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A Set-indexed Fractional Brownian Motion

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We define and prove the existence of a fractional Brownian motion indexed by a collection of closed subsets of a measure space. This process is a generalization of the set-indexed Brownian motion, when the condition of independance is relaxed. Relations with the Lévy fractional Brownian motion and with the fractional Brownian sheet are studied. We prove stationarity of the increments and a property of self-similarity with respect to the action of solid motions. Moreover, we show that there no "really nice" set indexed fractional Brownian motion other than set-indexed Brownian motion. Finally, behavior of the set-indexed fractional Brownian motion along increasing paths is analysed.

KEY WORDS: Fractional Brownian motion; Gaussian processes; stationarity; self-similarity; set-indexed processes.

AMS Classification: 62G05, 60G15, 60G17, 60G18.

1. INTRODUCTION

Recently, different developments around fractional Brownian motion were set up and widely used to describe complex or chaotic phenomena in several fields of sciences. For instance, let us mention theoretical aspects such as stochastic calculus with respect to fBm (e.g. Ref. 3), and some more applied aspects such as its use finance[(10,21)] or in data traffic modeling.[(17)] Here we define a new very natural set-indexed generalization of fractional Brownian motion. The set-indexed fractional Brownian motion studied

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here seems to be well-adapted to model problems in applied mathematics (see Ref. 19).

Fractional Brownian motion was defined by B. B. Mandelbrot and J. W. Van Ness, and extended essentially into two directions. One is generally called multifractional Brownian motion, replacing the index parameter of self-similarity (called also the Hurst parameter) by a real measurable function (see Refs. 4 and 18). The other one are multiparameter fractional Brownian motions in which the set of the indices is a subset of the Euclidean space $\mathbf{R}^N$ (see Refs. 14, 22 and 24 for trajectory properties).

At least two types of multiparameter fractional Brownian motions were introduced. One is called Lévy fractional Brownian motion because it extends Lévy Brownian motion and the other is called fractional Brownian sheet because it can be seen as an extension of the Brownian sheet. We refer to Herbin[7] for a survey on all these processes. Moreover, in Herbin[7] some multiparameter extensions of multifractional Brownian motion are well studied.

The frame of set-indexed processes is not only a new step of generalization of multiparameter processes, but it provides a real tool in modeling (e.g. Ref. 13). Then a set-indexed extension of fractional Brownian motion is hoped to provide a powerful model for multidimensional phenomena. The set-indexed fractional Brownian motion introduced here is a simple extension of set-indexed Brownian motion, also called white noise (see Refs. 2, 6, 16 and 20 for studies and applications), and possesses the main properties required for fractional Brownian motion. Moreover, by choosing the class of sets parametrizing the process, we get great flexibility and possibilities to obtain particular types of fractional Brownian motion.

In this paper, we prove that our definition of set-indexed fractional Brownian motion satisfies both self-similarity and a condition of stationarity. Moreover such a process is a time-changed classical fractional Brownian motion on each increasing path (flow).

Conversely, we compute the covariance function of any self-similar and stationary set-indexed process. For Gaussian processes, we get an extension of our set-indexed fractional Brownian motion.

This paper is divided as follow:
In the next section, we present the general framework needed for set-indexed processes. We define the concept of set-indexed fractional Brownian motion (sifBm). We prove existence of this process showing that its covariance function is positive definite. Moreover, we compare our definition to previous definitions given in some particular cases and our definition seems to be quite natural and satisfactory. The two fractal properties which are stationarity and self-similarity are studied in section

3. As it will be see, stationarity of increments can be defined in different non equivalent ways. In section 4, we discuss the possibility of finding a characterization of set-indexed fractional Brownian motion by the two main properties: stationarity and self-similarity. Moreover, we show that there no "really nice" set-indexed fractional Brownian motion other than set-indexed Brownian motion. More precisely, theorem 4.4 states that the only set-indexed Gaussian process satisfying $E\left[(\Delta X_C)^2\right] = m(C)^{2H} (\forall C \in \mathcal{C})$, where m is the Lebesgue measure, ΔX the increment process, and $H \in (0,1)$, is the white noise ($H = 1/2$). Section 5 deals with the problem of continuity. In fact, we show that sifBm is continuous when set-indexed Brownian motion is also continuous. Finally in the last section, we study sifBm on increasing paths.

2. FRAMEWORK AND DEFINITION

2.1. Indexing Collection, Set-indexed Processes

Let $\mathcal{T}$ be a locally compact complete separable metric and measure space with metric d and measure m. All processes will be indexed by a class $\mathcal{A}$ of compact connected subsets of $\mathcal{T}$.

In what follows, for any class of sets $\mathcal{D}$, the class of finite unions of sets from $\mathcal{D}$ will be denoted by $\mathcal{D}(u)$. In the terminology of (Hu and Øksendal[(10)]), we assume that $\mathcal{A}$ is an *indexing collection:*

Definition 2.1. A nonempty class $\mathcal{A}$ of compact, connected subsets of $\mathcal{T}$ is called an indexing collection if it satisfies the following:

1. $\emptyset \in \mathcal{A}$, and $A^\circ \neq A$ if $A \notin \{\emptyset, \mathcal{T}\}$. In addition, there exists an increasing sequence $(B_n)_{n\in N}$ of sets in $\mathcal{A}(u)$ such that $\mathcal{T} = \cup_{n\in N} B_n^\circ$.
2. $\mathcal{A}$ is closed under arbitrary intersections and if $A, B \in \mathcal{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathcal{A}$ and if there exists $n \in N$ s.t. for all i, $A_i \subseteq B_n$ then $\overline{\cup_i A_i} \in \mathcal{A}$.
3. The σ-algebra generated by $\mathcal{A}, \sigma(\mathcal{A}) = \mathcal{B}$, the collection of all Borel sets of $\mathcal{T}$.
4. Separability from above
   There exists an increasing sequence of finite subclasses $\mathcal{A}_n = \{A_1^n, \ldots, A_{k_n}^n\}$ of $\mathcal{A}$ closed under intersections and satisfying $\emptyset, B_n \in \mathcal{A}_n(u)$ and a sequence of functions $g_n: \mathcal{A} \to \mathcal{A}_n(u) \cup \{\mathcal{T}\}$ such that

   (a) g_n preserves arbitrary intersections and finite unions (i.e. $g_n(\cap_{A\in\mathcal{A}'} A) = \cap_{A\in\mathcal{A}'} g_n(A)$ for any $\mathcal{A}' \subseteq \mathcal{A}$, and if $\cup_{i=1}^k A_i = \cup_{j=1}^m A'_j$, then $\cup_{i=1}^k g_n(A_i) = \cup_{j=1}^m g_n(A'_j)$),

(b) for each $A \in \mathcal{A}$, $A \subseteq (g_n(A))^\circ$,
(c) $g_n(A) \subseteq g_m(A)$ if $n \geqslant m$,
(d) for each $A \in \mathcal{A}$, $A = \cap_n g_n(A)$,
(e) if $A, A' \in \mathcal{A}$ then for every n, $g_n(A) \cap A' \in \mathcal{A}$, and if $A' \in \mathcal{A}_n$ then $g_n(A) \cap A' \in \mathcal{A}_n$.
(f) $g_n(\emptyset) = \emptyset$ for all n.

5. Every countable intersection of sets in $\mathcal{A}(u)$ may be expressed as the closure of a countable union of sets in $\mathcal{A}$.

(Note: '$\subset$' indicates strict inclusion and '$\overline{(\cdot)}$' and '$(\cdot)^\circ$' denote respectively the closure and the interior of a set.)

We shall define the semi-algebra $\mathcal{C}$ to be the class of all subsets of $\mathcal{T}$ of the form

$$C = A \backslash B,\ A \in \mathcal{A},\ B \in \mathcal{A}(u).$$

$\mathcal{C}$ is closed under intersections and any set in $\mathcal{C}(u)$ may be expressed as a finite disjoint union of sets in $\mathcal{C}$. Note that if $B = \cup_{i=1}^k A_i \in \mathcal{A}(u)$, without loss of generality we can require that for each i, $A_i \nsubseteq \cup_{j \neq i} A_j$. Such a representation of $B \in \mathcal{A}(u)$ will be called *extremal*. If $C = A \backslash B$, $A \in \mathcal{A}$, $B \in \mathcal{A}(u)$, then the representation of C is called extremal if that of B is. Unless otherwise stated, it will always be assumed that all representations of sets in $\mathcal{A}(u)$ and C are extremal.

