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# The Multiparameter Fractional Brownian Motion

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**Summary.** We define and study the multiparameter fractional Brownian motion. This process is a generalization of both the classical fractional Brownian motion and the multiparameter Brownian motion, when the condition of independence is relaxed. Relations with the Lévy fractional Brownian motion and with the fractional Brownian sheet are discussed. Different notions of stationarity of the increments for a multiparameter process are studied and applied to the fractional property. Using self-similarity we present a characterization for such processes. Finally, behavior of the multiparameter fractional Brownian motion along increasing paths is analysed.

## 1 Introduction

The aim of this paper is to give a satisfactory definition of the concept of Multiparameter Fractional Brownian Motion (MpfBm). The definition given here is a particular case of the Set-indexed Fractional Brownian Motion studied in [HeMe06], but in the multiparameter case, some new interesting properties can be obtained.

In the last decade, two other definitions for the MpfBm appeared in the literature (see [He05] for a review of their properties). Both are problematic as extensions of the classical fractional Brownian motion. In this work, we hope to persuade the reader that our definition is natural, is the “right” generalization of the fractional Brownian motion (fBm) and can be applied directly to real applied problems.

## 2 Definition of the MpfBm

In [HeMe06], a set-indexed extension of fractional Brownian motion was defined and some extensions of fractal properties were established. Let  $\mathcal{A}$  be an indexing collection of compact subsets of a metric measure space (metric  $d$  and measure  $m$ ) satisfying certain assumptions, the *Set-indexed fractional Brownian motion (SifBm)* was defined as the centered Gaussian process  $\mathbf{B}^H = \{\mathbf{B}_U^H; U \in \mathcal{A}\}$  such that

$$\forall U, V \in \mathcal{A}; E[\mathbf{B}_U^H \mathbf{B}_V^H] = \frac{1}{2} [m(U)^{2H} + m(V)^{2H} - m(U \Delta V)^{2H}], \quad (1)$$

where  $0 < H \leq \frac{1}{2}$  and  $m$  is a measure defined on the  $\sigma$ -algebra generated by  $\mathcal{A}$ .

As the collection  $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$  is a particular indexing collection, definition (1) provides a multiparameter process which can be seen as a multiparameter extension of fractional Brownian motion. We get the following definition.

**Definition 1.** *The Multiparameter Fractional Brownian Motion (MpfBm) is defined as the centered Gaussian process  $\mathbf{B}^H = \{\mathbf{B}_t^H; t \in \mathbf{R}_+^N\}$  such that*

$$\forall s, t \in \mathbf{R}_+^N; E[\mathbf{B}_s^H \mathbf{B}_t^H] = \frac{1}{2} [m([0, s])^{2H} + m([0, t])^{2H} - m([0, s] \Delta [0, t])^{2H}] \quad (2)$$

where  $m$  is a measure on  $\mathbf{R}^N$  and  $H \in (0, 1/2]$  is called the index of similarity.

This definition looks very natural since it relies on a set-indexed process and thus, structure of the space  $\mathbf{R}^N$  is only present in indices and not in the shape of the covariance function.

Notice that the definition of the MpfBm depends on the measure  $m$ .

In the particular case of  $\mathbf{R}_+^2$  with the Lebesgue measure  $m$ , we can explicitly give the covariance between  $s = (s_1, s_2)$  and  $t = (t_1, t_2)$

$$E[\mathbf{B}_s^H \mathbf{B}_t^H] = \frac{1}{2} [(s_1 s_2)^{2H} + (t_1 t_2)^{2H} - (s_1 s_2 + t_1 t_2 - 2(s_1 \wedge t_1)(s_2 \wedge t_2))^{2H}].$$

Let us notice that parameter  $H$  is restricted to be in  $(0, 1/2]$ , on the contrary to standard fractional Brownian motion, in which  $H$  is in  $(0, 1)$ .

*Remark 1.* If the measure  $m$  is absolutely continuous with respect to the Lebesgue measure, the process  $\mathbf{B}^H$  is almost surely null on the axis.

