Analogue of the Bruhat–Chevalley Order for Reductive Monoids

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INTRODUCTION

The purpose of this paper is to describe the Adherence Order of $B \times B$-orbits on a reductive algebraic monoid. By the results of [10] there is already a perfect analogue of the Bruhat decomposition for reductive monoids. Indeed, by [10, Corollary 5.8], if $M$ is reductive with unit group $G$ and maximal torus $T \subseteq G$ with Borel subgroup $B \supseteq T$, then the set of two sided $B$-orbits, $B \backslash M / B$ is canonically identified with a certain finite inverse monoid $R$. In fact, $R = N_G(T)/T$, the orbit monoid of the Zariski closure of $N_G(T)$ in $M$. By definition, for $\sigma, \tau \in R$ we define the Adherence Order by

$$\sigma \leq \tau \quad \text{if} \quad B\sigma B \subseteq B\tau B,$$

where $B\tau B$ denotes the Zariski closure in $M$ of $B\tau B$.
In the case of reductive groups, a similar order relation was first studied by Chevalley. Since then Tits [13] developed the notion of a “BN pair” or “Tits system” which provided the abstract framework for much subsequent work in group theory (including complete homogeneous spaces and Schubert varieties [2]). Furthermore, the much studied KL-polynomials are quantified in terms of the Bruhat–Chevalley order on $W$ [3].

For reductive monoids one would like to describe the Adherence Order on $\mathbb{R}$ in terms of the Bruhat order on $W$ along with some other necessary invariant. This other invariant turns out to be the $\Pi$-order. For $n = n$ matrices, $A \succeq B$ in the $\Pi$-order, if $\text{rank}(A) \geq \text{rank}(B)$. Furthermore, each element $\sigma$ of $\mathbb{R}$ can be expressed uniquely in standard form $\sigma = xey^{-1}$, where $e \in \Lambda$ the cross section lattice, and $x, y \in W$ are appropriately restricted (see Corollary 1.5 for details). Our main result is the following:

**Theorem.** Let $\sigma = xey^{-1}$ and $\tau = sft^{-1}$ be in standard form. Then the following are equivalent:

(a) $\sigma \leq \tau$

(b) $e \leq f$ and there exists $w \in W(f)W_e$ such that $x \leq sw$ and $tw \leq y$.

Here $e \leq f$ means simply that $ef = fe = e$. This is the $\Pi$-order condition mentioned above. As usual, $W(f) = \{w \in W|wf = fw\}$ and $W_e = \{w \in W|we = ew = e\}$.

In Section 2 we analyze further the Adherence Order. To do this, we introduce the $j$ order on $\mathbb{R}$. Indeed, we define

$$\sigma \leq_j \tau \quad \text{if} \quad B\sigma B \leq B\tau B.$$  

Clearly, $\sigma \leq_j \tau$ implies $\sigma \leq \tau$. The main result of Section 2 is essentially the following theorem.

**Theorem.** Suppose $\sigma \leq \tau$. Then there exists $\theta_0, \theta_1, \ldots, \theta_m \in \mathbb{R}$ such that $\sigma = \theta_0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m = \tau$, and for all $i \leq m$ either

(a) $\theta_i \leq_j \theta_{i+1}$ or else

(b) $\theta_i$ and $\theta_{i+1}$ differ by a Bruhat interchange (see Definition 2.9).

This determines the Adherence Order in terms of the two simpler relations. The relation of (a) is not encountered in group theory but is easily analysed via Theorem 2.7. The relation of (b) is described in more detail in Definition 2.9 and Theorem 2.11. (For $M_n(k)$ it implies that $\theta_i$ and $\theta_{i+1}$ have the same nonzero rows and columns.) It is similar to the situation encountered in the study of the Bruhat–Chevalley order on a Coxeter group [4, Chap. 5].

In Section 3 we apply the general results of Sections 1 and 2 to the monoid $M_n(k)$ of $n \times n$ matrices. In this example, $\mathbb{R} = \mathbb{R}_n$ can be identified with the finite monoid of zero-one matrices with at most one
nonzero entry in each row or column. These are sometimes referred to as partial permutation matrices. We use the results of Section 1 or 2 to find a purely combinatorial description of the Adherence Order on $\mathbb{R}_n$.