Numerous examples of topological spaces $\mathcal{T}$ and indexing collections $\mathcal{A}$ satisfying the preceding assumptions can be found in Ivanoff and Merzbach.[12] In particular, our framework generalizes the usual multiparameter setting: if $\mathcal{T} = \mathbf{R}_+^N$, then the class $\mathcal{A} = \{[0,t] : t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$ satisfies all the assumptions. More generally, we can allow $\mathcal{A}$ to consist of all the *lower layers* of $\mathbf{R}_+^N$: a set A is a lower layer if $[0,t] \subseteq A, \forall t \in A$.

Now, let $(\Omega, \mathcal{F}, P)$ be any complete probability space. A filtration (indexed by $\mathcal{A}$) is a class $\{\mathcal{F}_A : A \in \mathcal{A}\}$ of complete sub-σ-fields of $\mathcal{F}$ which satisfies the following conditions:

- $\forall A, B \in \mathcal{A}, \mathcal{F}_A \subseteq \mathcal{F}_B$, if $A \subseteq B$.
- Monotone outer-continuity: $\mathcal{F}_{\cap A_i} = \cap \mathcal{F}_{A_i}$ for any decreasing sequence (A_i) in $\mathcal{A}$.

Definition 2.2. A ($\mathcal{A}$-indexed) stochastic process $X = \{X_A : A \in \mathcal{A}\}$ is a collection of random variables indexed by $\mathcal{A}$ with $X_\emptyset = 0$, and is said to be adapted if X_A is $\mathcal{F}_A$-measurable, for every $A \in \mathcal{A}$. X is said to be integrable (square integrable) if $E[|X_A|] < \infty (E[(X_A)^2] < \infty)$ for every $A \in \mathcal{A}$.

Remark 2.3. Any multiparameter process $\tilde{X}$ can be considered as a set-indexed process X, setting $\mathcal{T}=\mathbf{R}_+^N, \mathcal{A}=\{[0,t]; t\in\mathbf{R}_+^N\}$ and $\tilde{X}_t=X_{[0,t]}$.

Definition 2.4. A ($\mathcal{A}$-indexed) stochastic process X is additive if it has an (almost sure) additive extension to $\mathcal{C}(u)$: i.e., $X_\emptyset = 0$ and if $C, C_1, C_2, \in\mathcal{C}(u)$ with $C=C_1\cup C_2$ and $C_1\cap C_2=\emptyset$, then almost surely

$$X_{C_1}+X_{C_2}=X_C.$$

In addition to assuming that $X_\emptyset=0$, to avoid technicalities we will generally assume as well that $X_{\emptyset'}=0$, where $\emptyset':=\cap_{A\in\mathcal{A}}A$.

Definition 2.5. An additive process X is increasing if $X_C\geqslant 0\ \forall C\in\mathcal{C}$ and if for any decreasing sequence (A_n) in $\mathcal{A}$, $X(\cap_n A_n)=\lim_n X(A_n)$.

It is observed in Corollary 1.4.11 of Ivanoff and Merzbach[12] that an increasing process in fact defines a measure on $\mathcal{B}$ for each $\omega\in\Omega$.

Definition 2.6. Let Λ be a non-negative increasing function defined on $\mathcal{A}$ with $\Lambda_\emptyset=0$. We say that an $\mathcal{A}$-indexed additive process X is a Brownian motion with variance measure Λ if $X_\emptyset=0$, and if for disjoint sets $C_1,\dots,C_n\in\mathcal{C}, X_{C_1},\dots,X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\Lambda C_1,\dots,\Lambda C_n$, respectively.

2.2. Increments of a Set-indexed Process

The notion of increments for a set-indexed process $X=\{X_U; U\in\mathcal{A}\}$ is not as simple as in the case of real indices, where it is only the difference between values of the process.

In the case of multiparameter processes, we use to define the increment between $s,t\in\mathbf{R}_+^N$ by

$$\Delta X_{s,t}=\sum_{r\in\{0,1\}^N}(-1)^{N-\sum_i r_i}X_{[s_i+r_i(t_i-s_i)]_i} \tag{1}$$

which is different from the simple difference X_t-X_s (see Ref. 7).

In the case of set-indexed processes, the increments are defined from the collection of subsets $\mathcal{C}$.

For all $C=U\setminus\bigcup_{1\leqslant i\leqslant n}U_i$, we define the increment of the process X on C by

$$\Delta X_C = X_U - \sum_{i=1}^{n} X_U \cap U_i + \sum_{i<j} X_{U\cap(U_i\cap U_j)} - \cdots + (-1)^n X_{U\cap(\cap_{1\leqslant i\leqslant n} U_i)} \quad (2)$$

According to remark 2.3, this expression, applied to the multiparameter case, gives the definition (1) of the increments.
In the following, it would be important to consider the particular increments $\mathcal{C}_0 = \{\mathcal{C} = U\setminus V; U, V \in \mathcal{A}\}$. Moreover the definition of the increment process ΔX can be extended to $\mathcal{C}(u)$, the finite unions of elements of $\mathcal{C}$. Particularly, for all $U, V \in \mathcal{A}$, $\Delta X_{U\triangle V}$ is well-defined.

Remark 2.7. The process ΔX could be seen as an extension of the process X for the set of indices $\mathcal{C}$. For all $A \in \mathcal{A} \subset \mathcal{C}$, we have $\Delta X_A = X_A$ (because $A = \mathcal{A} - \emptyset$).

In the case of an additive process, the definition of the increment ΔX_C coincides with the additive extension of X_C to $C \in \mathcal{C}$. However, if X is not additive, which is the case of a direct definition of the process for the set of indices $\mathcal{C}$, in general $\Delta X_C \neq X_C$ for $C \in \mathcal{C}$. For this reason, we use a different notation for the increments of X.

Remark 2.8. If $X = \{X_U; U \in \mathcal{A}\}$ is Gaussian, then $\Delta X = \{\Delta X_C; C \in \mathcal{C}\}$ is clearly Gaussian.

2.3. Definition of sifBm

Recall that the fractional Brownian motion B^H is defined to be a mean-zero Gaussian process such that

$$\forall s, t \in \mathbf{R}_+; \quad E\left[(B_t^H - B_s^H)^2\right] = |t-s|^{2H}$$

The natural set-indexed extension of this process is to substitute the term $|t-s|^{2H}$ with $d(U, V)^{2H}$, where d is some distance between two subsets of $\mathcal{T}$. In this paper, we consider the choice of $d(U, V) = m(U\triangle V)$, where $\triangle$ is the symmetric difference between two sets and m is a measure on $\mathcal{T}$.

Lemma 2.9. Let m be finite measure on $\mathcal{T}$. For all $\alpha \in (0, 1]$, the function

$$(U, V) \mapsto m(U)^\alpha + m(V)^\alpha - m(U\triangle V)^\alpha$$

is positive definite.

Proof. For all measurable subset U of $\mathcal{T}$ such that $m(U) < +\infty$, we define the elementary function $f = \mathbb{1}_U$. Finite linear combinations of elementary functions are called simple functions. It is well-known that simple functions are dense in $L^2(m)$.

Moreover, for all $U, V \subseteq \mathcal{T}$, we have

$$\begin{aligned}\mathbb{1}_{U\triangle V} &= \mathbb{1}_{(U\setminus V)\cup(V\setminus U)}\\ &= \mathbb{1}_U(1-\mathbb{1}_V)+\mathbb{1}_V(1-\mathbb{1}_U)\\ &= \mathbb{1}_U+\mathbb{1}_V-2\mathbb{1}_U\cdot\mathbb{1}_V\\ &= (\mathbb{1}_U-\mathbb{1}_V)^2\\ &= |\mathbb{1}_U-\mathbb{1}_V|\end{aligned}$$

Then we only have to show that the function

$$\begin{aligned}L^2(m)\times L^2(m) &\to \mathbf{R}\\ (f,g) &\mapsto m(f^2)^\alpha+m(g^2)^\alpha-m(|f-g|^2)^\alpha\end{aligned}$$

is positive definite.

Let $f_1, f_2, \ldots, f_n \in L^2(m)$ and $u_1, u_2, \ldots, u_n \in \mathbf{R}$. We have to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m(f_i^2)^\alpha+m(f_j^2)^\alpha-m(|f_i-f_j|^2)^\alpha\right\}u_iu_j \geqslant 0 \tag{3}$$

Setting $u_0 = -\sum_{i=1}^n u_i$ and $f_0 = \mathbb{1}_\emptyset$, we can write

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m(f_i^2)^\alpha+m(f_j^2)^\alpha-m(|f_i-f_j|^2)^\alpha\right\}u_iu_j\\ &\quad= -\sum_{i=0}^{n}\sum_{j=0}^{n} m(|f_i-f_i|^2)^\alpha u_iu_j \qquad (4)\end{aligned}$$

But, for all $\lambda > 0$, we have

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\lambda m(|f_i-f_j|^2)^\alpha}u_iu_j &= \sum_{i=0}^{n}\sum_{j=0}^{n}\left(e^{-\lambda m(|f_i-f_j|^2)^\alpha}-1\right)u_iu_j\\ &= -\lambda\sum_{i=0}^{n}\sum_{j=0}^{n} m(|f_i-f_j|^2)^\alpha u_iu_j+o(\lambda) \qquad (5)\end{aligned}$$

Then, (3) is equivalent to

$$\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\lambda m(|f_i-f_j|^2)^\alpha}u_iu_j \geqslant 0 \tag{6}$$

in the neighborhood of $\lambda = 0$.