Self-similarity is the first property of MpfBm. As a particular case of the set-indexed fractional Brownian motion, the multiparameter process inherits its properties. For all  $a \in \mathbf{R}_+$ ,

$$\{\mathbf{B}_{at}^H; t \in \mathbf{R}_+^N\} \stackrel{(d)}{=} \{a^{NH} \mathbf{B}_t^H; t \in \mathbf{R}_+^N\}.$$

where  $\stackrel{(d)}{=}$  denotes equality of finite dimensional distributions.

### 3 Comparisons with other multiparameter extensions of fBm

The following two multiparameter extensions of fractional Brownian motions are classical. Their definitions rely on a generalization of covariance structure of fBm based on euclidian structure of  $\mathbf{R}^N$ . The first definition uses the euclidian norm and the second one uses the canonical basis of  $\mathbf{R}^N$ .

#### 3.1 The Lévy fractional Brownian motion

The Lévy fractional Brownian motion (Lévy fBm) is defined as the mean-zero Gaussian process  $\mathcal{B}^H = \{\mathcal{B}_t^H; t \in \mathbf{R}_+^N\}$  such that

$$\forall s, t \in \mathbf{R}_+^N; \quad E[\mathcal{B}_s^H \mathcal{B}_t^H] = \frac{1}{2} [\|s\|^{2H} + \|t\|^{2H} - \|t - s\|^{2H}]$$

where  $H \in (0, 1)$ .

The structure of the covariance function of  $\mathcal{B}^H$  provides an extension of fractional Brownian motion where the absolute value in  $\mathbf{R}_+$  is substituted with the euclidian norm of the space  $\mathbf{R}_+^N$ . From this point of view, the Lévy fBm is usually called an isotropic extension of fBm. However, with this simple generalization, the process does not seem to be really a multiparameter process.

This simple definition allows to state directly the self-similarity property. For all  $a \in \mathbf{R}_+$ ,

$$\{\mathcal{B}_{at}^H; t \in \mathbf{R}_+^N\} \stackrel{(d)}{=} \{a^H \mathcal{B}_t^H; t \in \mathbf{R}_+^N\}.$$

#### 3.2 The fractional Brownian sheet

The fractional Brownian sheet is defined as the mean-zero Gaussian process  $\mathbb{B}^H = \{\mathbb{B}_t^H; t \in \mathbf{R}_+^N\}$  such that

$$\forall s, t \in \mathbf{R}_+^N; \quad E[\mathbb{B}_s^H \mathbb{B}_t^H] = \frac{1}{2} \prod_{i=1}^N [s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}]$$

where  $H = (H_1, \dots, H_N) \in (0, 1)^N$ .

In this definition, the euclidian structure of the space  $\mathbf{R}^N$  is strongly present in the shape of the covariance function of the fractional Brownian sheet. Particularly, this kind of tensor product of standard fractional Brownian motions along each direction of the canonical basis of  $\mathbf{R}^N$  seems quite artificial and lacks generality to be really efficient in concrete applications.

From the covariance structure of fractional Brownian sheet, the self-similarity property can be easily established. For all  $a \in \mathbf{R}_+$ ,

$$\{\mathbb{B}_{at}^H; t \in \mathbf{R}_+^N\} \stackrel{(d)}{=} \{a^{\sum_j H_j} \mathbb{B}_t^H; t \in \mathbf{R}_+^N\}$$

## 4 Different notions of stationarity

Stationarity of increments is one of the two characteristic properties of the classical fractional Brownian motion. In the framework of multiparameter processes, the notion of stationarity can take different forms:

- Stationarity against translation

$$\forall h \in \mathbf{R}_+^N; \quad (X_t - X_0)_{t \in \mathbf{R}_+^N} \stackrel{(d)}{=} (X_{t+h} - X_h)_{t \in \mathbf{R}_+^N} \quad (3)$$

- Stationarity in the strong sense

$$\forall g \in \mathcal{G}(\mathbf{R}^N); \quad (X_t - X_0)_{t \in \mathbf{R}_+^N} \stackrel{(d)}{=} (X_{g(t)} - X_{g(0)})_{t \in \mathbf{R}_+^N} \quad (4)$$

where  $\mathbf{R}^N$  is the set of rigid motions on  $\mathbf{R}^N$ ; see [Ta95, p. 392].