In Section 4 we discuss a natural length function on $\mathbb{R}$. Here we continue our policy of determining all “monoid quantities” in terms of “$\Lambda$-quantities” and “$W$-quantities.” For example, if $\sigma = xey^{-1} \in \mathbb{R}$ is in standard form we define the length of $\sigma$ by

$$l(\sigma) = l(x) + l(e) - l(y),$$

where $l(x), l(y)$ are the lengths of $x$ and $y$ as elements of the Coxeter group $(W,S)$ ($l(e)$ is defined in Section 4). We show in Section 2 that length is subadditive, namely

$$l(\sigma \tau) \leq l(\sigma) + l(\tau).$$

In the final section of the paper we pursue some further properties of the Adherence Order. While the results of this section could benefit from some further refinements, we include them for future reference.

1. THE ADHERENCE ORDER

Let $M$ be a reductive algebraic monoid with maximal torus $T$ and Borel subgroup $B \supseteq T$. $E(T) = \{e \in T | e^2 = e\}$. Let $\Lambda = \{e \in E(T) | Be \subseteq eB\}$ be a cross section lattice, and let $R = N_G(T)/T$. Recall from [10, Corollary 5.8] that $R$ can be canonically identified with the set of two sided $B$-orbits on $M$. For $\sigma, \tau \in \mathbb{R}$ we define

$$\sigma \leq \tau$$

if $B\sigma B \subseteq B\tau B$ (Zariski closure in $M$). It is easy to verify that $\leq$ determines a partial order on $\mathbb{R}$. The main result of this section describes this order relation in terms of

(i) the $I$-order on $M$

(ii) the Adherence Order on $W$.

The $I$-order (on $M$) has been determined explicitly in much detail by the second and third named authors for a large class of reductive monoids [8]. The Adherence Order on $W$ is the much studied Bruhat–Chevalley order.

Before stating our main result we recall some relevant background information. Let $G$ be a reductive group with $T \subseteq B \subseteq G$ as usual. Let $W$ be the Weyl group of $T$. By definition, if $x, y \in W$, then $x \leq y$ if and only if $BxB \subseteq B\gamma B$. 
1.1. Lemma. \( x \leq y \) if and only if \( B^- y B \subseteq B^- x B \) where \( B^- \) is the Borel subgroup opposite to \( B \).

Proof. Let \( w \in W \) be the longest element. Then by [4, Sect. 5.9], \( x \leq y \) iff \( wy \leq wx \) iff \( BwyB \subseteq BwxB \) iff \( wBwyB \subseteq wBwxB \) iff \( B^- y B \subseteq B^- x B \), since \( B^- = wBw \).

1.2. Lemma. \( x \leq y \) iff \( xBy = B \).

Proof. By [3, Corollary 1.2], \( x \leq y \) iff \( B^- x B \cap B y B \neq \emptyset \). But

\[
B^- x B \cap B y B \neq \emptyset \iff B^- x B \cap B \neq \emptyset
\]

\[
\iff B^- x B \cap B^- y B \neq \emptyset
\]

\[
\iff xBy^{-1} \cap B^- B \neq \emptyset.
\]

1.3. Lemma. For all \( x \in W \), \( B^- x B \subseteq B^- B x \cap xB^- B \).

Proof. Write \( xBx^{-1} = U^- UT \) where \( U^\prime \subseteq B^- \) and \( U \subseteq B \). Then \( xBx^{-1} = U^- (UT) \subseteq B^- B \). So \( xBx^{-1} \subseteq B^- Bx \). Thus, \( B^- x B \subseteq B^- Bx \). Similarly, \( x^{-1} B^- x \subseteq B^- B \). Thus, \( B^- x B \subseteq xB^- B \) as well.

For \( I \subseteq S \) let \( L_I \) be the associated Levi factor and \( B_L = L_I \cap B \). Let \( x \in W \). Then by [1, Proposition 2.3.3]

\[ \text{x has minimal length } l(x) \text{ in } W_I x \iff x^{-1}B_L x \subseteq B. \] (2)

Now let \( M \) be a reductive monoid with unit group \( G \) and cross section lattice \( \Lambda \). For \( e \in \Lambda \) let

\[
W(e) = C_W (e)
\]

\[
W_e = \{ w \in W | we = e \} \triangleleft W(e)
\]

\[
D_e = \{ x \in W | x \text{ has minimal length in } xW_e \}
\]

\[
D(e) = \{ x \in W | l(x) \text{ is minimal in } W(e)x \}
\]

\[
L(e) = C_G (e), \quad H(e) = eL(e)
\]

\[
B(e) = C_B (e)
\]

\[
B^- (e) = C_{B^-} (e).
\]