In $L^2(m)$, let us dene the bilinear form $\langle f, g\rangle = m(fg)$. If we identify the elements of $L^2(m)$ that are almost everywhere equal, then $L^2(m)$ is a complete separable metric space with this scalar product.
Let us show that there exists a random variable X taking its values in $L^2(m)$, such that

$$\forall f \in L^2(m); \quad E[e^{i\langle f, X\rangle}] = e^{-\lambda \|f\|^{2\alpha}} \tag{7}$$

Consider a α-stable real random variable

$$Y \overset{(d)}{=} S_\alpha\left(\left(\cos\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha}}, 1, 0\right)$$

where $0 < \alpha < 1$ (see Ref. 23, prop. 1.2.12, p 15).

As the function $f \mapsto e^{\frac{1}{2}\|f\|^2}$ is positive definite (see remark 2.10), by a theorem of Bochner–Minlos (see Refs. 9 and 15), there exists a random variable G, such that

$$\forall f \in L^2(m); \quad E[e^{i\langle f, G\rangle}] = e^{-\frac{1}{2}\|f^2\|}.$$

Moreover, we can suppose that G is independant from Y.
For all $f \in L^2(m)$, we compute

$$\begin{aligned} E\left[e^{i\langle f, Y^{1/2}G\rangle}\right] &= E\left\{E\left[e^{i\langle f, Y^{1/2}G\rangle}|Y\right]\right\} \\ &= E\left\{E\left[e^{iY^{1/2}\langle f, G\rangle}|Y\right]\right\} \\ &= E\left[e^{-\frac{Y}{2}\|f\|^2}\right] \end{aligned}$$

As $E[e^{-\gamma Y}] = e^{-\gamma^\alpha}$ for all $\gamma > 0$, we get

$$E\left[e^{i\langle f, Y^{1/2}G\rangle}\right] = e^{-2^{-\alpha}\|f\|^{2\alpha}}$$

Then $X = \begin{cases} \sqrt{2\lambda^{1/\alpha}}Y^{1/2}G & \text{if} \quad \alpha \neq 1 \\ \sqrt{2\lambda}G & \text{if} \quad \alpha = 1 \end{cases}$ satisfies (7), and $f \mapsto e^{-\lambda\|f\|^{2\alpha}}$ is non negative definite. That proves (6) and the result follows. □

Remark 2.10. For all family $(f_1, \ldots, f_n)$ of $L^2(m)$ provided with the scalar product previously defined, there exist $p \leqslant n$ and a family $(x_1, \ldots, x_n)$ of $\mathbf{R}^p$ such that

$$\forall_i, j; \quad \|x_i - x_j\|_{\mathbf{R}^p} = \|f_i - f_j\|_{L^2(m)} \tag{8}$$

To show this result, let us consider an orthonormal basis $(e_1, \dots, e_p)$ of $Vect(f_i)_{1\leqslant i\leqslant n}$, and the canonical basis $(\epsilon_1, \dots, \epsilon_p)$ of $\mathbf{R}^p$. For all i, there exists a family $(\lambda_1^i, \dots, \lambda_p^i)$ s.t. $f_i = \sum_k \lambda_i^k \cdot e_k$. Then the vectors $(x_1, \dots, x_n)$ defined by $x_i = \sum_k \lambda_i^k \cdot \epsilon_k$ satisfy (8). □

This remark allows to show directly (3) using lemma 2.10.8 in Ref. 23, for $2\alpha \in (0, 2]$ i.e. $\alpha \in (0, 1]$.

Since the existence of a mean-zero Gaussian process is equivalent to the positive definite property of its covariance function, we can define

Definition 2.11. A mean-zero Gaussian process $\mathbf{B}^H = \{\mathbf{B}_U^H; U \in \mathcal{A}\}$ satisfying

$$E\left[\mathbf{B}_U^H \mathbf{B}_V^H\right] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}\right] \tag{9}$$

where $H \in (0, 1/2]$, is called a set-indexed fractional Brownian motion (sifBm). H is the index of self-similarity of the process.

Lemma 2.9 only shows that the right side of equation (9) is positive definite for $\alpha \in (0, 1]$, which restricts the definition of the sifBm for $H \in (0, 1/2]$. Even in the simple case of $\mathcal{A} = \{[0, t]; t \in \mathbf{R}2_+\} \cup \{\emptyset\}$, some examples can be found where $H > 1/2$ leads to a non positive definite expression (9).

Remark 2.12. If $H = \frac{1}{2}$, the process $\mathbf{B}^{\frac{1}{2}}$ is the well known set-indexed Brownian motion. Indeed, let us compute the covariance function of this process

$$E\left[\mathbf{B}_U^{\frac{1}{2}} \mathbf{B}_V^{\frac{1}{2}}\right] = \frac{1}{2}\left[m(U) + m(V) - m(U \triangle V)\right]$$

As

$$\begin{aligned} m(U \triangle V) &= m(U \backslash V) + m(V \backslash U) \\ &= m(U \backslash U \cap V) + m(V \backslash U \cap V) \\ &= m(U) + m(V) - 2m(U \cap V) \end{aligned}$$

we have

$$E\left[\mathbf{B}_U^{\frac{1}{2}} \mathbf{B}_V^{\frac{1}{2}}\right] = m(U \cap V)$$

which is the covariance function of the set-indexed Brownian motion.

Remark 2.13. In the case of $\mathcal{T}=\mathbf{R}_+$ and $\mathcal{A}=\{[0,t];\, t\in\mathbf{R}_+\}\cup\{\emptyset\}$, the process $\mathbf{B}^H$ is the classical fBm. Indeed, the covariance function is

$$E\left[\mathbf{B}^H_{[0,s]}\mathbf{B}^H_{[0,t]}\right]=\frac{1}{2}\left[s^{2H}+t^{2H}-|t-s|^{2H}\right]$$

which is the covariance function of the fractional Brownian motion.

Remark 2.14. In the case of $\mathcal{T}=\mathbf{R}^N_+$ and $\mathcal{A}$ is the set of rectangles of the form $[0,t]$, the process $\mathbf{B}^H$ can be seen as a multiparameter process. Then it is interesting to compare it with the other known multiparameter fractional Brownian motions (see Ref. 7).

- the Lévy fractional Brownian motion is a mean-zero Gaussian process $\mathscr{B}^H=\{\mathscr{B}^H_t;\, t\in\mathbf{R}^N_+\}$ such that

$$\forall_s,t\in\mathbf{R}^N_+;\; E\left[\mathscr{B}^H_s\mathscr{B}^H_t\right]=\frac{1}{2}\left[\|s\|^{2H}+\|t\|^{2H}-\|t-s\|^{2H}\right]$$

  where $H\in(0,1)$.
- the fractional Brownian sheet is a mean-zero Gaussian process $\mathbb{B}^H=\{\mathbb{B}^H_t;\, t\in\mathbf{R}^N_+\}$ such that

$$\forall_s,t\in\mathbf{R}^N_+;\quad E\left[\mathbb{B}^H_s\mathbb{B}^H_t\right]=\frac{1}{2}\prod_{i=1}^{N}\left[s_i^{2H_i}+t_i^{2H_i}-|t_i-s_i|^{2H_i}\right]$$

  where $H=(H_1,\ldots,H_N)\in(0,1)^N$.

As for all s, $t\in\mathbf{R}^N_+$,

$$E\left[\mathbf{B}^H_{[0,s]}\mathbf{B}^H_{[0,t]}\right]=\frac{1}{2}\left[\prod_{i=1}^N s_i^{2H}+\prod_{i=1}^N t_i^{2H}-\left(\prod_{i=1}^N s_i+\prod_{i=1}^N t_i-2\prod_{i=1}^N s_i\wedge t_i\right)^{2H}\right]$$

we can see that, if m is the Lebesgue measure and $N>1$, $\mathbf{B}^H$ is different from the two processes $\mathscr{B}^H$ and $\mathbb{B}^H$.

This fact will be also shown in the next sections in the study of properties of sifbm, and its restriction on flows. It is therefore natural to wonder if the Lévy fBm and the fBs can have set-indexed extension. The answer seems to be negative.