For the next definitions, one needs the notion of the increment of a process  $X$  on a rectangle  $D = [s, t]$ ,  $s = (s_1, \dots, s_N)$  and  $t = (t_1, \dots, t_N)$  where  $s \prec t$  ( $s_i \leq t_i$ ,  $i = 1, \dots, N$ )

$$\Delta X(D) = \sum_{r \in \{0,1\}^N} (-1)^{N - \sum_i r_i} X_{[s_i + r_i(t_i - s_i)]_i}.$$

- Increment stationarity against translation

$$\forall h \in \mathbf{R}_+^N; \quad (\Delta X_{[0,t]})_{t \in \mathbf{R}_+^N} \stackrel{(d)}{=} (\Delta X_{[h,t+h]})_{t \in \mathbf{R}_+^N} \quad (5)$$

- Increment stationarity in the strong sense

$$\forall g \in \mathcal{G}(\mathbf{R}^N); \quad (\Delta X_{[0,t]})_{t \in \mathbf{R}_+^N} \stackrel{(d)}{=} (\Delta X_{[g(0),g(t)]})_{t \in \mathbf{R}_+^N} \quad (6)$$

- Measure stationarity (also called  $C_0$ -increment stationarity)

$$\forall t, \forall \tau > \tau' \in \mathbf{R}_+^N; \quad m([0, \tau]) - m([0, \tau']) = m([0, t]) \Rightarrow X_t - X_0 \stackrel{(d)}{=} X_\tau - X_{\tau'} \quad (7)$$

- Increment measure stationarity

$$\forall t, \forall \tau, \tau' \in \mathbf{R}_+^N; \quad m([0, t]) = m([\tau, \tau']) \Rightarrow \Delta X_{[0,t]} \stackrel{(d)}{=} \Delta X_{[\tau, \tau']}. \quad (8)$$

Notice that, among these 6 properties of stationarity, the first 4 are process properties, but the last 2 properties are pointwise properties.

The following result summarizes the connections between these different stationarity properties:

**Proposition 1.** *The following implications hold:*

- (4)  $\Rightarrow$  (3)  $\Rightarrow$  (5);
- (4)  $\Rightarrow$  (6)  $\Rightarrow$  (5);
- (8)  $\Rightarrow$  (7).

In [HeMe06], the following was proved:

**Proposition 2.** *The MpfBm is  $C_0$ -increment stationary.*

Let  $B^H$  be a MpfBm. The increment covariance between two rectangles  $D$  and  $D'$ ,  $E[\Delta B^H(D) \cdot \Delta B^H(D')]$  can be computed, but the formula is quite complicated.

In the particular case of  $\mathbf{R}_+^2$ , with the Lebesgue measure and  $D = D' = (s, t]$ , we get:

$$\begin{aligned} E[\Delta B^H(D)]^2 &= (t_1 t_2 - s_1 t_2)^{2H} + (t_1 t_2 - t_1 s_2)^{2H} - (t_1 t_2 - s_1 s_2)^{2H} \\ &\quad - (s_1 t_2 + t_1 s_2 - 2s_1 s_2)^{2H} + (s_1 t_2 - s_1 s_2)^{2H} + (t_1 s_2 - s_1 s_2)^{2H}. \end{aligned}$$

## 5 Characterization

In Sections 2 and 4, the multiparameter fractional Brownian motion was shown to be self-similar and  $C_0$ -increment stationary. As standard fractional Brownian motion is characterized by its two fractal properties, self-similarity and stationarity, it is natural to wonder what are the multiparameter processes satisfying the two properties.

As a particular case of set-indexed fractional Brownian motion, the multiparameter fractional Brownian motion satisfies a pseudo-characterization property.