If \( C_W (e) = \langle s | s \in I \rangle \) and \( x \in D_e \) then

\[
Bex = B(e)x
\]

\[
= ex^{-1} B(e) x
\]

\[ \subseteq exB, \quad \text{by (2)}.\]
Hence, $BexB \subseteq exB \subseteq BexB$ and so

$$BexB = exB \quad \text{for } x \in D_x. \quad (3)$$

Now let $e, f \in \Lambda$ with $e \leq f$. Then

$$W(e) = W_e [W(e) \cap W(f)]. \quad (4)$$

Clearly,

$$R = WAW = \bigcup_{e \in \Lambda} W_e D(e). \quad (5)$$

Let $x \in W$, $e \in \Lambda$, $y \in D(e)$, and assume $x' \leq x$. Then by Lemma 1.3 and (3),

$$Bx'eyB \subseteq Bx'BeyB$$

$$\subseteq Bx'BeyB$$

$$\subseteq Bx'BeyB$$

Thus

$$x'ey \leq xey \quad \text{if } y \in D(e) \text{ and } x' \leq x. \quad (6)$$

Finally, let $y \in D(e)$ and assume $x \leq y^{-1}$. Then by (6), $xey \leq y^{-1}ey$. Hence

$$xey \in B \quad \text{if } y \in D(e) \text{ and } x \in W, x \leq y^{-1}. \quad (7)$$

1.4. Theorem. Let $e, f \in \Lambda$, $x, s \in W$, $y \in D(e)$, and $t \in D(f)$. Then the following are equivalent:

(a) $xey \leq sft$

(b) $ef = e$ and there exist $w \in W(f)W_e$, $z \in W_e$ such that $w^{-1}t \leq y$ and $x \leq swz$ in $W$.

Proof. Let $e, f, s, t, x$, and $y$ be as above and assume (b) holds for $w$ and $z$ as indicated. Now by (7), $t^{-1}wey \in B$. So $sfwey = sfft^{-1}wey \in sftB \subseteq BsfIB$. So $sfwey \leq sft$. By assumption, $w = w_1w_2$ with $w_1 \in W(f)$ and $w_2 \in W_e$. Thus, $sfwey = sfw_1w_2ey = sfw_1ey = sw_1ey = sw_1w_2ey = sw_1w_2ey = sw_1w_2ey$. But by (6), $xey \leq swzey \leq swey \leq sfwey \leq sft$.

Conversely, suppose $xey \leq sft$. Then clearly $e \leq f$ and $xey \in BsfIB$. Hence, $e \in x^{-1}BsfitB^{-1}$. Now, for $w \in W$, let

$$A_w = x^{-1}Bs \cap B^{-w}B. \quad (8)$$
So \( x^{-1}B_s = \bigcup_{w \in W} A_w \). Since this is a finite disjoint union of subvarieties, there exists a unique \( w \in W \) such that \( A_w \subseteq x^{-1}B_s \) is open and dense. So

\[
e \in \overline{A}_w \text{fitBy}^{-1}.
\]

Thus, \( \overline{e} \in \overline{eA}_w \text{fitBy}^{-1}e \subseteq \overline{eA}_w \text{fitBy}^{-1}e \subseteq \overline{eB^-wBfBy}^{-1}e \subseteq \overline{eB^-ewfBfBy}^{-1}e \).

Hence, \( ewf \in e \). So \( wfw^{-1}e \geq e \). But \( f \geq e \) by assumption. Thus, there exists \( v \in C_w(e) \) such that \( vfv^{-1} = wfw^{-1} \). But then \( v \in W(f) \cap W(e) \neq \emptyset \), say \( w = ve \) with \( e \in W(f) \). Hence, by (4), \( w = ve \in W(e)W(f) \subseteq W[fW(e) \cap W(f)]W(f) \subseteq W_eW(f) \). Conclude that

\[
w = w_1w_2 \text{ for some } w_1 \in W_e, w_2 \in W(f). \tag{9}
\]

Since \( A_w \neq 0 \), we see by Lemma 1.3 that \( \emptyset \neq x^{-1}B_s \cap B^-wB \subseteq x^{-1}B_s \cap B^-Bw \). Thus \( x^{-1}Bs^{-1} \cap B^-B \neq \emptyset \). So by Lemma 1.2 and (9)

\[
x \leq sw^{-1} = sw_1^{-1}w_1^{-1}. \tag{10}
\]

By (10)

\[
e \in \overline{B^-w_1w_2BfBy}^{-1}
= \overline{B^-w_1w_2fBy}^{-1}, \text{ by (3)}
= \overline{B^-w_1w_2fBy}^{-1}.
\]

For \( u \in W \), let

\[
C_u = w_2tBy^{-1} \cap B^-uB.
\]