Actually the definition of the sheet is strongly associated with the Euclidean structure of $\mathbf{R}^N$. Therefore it is incompatible with a set-indexed viewpoint. Moreover the Lévy fBm can be seen as a simple one parameter process where the increment between two points only depends from distance between them.

3. FRACTAL PROPERTIES

The fractional Brownian motion B^H has two important properties which make it the most natural fractal process:

- its increments are stationary

$$\forall h \in \mathbf{R}_+; \quad \left(B^H_{t+h} - B^H_h\right)_{t\in\mathbf{R}_+} \stackrel{(d)}{=} \left(B^H_t - B^H_0\right)_{t\in\mathbf{R}_+}$$

- it is self-similar

$$\forall a \in \mathbf{R}_+; \quad \left(B^H_{at}\right)_{t\in\mathbf{R}_+} \stackrel{(d)}{=} \left(a^H B^H_t\right)_{t\in\mathbf{R}_+}$$

Moreover, the fBm is the only Gaussian process which has these two properties.

In this section, we show that in some sense these properties still hold for the set-indexed fractional Brownian motion. Moreover they characterize the covariance function of the process between two sets U and V such that $U \subseteq V$.

3.1. Stationarity of the Increments

Stationarity of increments of a set-indexed process can be defined in various ways. The set-indexed Brownian motion satisfies all of them, but the different extensions of *fractional* Brownian motion do not.

In the case of $\mathcal{T} = \mathbf{R}^N_+$ and $\mathcal{A}$ is the collection of rectangles, the classical definition of stationarity of increments can be studied.

A process $X = \left\{X_{[0,t]}; t \in \mathbf{R}^N_+\right\}$ is said to have *stationary increments against translations* if for all $\tau \in \mathbf{R}^N_+$, the two processes $\left\{\Delta X_{[\tau,t+\tau]}; t \in \mathbf{R}^N_+\right\}$ and $\left\{\Delta X_{[0,t]}; t \in \mathbf{R}^N_+\right\}$ have the same law.

Both Lévy fractional Brownian motion and fractional Brownian sheet satisfy this property of stationarity (see Ref. 7).

Remark 3.1. This definition is weaker than stationarity of increments against isometries of $\mathbf{R}^N_+$, i.e. for all $g \in \mathcal{G}(\mathbf{R}^N)$,

$$\left\{\Delta X_{g([0,t])}; \quad t \in \mathbf{R}^N_+\right\} \stackrel{(d)}{=} \left\{\Delta X_{[0,t]}; \quad t \in \mathbf{R}^N_+\right\}$$

where $\mathcal{G}(\mathbf{R}^N)$ is the group of isometries of $\mathbf{R}^N$.

On the contrary to stationarity against translations, the context of set-indexed processes imposes additional assumptions to the strict context of multiparameter processes. Actually as the image of $C \in \mathcal{C}$ by any isometry of $\mathbf{R}^N$ does not necessarily belong to $\mathcal{C}$, a stability assumption is needed for the definition to make sense.

However in the strict context of multiparameter processes, this assumption does not need to be considered. The Lévy fractional Brownian motion satisfy this property of increment stationarity in the strong sense (see Ref. 23).

However in general, there is no reason that the sifBm possesses the stationarity increments property against translations. In some particular cases, we can show directly using the next lemma that this property is not satisfied.

Lemma 3.2. Let $\mathbf{B}^H = \{\mathbf{B}^H_U;\quad U \in \mathcal{A}\}$ be a sifBm of index $H \in (0, 1/2]$. For all $C = U \setminus \left(U_{1\leqslant i\leqslant n} U_i\right) \in \mathcal{C}$, where $\forall_i \in \{1, \ldots, n\}; U_i \subset U$, we have

$$E\left[\left(\triangle \mathbf{B}^H_C\right)^2\right] = -\sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} m\left(U \triangle \left[\bigcap_{p\in\{i_1,\ldots,i_k)} U_p\right]\right)^{2H}$$
$$+\frac{1}{2}\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_l}} m\left(\left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right] \triangle \left[\bigcap_{p\in\{j_i,\ldots,j_l\}} U_p\right]\right)^{2H} \quad (10)$$

Proof. By definition,

$$\triangle \mathbf{B}^H_C = \mathbf{B}^H_U + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbf{B}^H_{\cap p\in\{i_1,\ldots,i_k\}U_p}$$

Then,

$$E\left[\left(\triangle \mathbf{B}^H_C\right)^2\right] = E\left[\left(\mathbf{B}^H_U\right)^2\right] + 2\sum_{k}(-1)^k \sum_{i_1<\cdots<i_k} E\left[\mathbf{B}^H_U \cdot \mathbf{B}^H_{\cap p\in\{i_1,\ldots,i_k\}U_p}\right]$$
$$+\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_l}} E\left[\mathbf{B}^H_{\cap p\in\{i_1,\ldots,i_k\}U_p} \cdot \mathbf{B}^H_{\cap p\in\{j_1,\ldots,j_k\}U_p}\right]$$

and, using the covariance function of $\mathbf{B}^H$

$$E\left[\left(\triangle \mathbf{B}^H_C\right)^2\right] = m(U)^{2H}$$
$$+\sum_{k}(-1)^k \sum_{i_1,<\cdots,<i_k} \left\{ m(U)^{2H} + m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} \right.$$
$$\left. -m\left(U \triangle \left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right]\right)^{2H} \right\}$$

$$+\frac{1}{2}\sum_{k,l}(-1)^{k+l}\sum_{\substack{i_1,<\cdots,<i_k\\ j_1,<\cdots,<j_l}}\left\{m\left(\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right)^{2H}+m\left(\bigcap_{p\in\{j_1,\dots,j_l\}}U_p\right)^{2H}\right.$$
$$\left.-m\left(\left[\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right]\triangle\left[\bigcap_{p\in\{j_1,\dots,j_l\}}U_p\right]\right)^{2H}\right\} \tag{11}$$

Let us consider the two following terms in expression (11):

- term in $m(U)^{2H}$

$$m(U)^{2H}+\sum_k(-1)^k\sum_{i_1<\cdots<i_k}m(U)^{2H}=\sum_{k=0}^{n}C_n^k m(U)^{2H}=0$$

- term in $m\left(\cap_{p\in\{i_1,\dots,i_k\}U_p}\right)^{2H}$

$$\underbrace{\sum_k(-1)^k\sum_{i_1<\cdots<i_k}m\left(\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right)^{2H} +\sum_{k,l}(-1)^{k+l}\sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_l}}m\left(\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right)^{2H}}_{\sum_k(-1)^k\sum_{i_1<\cdots<i_k}m\left(\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right)^{2H}\sum_{l=1}^{n}(-1)^l C_n^l}$$

  therefore this term is equal to

$$\sum_k(-1)^k\sum_{i_1<\cdots<i_k}m\left(\bigcap_{p\in\{i_1,\dots,i_k\}}U_p\right)^{2H}\sum_{l=0}^{n}(-1)^l C_n^l=0$$

The two other terms of expression (11) give the result. □

The main idea to define a set-indexed extension of the fractional Brownian motion, was to extend

$$\forall s,t\in\mathbf{R}_+;\quad E\left[\left(B_t^H-B_s^H\right)^2\right]=|t-s|^{2H}$$

in

$$\forall U,V\in\mathcal{A};\quad E\left[(X_U-X_V)^2\right]=m(U\triangle V)^{2H}$$

However, it should be more interesting to get

$$\forall C \in \mathcal{C}; \quad E\left[(\triangle X_C)^2\right] = m(C)^{2H} \tag{12}$$

According to lemma 3.2, the set-indexed fractional Brownian motion $\mathbf{B}^H$ satisfies

$$\forall U, V \in \mathcal{A}; \quad E\left[\left(\triangle \mathbf{B}^H_{U \setminus V}\right)^2\right] = m(U \setminus V)^{2H}$$

but the property (12) does not hold.

Moreover, we will see that there is no set-indexed process satisfying (12) for $H \neq \frac{1}{2}$ (theorem 4.4).

Proposition 3.3. Let $\mathbb{B}^H = \{\mathbb{B}^H_t; t \in \mathbf{R}^N_+\}$ be a fractional Brownian sheet of constant parameter H in every axis.