**Proposition 3.** *Let  $X = \{X_t; t \in \mathbf{R}_+^N\}$  be a multiparameter process satisfying the following two properties:*

1. *self-similarity of index  $\alpha \in (0, N/2)$ ,*
2.  *$C_0$ -increment stationarity.*

*Then, the covariance function between  $s$  and  $t$  such that  $s \prec t$  is*

$$E[X_s \cdot X_t] = K \left[ m([0, s])^{2\alpha/N} + m([0, t])^{2\alpha/N} - m([0, t] \setminus [0, s])^{2\alpha/N} \right].$$

*Proof.* The result simply relies on Proposition 4.1 of [HeMe06], where we consider the operation of  $\mathbf{R}_+$  on  $\{[0, t]; t \in \mathbf{R}_+^N\}$  such that

$$\forall a > 0, \forall t \in \mathbf{R}_+^N; \quad a \cdot [0, t] = [0, at].$$

In that framework, we have

$$\forall a > 0, \forall t \in \mathbf{R}_+^N; \quad m(a \cdot [0, t]) = a^N m([0, t])$$

and then,  $\mu$  is the function  $a \mapsto a^N$ , which is surjective.

A consequence of Proposition 3 is that the fractal properties of self-similarity and  $C_0$ -increments stationarity prescribe the covariance between points  $s$  and  $t$  that are comparable for partial order  $\prec$  of  $\mathbf{R}^N$ . Since there are non ordered points, we cannot get a complete characterization of the MpfBm by the two properties of self-similarity and stationarity.

A natural question is then, what are the self-similar processes which are stationary in the different definitions of Section 4? The following result shows that for some choice of stationarity definition, we obtain characterization of the Lévy fBm.

**Proposition 4.** *Let  $H \in (0, 1)$ . The Lévy fBm is the only Gaussian process which is self-similar of index  $H$  and whose increments are stationary in the strong sense (property (6)).*

*Proof.* (cf. [SaTa95, p. 393])

It is known (Sections 3 and 4) that the Lévy fBm is self-similar and has stationary increments in the strong sense.

Conversely, let  $X = \{X_t; t \in \mathbf{R}_+^N\}$  be a Gaussian process such that

$$\forall a \in \mathbf{R}_+; \quad \{X_{at}; t \in \mathbf{R}_+^N\} \stackrel{(d)}{=} \{a^H X_t; t \in \mathbf{R}_+^N\}$$

and

$$\forall g \in \mathcal{G}(\mathbf{R}^N); \quad \{X_t - X_0; t \in \mathbf{R}_+^N\} \stackrel{(d)}{=} \{X_{g(t)} - X_{g(0)}; t \in \mathbf{R}_+^N\}$$

First of all, considering the canonical basis  $(\epsilon_i)_{1 \leq i \leq N}$  of  $\mathbf{R}^N$ , and the rotation  $g_u$  that maps  $\epsilon_1$  onto any unit vector  $u$ , the stationarity property leads to

$$E[X_u] = E[X_{g_u(\epsilon_1)} - X_0] = E[X_{\epsilon_1}]$$

For any  $s$  and  $t$  in  $\mathbf{R}_+^N$ , the self-similarity property leads to

$$E[X_t - X_s] = E[X_{t-s}] - E[X_0] = \|t - s\|^H E[X_{\epsilon_1}].$$

As we also have

$$E[X_t - X_s] = (\|t\|^H - \|s\|^H) E[X_{\epsilon_0}],$$

we get  $E[X_t] = 0$  for all  $t \in \mathbf{R}_+^N$ .

In the same way, we prove that for any  $s$  and  $t$  in  $\mathbf{R}_+^N$ ,

$$E[(X_t - X_s)^2] = E[(X_{t-s} - X_0)^2] = \|t - s\|^{2H} E[X_{\epsilon_1}^2].$$

The result follows.

In the fractional Brownian sheet case, several supplementary assumptions are needed to obtain a characterization of the process. Particularly, a null value of the process on each axis must be imposed as well as a condition of self-similarity for each parameter, when the  $N - 1$  other ones are fixed (see [LePo99]). From that point of view, the fractional Brownian sheet has no real motivation to be considered, although it satisfies the two properties of stationarity and self-similarity.