Then

\[
w_2tBy^{-1} = \bigcup_{u \in W} C_u.
\]

Since this is a finite union we see that for some \( u \in W \), \( C_u \subseteq w_2tBy^{-1} \) is open and dense. Thus,

\[
e \in \overline{B^-w_1fC_u}.
\]

So

\[
e \in \overline{eB^-w_1fC_u}e \subseteq \overline{eB^-w_1fC_u}e \subseteq \overline{eB^-w_1fBy}^{-1}e \subseteq \overline{eB^-ew_1fB^-uB}e
\subseteq \overline{eB^-ew_1fB^-uB}e = eB^-eB^-uB = eB^-euB.
\]

Thus, \( eu \in e \). Hence \( u \in W(e) \). So in \( H(e) \), \( e \in \overline{eB^-euB} \). By Lemma 1.1 applied to \( H(e) \), \( eu \leq e \) in \( W(H(e)) \). Hence, \( eu = e \). So \( u \in W_e \). Since
Let \( \tilde{w} = w_2^{-1}u \in W(f)W_e \), \( z = u^{-1}w_1^{-1} \in W_e \). By (10) and (11)
\[
sw_2^{-1}w_1^{-1} = sw_2^{-1}uu^{-1}w_1^{-1} = swz, \quad \text{and} \quad \tilde{w}^{-1}t = u^{-1}w_2t \leq y.
\]

One can streamline the theorem somewhat as follows.

It is easy to check that each element \( \sigma \in WeW \), with \( e \in \Lambda \), can be written uniquely as
\[
\sigma = xey^{-1}, \quad x \in D_e, y \in D(e).
\]

We call this the **standard form** for \( \sigma \).

**Example.** Let \( M = M_3(k) \), and let \( T \) be the group of diagonal invertible matrices, \( B \) the group of upper triangular invertible matrices. One checks that
\[
\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

So let
\[
\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in R_3.
\]

Then
\[
\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (132)e(1).
\]

One easily checks that this is a standard form since \( D_e = W \) and \( D(e) = (1,23,123) \).

1.5. **Corollary.** Let \( \sigma = xey^{-1} \) and \( \tau = sft^{-1} \) be in standard form. Then the following are equivalent:

(a) \( \sigma \leq \tau \).

(b) \( e \leq f \) and there exists \( w \in W(f)W_e \) such that \( x \leq sw \) and \( tw \leq y \).
Proof. This is straightforward.

Remark 1. Let $\sigma = xey^{-1}$, $\tau = sft^{-1}$ be in standard form. The characterization in Corollary 1.5(b) is “best possible” in the sense that $W(f)W_e$ cannot be replaced by $W(f)$. For example, let

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Coincidently, the same example shows that $\sigma \leq \tau \iff l(\sigma) - l(e) \leq l(\tau) - l(f)$. On the other hand, if $\sigma \leq \tau$ and $\sigma \in WfW$ then $l(\sigma) \leq l(\tau)$.

Remark 2. It is interesting to notice that $\{\sigma \in R \mid \sigma \leq 1\} = \{\sigma \in R \mid \sigma \in \bar{B}\}$ can be identified with a subset of $((x, y) \in W \times W \mid x \leq y)$. Indeed, write $\sigma = xey^{-1}$ in standard form. Then by Lemma 2.3 below $x \leq y$. One might hope for an interesting relationship with Kazhdan–Lusztig polynomials. Indeed, the KL-polynomials are indexed by ordered pairs in $W$. There is already some connection between KL-polynomials and the Bruhat decomposition established by the second named author [7].

2. THE $j$-ORDER ON $R$

2.1. Definition. Let $r, s \in R$. We say $r \leq_j s$ if $BrB \subseteq BsB$.

One checks easily that $\leq_j$ is a partial order on $R$. The major purpose of this section is to find a description of $\leq_j$ entirely within $R$ (see Theorem 2.7 below). Along the way, we shall need to establish several other important results about $\leq$ and $\leq_j$.

2.2. Definition. Define $R^0 \subseteq R$ as $R^0 = \{r \in R \mid BrB \subseteq \bar{B}\}$.

One checks that this is well defined. For $M_n(k)$, $R^0$ is the set of upper triangular partial permutation matrices.

2.3. Lemma. Let $r = xey^{-1}$ be in standard form. Then $r \in R^0$ if and only if $x \leq y$.

Proof. If $r \in R^0$ then $r \leq 1$. Applying Corollary 1.5 we find that there exists $w \in W$ such that $x \leq w \leq y$. In particular, $x \leq y$. Conversely, if $x \leq y$, then plainly the criteria of Corollary 1.5(b) are satisfied for $\sigma = r$ and $\tau = 1$.