For all $a, b \in \mathbf{R}^H_+$ such that $a \prec b$ (i.e. $\forall_i = 1, \ldots N;\ a_i < b_i$), we have

$$E\left[\left(\Delta(\mathbb{B}^H)_{[a,b]}\right)^2\right] = m([a,b])^{2H}$$

and consequently, For all $a, b, a', b' \in \mathbf{R}^N_+$, such that $m([a,b]) = m([a',b'])$, we have

$$\Delta(\mathbb{B}^H)_{[a,b]} \stackrel{(d)}{=} \Delta(\mathbb{B}^H)_{[a',b']}$$

Proof. For all $a, b \in \mathbf{R}^N_+$ with $a \prec b$, as X has stationary increments against translations, we have

$$\begin{aligned}
E\left[\left(\Delta(\mathbb{B}^H)_{[a,b]}\right)^2\right] &= E\left[\left(\Delta(\mathbb{B}^H)_{[0,b-a]}\right)^2\right] \\
&= E\left[\left(\mathbb{B}^H_{[b-a]}\right)^2\right] \\
&= \prod_{i=1}^{N} |b_i - a_i|^{2H} \\
&= m([a,b])^{2H}
\end{aligned}$$

As $\Delta(\mathbb{B}^H)$ is a mean-zero Gaussian process, the result follows. □

Remark 3.4. • If a fractional Brownian sheet of index $H = (H_1, \ldots, H_N)$ satisfies the property of proposition 3.3, we have $H_1 = \cdots = H_N$.

- The Lévy fractional Brownian motion does not satisfy this property.

Definition 3.5. A set-indexed process X is said to be $\mathcal{C}_0$-increment stationary if for all $C, C' \in \mathcal{C}_0$ such that $m(C)=m(C')$, we have $\Delta X_C \stackrel{(d)}{=} \Delta X_{C'}$.

In the case of one-parameter fractional Brownian motion, the increment stationarity property gives an equality between laws of some increment processes. However, in the set-indexed case, definition 3.5 does not tell anything about correlation between increments.

In section 4, we see that a stronger property is not worth to be considered.

Proposition 3.6. The set-indexed fractional Brownian motion $\mathbf{B}^H$ is $\mathcal{C}_0$-increment stationary.

Proof. For all $C \in \mathcal{C}_0$, there exist $U, V \in \mathcal{A}$ where $V \subset U$, such that $C = U \setminus V$. Then $\Delta \mathbf{B}^H_C = \mathbf{B}^H_U - \mathbf{B}^H_{U \cap V} = \mathbf{B}^H_U - \mathbf{B}^H_V$. We compute

$$\begin{aligned} E\left[\left(\Delta \mathbf{B}^H_C\right)^2\right] &= E\left[\left(\mathbf{B}^H_U - \mathbf{B}^H_V\right)^2\right] \\ &= m(U \triangle V)^{2H} \\ &= m(C)^{2H} \end{aligned}$$

Thus, as $\Delta \mathbf{B}^H$ is a Gaussian process, for all $C, C' \in \mathcal{C}_0$ such that $m(C) = m(C')$, we have $\Delta \mathbf{B}^H_C \stackrel{(d)}{=} \Delta \mathbf{B}^H_{C'}$. □

Remark 3.7. In the proof of proposition 3.6, we saw that

$$\forall C \in \mathcal{C}_0; \quad E\left[\left(\Delta \mathbf{B}^H_C\right)^2\right] = m(C)^{2H}$$

However, in general

$$E[\Delta \mathbf{B}^H_C \cdot \Delta \mathbf{B}^H_{C'}] \neq \frac{1}{2}\left[m(C)^{2H} + m(C')^{2H} - m(C \triangle C')^{2H}\right]$$

In fact, it can be shown that for $C = U \setminus V$ where $V \subset U$ and $C' = U' \setminus V'$ where $V' \subset U'$, and $U, V, U', V' \in \mathcal{A}$

$$\begin{aligned} E\left[\Delta \mathbf{B}^H_C \cdot \Delta \mathbf{B}^H_{C'}\right] = \frac{1}{2}\Big[& m(U \triangle V')^{2H} + m(V \triangle U')^{2H} \\ & - m(U \triangle U')^{2H} - m(V \triangle V')^{2H}\Big] \end{aligned}$$

3.2. Self-similarity

To study a set-indexed version of the notion of self-similarity for a set-indexed process, we need some assumptions about the set $\mathcal{A}$.

We suppose that $\mathcal{A}$ is provided with the operation of a non trivial group G that can be extended satifying

$$\forall U, V \in \mathcal{A}, \forall g \in G; \quad g \cdot (U \cup V) = g \cdot U \cup g \cdot V \qquad (13)$$
$$g \cdot (U \backslash V) = g \cdot U \backslash g \cdot V$$

and assume there exists a non constant function $\mu : G \to \mathbf{R}_+^*$

$$\forall U \in \mathcal{A}, \forall g \in G; \quad m(g \cdot U) = \mu(g) \cdot m(U) \qquad (14)$$

Remark 3.8. We can see easily that μ is an group-homomorphism from G into the multiplicative group $\mathbf{R}_+ \backslash \{0\}$.

Example 3.9. In the case of $\mathcal{T} = \mathbf{R}_+^N$ and $\mathcal{A} = \{[0, t]; t \in \mathbf{R}_+^N\} \cup \{\emptyset\}$ we can consider the multiplication by elements of $\mathbf{R}_+$

$$\forall g \in \mathbf{R}_+, \forall t \in \mathbf{R}_+^N; \quad g \cdot [0, t] = [0, g \cdot t]$$

Moreover,

$$\forall g \in \mathbf{R}_+, \forall t \in \mathbf{R}_+^N; \quad m\big(g \cdot [0, t]\big) = g^N m\big([0, t]\big)$$

The following result will be useful in the next section.

Lemma 3.10. Under the assumptions about the group G, the cardinal of G is not finite.

Proof. As the function μ is not constant, there exists $g \in G$ such that $\mu(g) > 1$ (take $\tilde{g}$ s.t. $\mu(g) \neq 1$ and then $g = \tilde{g}$ or $g = \tilde{g}^{-1}$).

For all integer n, we have $\mu(g^n) = [\mu(g)]^n$. If G is finite, the set $\{g^n; n \in \mathbf{N}\}$ is finite, which is in conflict with $\lim_{n\to\infty} \mu(g^n) = \infty$. $\square$

Definition 3.11. A set-indexed process $X = \{X_U; U \in \mathcal{A}\}$ is said to be selfsimilar of index H, if there exists a group G which operates on $\mathcal{A}$, and satisfies (13) and (14), such that for all $g \in G$,

$$\{X_{g \cdot U}; U \in \mathcal{A}\} \overset{(d)}{=} \{\mu(g)^H \cdot X_U; U \in \mathcal{A}\} \qquad (15)$$

Proposition 3.12. Assuming the existence of a group G which operates on $\mathcal{A}$, and satisfies (13) and (14), the set-indexed fractional Brownian motion $\mathbf{B}^H$ is self-similar of index H.

Proof. Let g be an element of the group G. For all $U, V \in \mathcal{A}$, we have

$$E[\mathbf{B}^H_{g \cdot U}\mathbf{B}^H_{g \cdot V}] = \frac{1}{2}\left[m(g \cdot U)^{2H} + m(g \cdot V)^{2H} - m(g \cdot U \triangle g \cdot V)^{2H}\right]$$

As $g \cdot (U \triangle V) = g \cdot U \triangle g \cdot V$, we get

$$\begin{aligned} E[\mathbf{B}^H_{g \cdot U}\mathbf{B}^H_{g \cdot V}] &= \frac{\mu(g)^{2H}}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \triangle V)^{2H}\right] \\ &= \mu(g)^{2H} E[\mathbf{B}^H_U \mathbf{B}^H_V] \end{aligned}$$

Therefore, the two mean-zero Gaussian processes $\{\mathbf{B}^H_{g \cdot U}; U \in \mathcal{A}\}$ and $\{\mu(g)^H \cdot \mathbf{B}^H_U; U \in \mathcal{A}\}$ have the same law. □

4. PSEUDO-CHARACTERISATION OF sifBm

Recall that fractional Brownian motion is the only mean-zero Gaussian process which is self-similar and has stationary increments. In the same way, the only multiparameter mean-zero Gaussian process which is self-similar and whose increments are stationary in the strong sense (under isometries of $\mathbf{R}^N$), is the Lévy fractional Brownian motion.[23]

In the case of set-indexed processes, there is not such a characterisation. However, the two properties of self-similarity and stationarity of increments characterise the covariance function of the process between any U and V such that $U \subseteq V$.

Proposition 4.1. Suppose the existence of a non trivial group G operating on $\mathcal{A}$, and satisfying (13) and (14). Moreover, assume the function μ is surjective.