## 6 Integral representation

Existence of such integral representations allows to define multifractional generalization of these two processes (see [He05]). The aim of this kind of extension is to substitute the constant parameter  $H$  with a function  $H : \mathbf{R}_+^N \rightarrow (0, 1)$ , in order to allow the regularity to vary along the paths. These processes with prescribed regularity bring a great flexibility for modelling random phenomena.

## 7 Projection on flows

The notion of flow is the key to reduce the proof of many theorems. It was extensively studied in [Iv03] and [IvMe00].

**Definition 2.** Let  $S = [a, b] \subseteq \mathbf{R}$ . An increasing function  $f : S \rightarrow \mathbf{R}_+^N$  ( $x < y \Rightarrow f(x) \prec f(y)$ ) is called a flow.

The following results, proved in [HeMe06], give a good justification of the definition of the MpfBm.

**Proposition 5.** Let  $B^H$  be a MpfBm and  $f$  be a flow. Then the process  $(B^H)^f = \{B_{f(t)}^H, t \in [a, b]\}$  is a time changed fractional Brownian motion.

However, in general, the projection of a multiparameter process does not inherit its different properties.

**Proposition 6.** Let  $f$  be a flow, and  $X$  be a multiparameter process.

1. If  $X$  is a Lévy  $f$ Bm, then  $(X)^f$  is a classical fractional Brownian motion iff  $f(t) = \alpha t$  where  $\alpha \in \mathbf{R}_+^N$ .
2. If  $X$  is a fractional Brownian sheet, then  $(X)^f$  is a classical fractional Brownian motion iff  $f$  is a line parallel to one axis of  $\mathbf{R}_+^N$ .

We conclude this section by giving an integral representation.

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## References

- [Ad90] Adler, R. J.: An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. IMS Lect. Notes, Monograph Series, vol. 12, Hayward, California (1990)
- [BJR98] Benassi, A., Jaffard, S., Roux, D.: Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana* **13**, 19-89 (1998)
- [Du73] Dudley, R. M.: Sample functions of the Gaussian process. *Ann. Probab.* **1**, no. 1, 66-103 (1973)
- [He05] Herbin, E.: From  $N$ -parameter fractional Brownian motions to  $N$ -parameter multifractional Brownian motions. *Rocky Mountain J. Math.* (2005)
- [HeLe04] Herbin, E., Lévy-Véhel, J.: Fine analysis of the local regularity of some Gaussian processes. Preprint (2004)
- [HeMe06] Herbin, E., Merzbach, E.: A set-indexed fractional Brownian motion. *J. Theoret. Probab.*, to appear (2006)
- [Iv03] Ivanoff, G.: Set-indexed processes: distributions and weak convergence. In: *Topics in Spatial Stochastic Processes, Lecture Notes in Mathematics*, 1802, 85-126, Springer (2003)
- [IvMe00] Ivanoff, G., Merzbach, E.: *Set-Indexed Martingales*, Chapman & Hall/CRC, 2000.
- [Ka96] Kamont, A.: On the fractional anisotropic Wiener field. *Prob. and Math. Stat.* **16**, fasc. 1, 85-98 (1996)
- [Kh02] Khoshnevisan, D.: *Multiparameter Processes: An Introduction to Random Fields*. Springer (2002)
- [LePo99] Leger, S., Pontier, M.: Drap brownien fractionnaire. *CRAS 329 srie I*, 893-898 (1999)
- [PeLe95] Peltier, R., Lévy-Véhel, J.: Multifractional Brownian motion: definition and preliminary results. *Rapport de recherche INRIA 2645* (1995)
- [Pes98] Pesquet-Popescu, B., *Modélisation bidimensionnelle de processus non stationnaires et application à l'étude du fond sous-marin*. Thèse de l'ENS Cachan (1998)
- [Ta95] Talagrand, M.: Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23**, no. 2, 767-775 (1995)
- [SaTa95] Samorodnitsky, G., Taqqu, M. S.: *Stable Non-Gaussian Random Processes*. Chapman & Hall/CRC (1994)
- [Xi97] Xiao, Y.: Hausdorff measure of the graph of fractional Brownian motion. *Math. Proc. Camb. Phil. Soc.* **122**, 565-576 (1997)