \tag{29}$$

where $U_j=(U_j^{(1)},U_j^{(2)}),j=1,2,..$ is a sequence of iid random vectors on $[0,1]^2\subset\mathbb{R}^2_+$ with the uniform distribution, that is, for any Borel $A\subset[0,1]^2$

$$\mathrm{P}\left(U_j\in A\right)=|A|,$$

and

$$0<\alpha_i<1,\quad i=1,2.$$

This definition is motivated by Ivanoff, Merzbach [12], Herbin, Merzbach [10] and Merzbach, Shaki [25]. Formula (29) is useful for computations as well as simulation. Notice that all components $U_j^{(1)}, U_j^{(2)}$, $j = 1, 2, ..$ are independent, and

$$\begin{aligned} \mathrm{I}\{U_j \in [0, G_{\alpha_1}(t_1)] \times [0, G_{\alpha_2}(t_2)]\} &= \mathrm{I}\left\{G_{\alpha_1}(U_j^{(1)}) \le t_1, G_{\alpha_1}(U_j^{(2)}) \le t_2\right\} \\ &= \mathrm{I}\left\{G_{\alpha_1}(U_j^{(1)}) \le t_1\right\} \mathrm{I}\left\{G_{\alpha_1}(U_j^{(2)}) \le t_1\right\} \\ &= \mathrm{I}\left\{T_1^{(1)} + ... + T_j^{(1)} \le t_1\right\} \mathrm{I}\left\{T_1^{(2)} + ... + T_j^{(2)} \le t_2\right\}, \end{aligned}$$

where $\left\{T_j^{(i)}\right\}$, $j = 1, 2, ...$ are iid random variables with the strictly monotone Mittag-Leffler distribution function

$$F_{\alpha_i}(t) = \mathrm{P}\left(T_j^{(i)} \le t\right) = 1 - E_{\alpha_i}(-\lambda t^{\alpha_i}),\ t \ge 0,\ 0 < \alpha_i < 1,\ j = 1, 2, ...,\ i = 1, 2.$$

Thus, the fractional Poisson-like random field (29) can be written as

$$\tilde{N}_F(t_1, t_2) = \sum_{j=1}^{\infty} \mathrm{I}\left\{T_1^{(1)} + ... + T_j^{(1)} \le t_1\right\} \mathrm{I}\left\{T_1^{(2)} + ... + T_j^{(2)} \le t_2\right\},\ (t_1, t_2) \in \mathbb{R}_+^2,$$

and $\mathrm{P}\left\{N_F^{(0)}(t_1, t_2) = k\right\}$ can be computed, while its capacity functional $T_{\tilde{N}}(K) = 1 - \mathrm{P}(\tilde{N}_F(K) = 0)$ is different from (24).

One can also define the Poisson-like random fields $N_F^{(k)}(A), k \in \{1, 2, 3\}$ by their capacity functionals

$$T_{N^{(1)}}(A) = 1 - \mathrm{P}(N_F^{(1)}(A) = 0) = 1 - \frac{1}{E_\alpha(\lambda |A|)},\ 0 < \alpha \le 1,\ \lambda > 0,$$

or

$$T_{N^{(2)}}(A) = 1 - \mathrm{P}(N_F^{(2)}(A) = 0) = 1 - E_\alpha(-\lambda |A|^\alpha),$$

for any Borel set $A \subset \mathbb{R}^2$ with finite Lebesgue measure $|A| < \infty$. The above constructions coincide with the definition of the homogeneous Poisson random field for $\alpha = 1$, but for $0 < \alpha < 1$ many basic properties (such as independence of increments) are lost and we even do not know if there exist Poisson-like fields with the properties

$$\mathrm{P}(N_F^{(1)}(A) = k) = \frac{(\lambda |A|)^k}{E_\alpha(\lambda |A|)\Gamma(ak + 1)},\ k = 0, 1, 2, ...$$

or

$$\mathrm{P}(N_F^{(2)}(A) = k) = \frac{(\lambda |A|^\alpha)^k}{k!} \sum_{r=0}^{\infty} \frac{(r + k)!}{r!} \frac{(\lambda |A|^\alpha)^r}{\Gamma(\alpha(r + k) + 1)},\ k = 0, 1, 2, ...,$$

or

(30)

$$\mathrm{P}(N_F^{(3)}(A) = k) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda |A|^\alpha)^r}{r!} \frac{(r + k)!}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - k)},\ k = 0, 1, 2, ...,$$

see however [3], [4],[26] for some discussion of the problem for $d = 1$.

These problems will be discussed elsewhere.

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