Let $X = \{X_U; U \in \mathcal{A}\}$ be a set-indexed process satisfying the two following properties

1. self-similarity of index with respect to G, $H \in (0, 1)$
2. $\mathcal{C}_0$-increment stationarity

Then, the covariance function between two subsets U and V such that $U \subseteq V$ is

$$E[X_U X_V] = K[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}] \tag{16}$$

Proof. Let U_0 be a non m-null fixed element of $\mathcal{A}$. For all $U, V \in \mathcal{A}$ such that $U \subset V$, as μ is surjective, there exists $g \in G$ such that $\mu(g) = \frac{m(V\setminus U)}{m(U_0)}$, i. e. $m(g \cdot U_0) = m(V \setminus U)$. Then using $\mathcal{C}_0$-increment stationarity property, we have

$$\begin{aligned} E\left[(X_V - X_U)^2\right] &= E\left[(\Delta X_{V\setminus U})^2\right] \\ &= E\left[(\Delta X_{g\cdot U_0})^2\right] \end{aligned}$$

As $g \cdot U_0 \in \mathcal{A}$, we have $\Delta X_{g\cdot U_0} = X_{g\cdot U_0}$ and by self-similarity

$$\begin{aligned} E\left[(X_V - X_U)^2\right] &= E\left[(X_{g\cdot U_0})^2\right] \\ &= [\mu(g)]^{2H} E\left[(X_{U_0})^2\right] \\ &= [m(V\setminus U)]^{2H} \frac{E\left[(X_{U_0})^2\right]}{m(U_0)^{2H}} \end{aligned} \tag{17}$$

The result follows from (17). □

Remark 4.2. If $G = (\mathbf{Q}^*, \cdot)$, $\mathcal{A} = \{[0, t]; t \in \mathbf{R}2_+\} \cup \{\emptyset\}$ and m is the Lebesgue measure, we have

$$\forall g \in \mathbf{R}^*, \forall t \in \mathbf{R}2_+; \quad m(q \cdot [o, t]) = q^2 \cdot m([o, t])$$

Then, the function μ is $q \mapsto q^2$, which is not surjective. The previous result does not hold in this case.

Remark 4.3. Proposition 4.1 shows that our set-indexed extension of the fractional Brownian motion is very natural provided the two properties of self-similarity and stationarity of the increments.

However, if there exists a mean-zero Gaussian process with covariance function

$$E[X_U X_V] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(V\setminus U)^{2H} - m(U\setminus V)^{2H}\right] \tag{18}$$

it satisfies proposition 4.1 as well.

To determine completely the covariance function of a self-similar, $\mathcal{C}_0$-increment stationary, set-indexed process, we need assumptions about $E[\Delta X_{U\setminus V} \cdot \Delta X_{V\setminus U}]$, where $U, V \in \mathcal{A}$.

For all $U, V \in \mathcal{A}$, we have $\Delta X_{U\setminus V} = X_U - X_{U\cap V}$ and $\Delta X_{V\setminus U} = X_V - X_{U\cap V}$.

Then,

$$\begin{aligned}E[\Delta X_{U\setminus V}\cdot\Delta X_{V\setminus U}] &= E[X_U\cdot X_V]\\ &\quad -E[X_U\cdot X_{U\cap V}]-E[X_V\cdot X_{U\cap V}]+E[X_{U\cap V}]^2\\ &= E[X_U\cdot X_V]\\ &\quad -\frac{1}{2}[m(U)^{2H}+m(V)^{2H}-m(V\setminus U)^{2H}m(U\setminus V)^{2H}]\end{aligned}$$

Particularly, assuming the independance of $\Delta X_{U\setminus V}$ and $\Delta X_{V\setminus U}$, is equivalent to (18), provided that such a process X exists.

The property of $\mathcal{C}_0$-increment stationarity seems too weak to characterize completely the covariance of a self-similar process. It can be tempting to define a process which would have a stronger kind of increment stationarity. For instance, does it exist a self-similar process which satisfies $E[\Delta X_C]^2=m(C)^{2H}$ for all $C\in\mathcal{C}$? The following important result gives a negative answer to the problem of existence of such processes.

Theorem 4.4. Suppose the existence of a non trivial group G operating on $\mathcal{A}$, and satisfying (13) and (14). Moreover, assume the function μ is surjective and that there exist at least two incomparable sets for the partial order $\subset$.

The only Gaussian set-indexed process $X=\{X_U;U\in\mathcal{A}\}$ such that

$$\forall C\in\mathcal{C};\quad E\left[(\Delta X_C)^2\right]=K\cdot m(C)^{2H} \tag{19}$$

where $K>0$ and $H\in(0,1)$, is the set-indexed Brownian motion.

Proof. Let X be a set-indexed Gaussian process satisfying (19).

First, we can see that X is $\mathcal{C}_0$-increment stationary. Moreover, for all $U\in\mathcal{A}$ and $g\in G$, we have $E\left[(X_g\cdot U)^2\right]=K\cdot\mu(g)^{2H}\cdot m(U)^{2H}=\mu(g)^{2H}\cdot E\left[(X_U)^2\right]$. Then, as X is Gaussian, we conclude that X is self-similar. Proposition 4.1 implies that for all $U,V\in\mathcal{A}$, such that $U\subseteq V$,

$$E[X_U\cdot X_V]=K[m(U)^{2H}+m(V)^{2H}-m(V\setminus U)^{2H} \tag{20}$$

For all U_1 and U_2 in $\mathcal{A}$ such that $U_1\nsubseteq U_2$ and $U_2\nsubseteq U_1$, let us consider $U\in\mathcal{A}$ such that $U_1\subset U$ and $U_2\subset U$. The subset of $\mathcal{T}$, $C=U\setminus(U_1\cup U_2)$ belongs to $\mathcal{C}$ and $\Delta X_C=X_U-X_{U_1}-X_{U_2}+X_{U_1\cap U_2}$. Then we have

$$\begin{aligned}E\left[(\Delta X_C)^2\right] &= E\left[(X_U)^2\right]+E\left[(X_{U_1})^2\right]+E\left[(X_{U_2})^2\right]+E\left[(X_{U_1\cap U_2})^2\right]\\ &\quad -2E\left[X_U\cdot X_{U_1}\right]-2E\left[X_U\cdot X_{U_2}\right]+2E\left[X_U\cdot X_{U_1\cap U_2}\right]\\ &\quad +2E[X_{U_1}\cdot X_{U_2}]-2E[X_{U_1}\cdot X_{U_1\cap U_2}]-2E[X_{U_2}\cdot X_{U_1\cap U_2}]\end{aligned}$$

Hence

$$\begin{aligned}2E[X_{U_1}\cdot X_{U_2}] = {} & E\left[(\Delta X_C)^2\right]-E\left[(X_U)^2\right]-E\left[(X_{U_1})^2\right]-E\left[(X_{U_2})^2\right]\\ & -E\left[X_{U_1\cap U_2})^2\right]+2E[X_U\cdot X_{U_1}]+2E[X_U\cdot X_{U_2}]\\ & -2E[X_U\cdot X_{U_1\cap U_2}]+2E[X_{U_1}\cdot X_{U_1\cap U_2}]+2E[X_{U_2}\cdot X_{U_1\cap U_2}]\end{aligned}$$

Using (20), we get

$$\begin{aligned}2E[X_{U_1}\cdot X_{U_2}] = {} & K\left\{m(U\backslash(U_1\cup U_2))^{2H}-m(U\backslash U_1)^{2H}-m(U\backslash U_2)^{2H}\right.\\ & \left.+m(U\backslash(U_1\cap U_2))^{2H}\right\}+K\left[m(U_1)^{2H}+m(U_2)^{2H}\right.\\ & \left.-m(U_1\backslash U_2)^{2H}-m(U_2\backslash U_1)^{2H}\right] \qquad (21)\end{aligned}$$

Taking $V\in\mathcal{A}$ such that $U\subsetneq V$, we get an expression of 2. $E[X_{U_1}\cdot X_{U_2}]$ different from (21) if $H\neq\frac{1}{2}$. □

Corollary 4.5. Suppose the existence of a non trivial group G operating on $\mathcal{A}$, and satisfying (13) and (14). Moreover, assume the function μ is surjective and that there exist at least two incomparable sets for the partial order $\subset$.

There exists no set-indexed process which is H-self-similar (for $H\neq\frac{1}{2}$) and whose increments satisfy one of the following

1. $\mathcal{C}$-increment stationarity

$$\forall C, C'\in\mathcal{C};\quad m(C)=m(C')\Rightarrow\Delta X_C\overset{(d)}{=}\Delta X_{C'}$$

2. for all function f:$\mathcal{C}\to\mathcal{C}$, such that $\forall C\in\mathcal{C}; m(f(C))=m(C)$

$$\{\Delta X_{f(C)}; C\in\mathcal{C}\overset{(d)}{=}\{\Delta X_C; C\in\mathcal{C}\}$$

Proof. First, we can see easily that the second property implies $\mathcal{C}$-increment stationarity. Then we only need to consider the first property.

Let U_0 be fixed element of $\mathcal{A}$. For all $C\in\mathcal{C}$, as μ is surjective, there exists $g\in G$ such that $\mu(g)\cdot m(U_0)=m(C)$, i.e. $m(g\cdot U_0)=m(C)$. Therefore, by $\mathcal{C}$-increment stationarity,

$$\begin{aligned}E\left[(\Delta X_C)^2\right] &= E\left[(X_{g\cdot U_0})^2\right]\\ &= \mu(g)^{2H}E\left[(X_{U_0})^2\right]\\ &= m(C)^{2H}\frac{E\left[(X_{U_0})^2\right]}{m(U_0)^{2H}}\end{aligned}$$

By theorem 4.4, the result follows. □

5. Continuity of the sifBm

The results about the existence of a continuous version of set-indexed processes are not as simple as processes indexed by $\mathbf{R}_+$. Even in the simple case of the set-indexed Brownian motion, if the collection $\mathcal{A}$ is too rich, there does not exist any version which is continuous on the whole $\mathcal{A}$ (see Adler).[1]

To study the continuity of a set-indexed process X, we have to consider the behavior of $|X_U - X_V|$ when U and V are close. In order to do this, we provide $\mathcal{A}$ with some distance. In the classical case of Gaussian processes, we used to consider the canonical distance $d^2(U,V)=E\left[(X_U-X_V)^2\right]$ for $U,V\in\mathcal{A}$ (see Refs. 1 and 5). Let us mention two other distances that are also classical:

- The measure m on $\mathcal{T}$ induces the pseudo-metric d_m on $\mathcal{A}$

$$\forall U,V\in\mathcal{A};\quad d_m(U,V)=m(U\triangle V)$$

- we recall the definition of the Hausdorff metric d_{Haus} on $\mathcal{K}\backslash\emptyset$, the nonempty compact subsets of $\mathcal{T}$

$$\forall U,V\in\mathcal{K}\backslash\emptyset;\quad d_{Haus}(U,V)=\inf\{\epsilon>0: U\subseteq V^{\epsilon}\text{and} V\subseteq U^{\epsilon}\}$$

The notion of continuity depends of the distance considered. However, if $(\mathcal{A},d_m)$ (resp.$(\mathcal{A},d_{Haus})$) is compact, then d-continuity and d_m-continuity (resp. d_{Haus}-continuity) are equivalent (see Adler).[1]

For any function $x:\mathcal{A}\to\mathbf{R}$, define

$$\|x\|_{\mathcal{A}}=\sup_{A\in\mathcal{A}}|x(A)|$$

and let

$$B(\mathcal{A})=\{x:\mathcal{A}\to\mathbf{R};\ \|x\|_{\mathcal{A}}<\infty\}.$$

Let $C(\mathcal{A})$ denote the class of functions in $B(\mathcal{A})$ which are d_{Haus}-continuous on $\mathcal{A}$.

Then, studying d-continuity, we consider the balls $\mathcal{B}(U,\epsilon)=\{V\in\mathcal{A}; d(U,V)<\epsilon\}$, and the metric entropy $D(\bullet;\mathcal{A},d)$ of $(\mathcal{A},d)$, which gives the smallest number $D(\epsilon;\mathcal{A},d)$ of balls of radius $\epsilon>0$ required to cover $\mathcal{A}$.

If $(\mathcal{A},d)$ is totally bounded, i.e. $D(\epsilon;\mathcal{A},d)$ finite for all $\epsilon>0$, then under the assumption

$$\int_0^1\sqrt{\ln D(\epsilon;\mathcal{A},d)}.d\epsilon<\infty$$

Dudley's theorem states that the process X has a continuous modification (see Refs. 1, 5 and 16).

Theorem 5.1. Let $\mathbf{B}^H = \{\mathbf{B}^H_U; U \in \mathcal{A}\}$ be a set-indexed fractional Brownian motion.

The two following statements are equivalent

1. $\mathbf{B}^H$ is almost surely continuous on $\mathcal{A}$.
2. the set-indexed Brownian motion $\mathbb{W}$ is almost surely continuous on $\mathcal{A}$.

Proof. As

$$\forall U, V \in \mathcal{A}; \quad E\left[(\mathbf{B}^H_U - \mathbf{B}^H_V)^2\right] = \left(E\left[(\mathbb{W}_U - \mathbb{W}_V)^2\right]\right)^{2H} \tag{22}$$

the two canonical pseudo-metrics associated to the set-indexed fractional Brownian motion $\mathbf{B}^H$ and the set-indexed Brownian motion $\mathbb{W}$, are equivalent. □

A simple consequence of this result is that the sifBm has a continuous modification on rectangles of $\mathbf{R}^N_+$.

A deeper study of regularity of the set-indexed fractional Brownian motion is the object of a forthcoming article. In particular, Hölder continuity will be investigated.

6. sifBm ON INCREASING PATHS

The notion of flows is the key to reducing the proofs of many of theorems on characterization and weak convergence to a one-dimensional problem (see Ref. 12)

6.1. Generality on Flows

In general, $\mathcal{A}(u)$ is not closed under countable intersections, so we will occasionally require a larger class $\tilde{\mathcal{A}}(u)$, which is the class of countable intersections of sets in $\mathcal{A}(u)$: i.e. $U \in \tilde{\mathcal{A}}(u)$ if there exists a sequence $(U_n)_{n\in\mathbf{N}}$ in $\mathcal{A}(u)$ such that $\cap_n U_n = U$.

Definition 6.1. *Let* $S = [a, b] \subseteq \mathbf{R}$. An increasing function $f : S \to \tilde{\mathcal{A}}(u)$ is called a flow.

- A flow f is right-continuous if

$$f(s) = \bigcap_{v>s} f(v), \forall_s \in [a, b),$$

and $f(b) = \overline{\bigcup_{u<b} f(u)}$.
- A flow f is continuous if it is right-continuous and

$$f(s) = \overline{\bigcup_{u<s} f(u)} \forall_s \in (a, b).$$

- A flow f is simple if there exists finite sequence $(t_0, \dots, t_n)$ with $a = t_0 \leqslant \cdots \leqslant t_n = b$ and flows $f_i : [t_{i-1}, t_i] \to \mathcal{A}, i = 1, \dots, n$ such that for $s \in [t_{i-1}, t_i]$, $f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j)$.

Any process X indexed by $\mathcal{A}$ can be projected by a simple flow f onto a process indexed by a subset of **R**:

Definition 6.2. Let X be an $\mathcal{A}$-indexed process and f a simple flow on $S = [a, b]$. Then the S-indexed process X^f is defined as follows:

$$X^f(s) := X_{f(s)}, \forall_s \in S.$$

X^f is called the projection of X along f.

In the case that X can be extended to an additive process on $\tilde{\mathcal{A}}(u)$, X can be projected by any arbitrary flow according to the preceding definition.

The following lemma shows the importance of the concept of flows for set-indexed processes.

Let $S(\mathcal{A})$ denote the class of simple continuous flows defined on [0, 1].

Lemma 6.3.[11] The finite dimensional distributions of an (additive) $\mathcal{A}$-indexed process X determine and are determined by the finite dimensional distributions of the class $\{X^f : f \in S(\mathcal{A})\}$.

In this section, we study the set-indexed fractional Brownian motion on flows.

6.2. Lévy Fractional Brownian Motion and Fractional Brownian Sheet on Flows

As a preliminary, let us study the classical cases of Lévy fractional Brownian motion and fractional Brownian sheet.

Let us consider $\mathcal{T}=\mathbf{R}_+^N$ and $\mathcal{A}=\{[0,t];\, t\in\mathbf{R}_+^N\}\cup\{\emptyset\}$. We can associate to any flow f, an increasing function $\tilde{f}:[0,1]\to\mathbf{R}_+^N$ such that

$$\forall t\in[0,1];\quad f(t)=[0,\tilde{f}(t)]$$

- If $\mathscr{B}^H$ is a Lévy fBm of index $H\in(0,1)$,

$$\forall s,t\in[0,1];\quad E\left[(\mathscr{B}^H_{\tilde{f}(t)}-\mathscr{B}^H_{\tilde{f}(s)})^2\right]=\|\tilde{f}(t)-\tilde{f}(s)\|^{2H}$$

  Then, if $\tilde{f}(t)=t.\alpha$, where $\alpha\in\mathbf{R}_+^N$, $(\mathscr{B}^H)^{\tilde{f}}$ is a classical fractional Brownian motion, otherwise it is not.
- If $\mathbb{B}^H$ is a fBs of index $H\in(0,1)$,

$$\forall s,t\in[0,1];\, E\left[\mathbb{B}^H_{\tilde{f}(t)}.\mathbb{B}^H_{\tilde{f}(s)}\right]=\prod_{i=1}^N\frac{1}{2}\left[\tilde{f}_i(s)^{2H}+\tilde{f}_i(t)^{2H}\right.$$
$$\left.-|\tilde{f}_i(t)^{2H}-\tilde{f}_i(s)|^{2H}\right]$$

  Then, if the function $\tilde{f}$ is a line parallel to one axis of $\mathbf{R}_+^N$, the process $(\mathbb{B}^H)^{\tilde{f}}$ is a fractional Brownian motion. However, if $\tilde{f}(t)=t.\alpha$, where $\alpha\in\mathbf{R}_+^N$,

$$\forall s,t\in[0,1];\quad E\left[\mathbb{B}^H_{\tilde{f}(t)}.\mathbb{B}^H_{\tilde{f}(s)}\right]=[s^{2H}+t^{2H}-|t-s|^{2H}]^N\prod_{i=1}^N\frac{\alpha_i^{2H}}{2}$$

  which is not a fBm.
  In the two cases of the classical multiparameter extensions of the fractional Brownian motion, we saw that the projection of the process along a flow, is not in general a real-indexed fBm.

6.3. SifBm on Flows is a Standard FBm

Our definition for a set-indexed fractional Brownian motion is also justified by the following proposition.

Proposition 6.4. Let $\mathbf{B}^H$ be a set-indexed fractional Brownian motion, and f be a flow on [0,1]. Then the process $(\mathbf{B}^H)^f=\{\mathbf{B}^H_{f(t)};\, t\in[0,1]\}$ is a time-changed fractional Brownian motion.

Proof. The process $(\mathbf{B}^H)^f$ is clearly a mean zero Gaussian process indexed by [0, 1]. Moreover, its covariance function can be computed

$$E\left[\mathbf{B}^H_{f(s)}\mathbf{B}^H_{f(t)}\right]=\frac{1}{2}\left\{m[f(s)]^{2H}+m[f(t)]^{2H}-m[f(s)\triangle f(t)]^{2H}\right\}$$

For all $s \leqslant t$, we have $f(s) \subseteq f(t)$ and then

$$E\left[\mathbf{B}^H_{f(s)}\mathbf{B}^H_{f(t)}\right] = \frac{1}{2}\left\{m[f(s)]^{2H} + m[f(t)]^{2H} - m[f(t)\setminus f(s)]^{2H}\right\}$$
$$= \frac{1}{2}\left\{m[f(s)]^{2H} + m[f(t)]^{2H} - (m[f(t)] - m[f(s)])^{2H}\right\}$$

The function $\theta : [0,1] \to \mathbf{R}_+$ such that for all $t \in [0,1], \theta(t) = m[f(t)]$ is clearly increasing. Thus it defines a time change and we have

$$\forall s,t \in [0,1]; \quad E\left[\mathbf{B}^H_{f(s)}\mathbf{B}^H_{f(t)}\right] = \frac{1}{2}\left\{\theta(s)^{2H} + \theta(t)^{2H} - |\theta(t)-\theta(s)|^{2H}\right\} \tag{23}$$

Then $\left\{\mathbf{B}_{f\circ\theta^{-1}(t)}; t \in \mathbf{R}_+\right\}$ is a classical fractional Brownian motion. □

Proposition 6.4 allows to identify the self-similarity index of the si-fBm, as the Hölder exponent of the projection along any flow.

Let us recall the definition of the two classical Hölder exponents of a stochastic process X at $t_0 \in \mathbf{R}_+$:

- the pointwise Hölder exponent

$$\boldsymbol{\alpha}_X(t_0) = \sup\left\{\alpha : \quad \limsup_{\rho\to 0} \sup_{s,t\in\mathcal{B}(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\alpha} < \infty\right\}$$

- the local Hölder exponent

$$\tilde{\boldsymbol{\alpha}}_X(t_0) = \sup\left\{\alpha : \quad \limsup_{\rho\to 0} \sup_{s,t\in\mathcal{B}(t_0,\rho)} \frac{|X_t - X_s|}{|t-s|^\alpha} < \infty\right\}$$

Corollary 6.5. Let $\mathbf{B}^H$ be a set-indexed fractional Brownian motion with self-similarity index $H \in (0, 1/2]$. The pointwise and local Hölder exponents of the projection $(\mathbf{B}^H)^f$ along flows f at $t_0 \in [0,1]$, satisfy almost surely

$$\boldsymbol{\alpha}_{(\mathbf{B}^H)^f}(t_0) = \begin{cases} \alpha_\theta t_0.H & if\, \alpha_\theta(t_0) < 1 \\ H & \text{otherwise} \end{cases}$$

$$\tilde{\boldsymbol{\alpha}}_{(\mathbf{B}^H)^f}(t_0) = \begin{cases} \tilde{\alpha}_\theta t_0.H & if\, \alpha_\theta(t_0) < 1 \\ H & \text{otherwise} \end{cases}$$

where θ is the real function such that $\theta(t) = m[f(t)](\forall \in [0,1])$, and $\alpha_\theta(t_0)$ (resp. $\tilde{\alpha}_\theta)(t_0))$ is the pointwise (resp. local) Hölder exponent of θ at t_0.

Proof. Let f be a flow and $(\mathbf{B}^H)^f$ the projection of $\mathbf{B}^H$ along f. By (23), we have

$$\forall s,t\in[0,1];\quad E\left[\left((\mathbf{B}^H)^f_t-(\mathbf{B}^H)^f_s\right)^2\right]=|\theta(t)-\theta(s)|^{2H} \tag{24}$$

- If θ is differentiable on [0,1], for all $t_0\in[0,1]$ and $\rho>0$,

$$\forall s,t\in\mathcal{B}(t_0,\rho);\quad |\theta(t)-\theta(s)|\sim K.|t-s|$$

as ρ tends to 0. Then,

$$\forall s,t\in\mathcal{B}(t_0,\rho);\quad E\left[\left((\mathbf{B}^H)^f_t-(\mathbf{B}^H)^f_s\right)^2\right]\sim K.|t-s|^{2H} \tag{25}$$

In Herbin and Lévy-Véhel,[8] we see that equation (25) implies

$$P\{\forall t_0\in[0,1];\quad \boldsymbol{\alpha}_{(\mathbf{B}^H)^f}(t_0)=\tilde{\boldsymbol{\alpha}}_{(\mathbf{B}^H)^f}(t_0)=H\}=1$$

- If θ is not differentiable in $t_0\in[0,1]$ (i.e. if $\tilde{\alpha}_\theta(t_0)<1$),

$$\forall\alpha<\tilde{\alpha}_\theta(t_0);\quad \limsup_{\rho\to0}\ \sup_{s,t\in\mathcal{B}(t_0,\rho)}\frac{E\left[\left((\mathbf{B}^H)^f_t-(\mathbf{B}^H)^f_s\right)^2\right]}{|t-s|^{2\alpha.H}}=0$$

and

$$\forall\alpha>\tilde{\alpha}(t_0);\quad \limsup_{\rho\to0}\ \sup_{s,t\in\mathcal{B}(t_0,\rho)}\frac{E\left[\left((\mathbf{B}^H)^f_t-(\mathbf{B}^H)^f_s\right)^2\right]}{|t-s|^{2\alpha.H}}=+\infty$$

then, $\tilde{\boldsymbol{\alpha}}_{(\mathbf{B}^H)^f}(t_0)=\tilde{\alpha}_\theta(t_0).H$ almost surely (see Ref. 8).
In the same way, we get $\boldsymbol{\alpha}_{(\mathbf{B}^H)^f}(t_0)=\alpha_\theta(t_0).H$ almost surely. □

Remark 6.6. As a set-indexed process on Flow only depends on its covariance between subsets U and V such that $U\subset V$, the result stated in proposition 6.4 still holds for the set-indexed mean-zero Gaussian process defined by (18). More generally, it holds for all process which is self-similar and $\mathcal{C}_0$-increment stationary. Therefore this result lends additional support that such a process could be called a set-indexed fractional Brownian motion.

7. CONCLUDING REMARKS

The preceeding discussions permit the following general definition

Definition 7.1. A set-indexed Gaussian process which is self-similar of index $H \in (0, 1)$ and $\mathcal{C}_0$-increment stationary, is called general set-indexed fractional Brownian motion.

Corollary 7.2 Let $X = \{X_U; U \in \mathcal{A}\}$ be a general set-indexed fractional Brownian motion. Then,

1. For all $H \in (0, 1/2]$, such a process exists (Definition 2.11),
2. for any flow f, X^f is a time-changed real-indexed fractional Brownian motion,
2. the covariance function between two subsets U and V such that $U \subseteq V$ is

$$E[X_U X_V] = K[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}].$$

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