It was established in [10] that if $s \in S$, the simple reflections, and $r \in R$, then

$$sBr \subseteq BrB \cup BsrB$$
and

\[ rBs \subseteq BrB \cup BrsB. \]

We use this to help prove Theorems 2.4, 2.5, and 2.6 below.

2.4. Theorem. If \( r, g \in R \) and \( x \in W \) then \( rBx \subseteq \bigcup_{y \leq x} BryB \) and \( xBr \subseteq \bigcup_{y \leq x} ByrB \).

Proof. We proceed by induction on the length of \( x \). By the results of [13], we know that both inclusions hold if \( l(x) = 1 \), since \( s = \{ x \in W | l(x) = 1 \} \). Assume the results hold if \( l(x) < k \), and suppose \( l(x) = k \). Then there exists \( s_1, s_2, \ldots, s_k \in S \) such that \( x = s_1s_2 \cdots s_k \). By induction we know that \( rBx \subseteq \bigcup_{y \leq x} BryB \) and \( xBr \subseteq \bigcup_{y \leq x} ByrB \). Hence \( rBx \subseteq \bigcup_{y \leq x} BryB \). The other case is similar.

2.5. Theorem. If \( r, r_1 \in R \) then \( rBr_1 \subseteq \bigcup_{y \leq r_1} Br_1B \).

Proof. Write \( r = xeyy^{-1} \) in standard form. Then by Theorem 2.4 we have that \( rBr_1 = rBxyy^{-1} \subseteq \bigcup_{y \leq y} Brx_1Byy^{-1} \).

Now let \( L = \{ x \in G | xx = x \} = C_G(e) \), \( B_L = B \cap L = C_B(e) \). Then \( B_1 = eB_1 = C_B(e)e \) by [6, Theorem 6.16, Corollary 6.34]. Hence

\[ Brx_1Byy^{-1} = Brx_1C_B(e)Byy^{-1} = Brx_1Byy^{-1}. \]

But, \( y \in D(e) \) and so \( yC_B(e)y^{-1} \subseteq B \). Thus, \( Brx_1C_B(e) \) is \( Brx_1Byy^{-1} \subseteq Brx_1Byy^{-1}B \). But notice that \( x_1 \leq x \) and \( y \in D(e) \), so that \( r_2 = x_1eyy^{-1} \leq xeyy^{-1} = r_1 \). Hence, \( rBr_1 \subseteq \bigcup_{y \leq x} Brx_1Byy^{-1} \subseteq \bigcup_{y \leq x} Brx_1Byy^{-1}B \). The other case is similar.

2.6. Theorem. If \( r, r_1 \in R \) then \( r_1Br \subseteq \bigcup_{y \leq r_1} Br_1B \).

Proof. By [9, Theorem 8.2], there exists an involution \( \tau: M \to M \) so that

\[ \tau^2(x) = x \quad \text{for all } x \in M, \]
\[ \tau(xy) = \tau(y)\tau(x) \quad \text{for all } x, y \in M, \]
\[ \tau|T = id \]

and

\[ \tau(B) = B^{-}. \]

Now let \( w \in N_G(T) \) represent the longest element of \( W \) (so that \( wBw^{-1} = B^{-} \)), and define \( \theta = \text{int}(w) \circ \tau \). In particular, \( \theta(B) = B \) and \( \theta(xy) = \theta(y)\theta(x) \) for \( x, y \in R \).
If \( r_1, r \in \mathbb{R} \), then by Theorem 2.5, \( rB_r \subseteq \bigcup_{r \leq r_1} B_{r_2}B \). So also, \( \theta(r_1)B \theta(r) \subseteq \bigcup_{r \leq r_1} \theta(B(r)) \theta(r)B \). But \( r_2 \leq r_1 \) if and only if \( \theta(r_2) \leq \theta(r_1) \) since \( \theta(B) = B \) and \( \theta \) is a homeomorphism in the Zariski topology. So our conclusion follows by applying \( \theta \) directly to the inclusion \( \theta(r)B \theta(r_1) \subseteq \bigcup_{r \leq \theta(r_1)} \theta(B(r)) \theta(r_1)B \) of Theorem 2.5.

2.7. Theorem. Let \( \sigma, \theta \in \mathbb{R} \). Then \( \sigma \in \mathbb{R}^+ \theta \mathbb{R}^+ \) if and only if \( B\sigma B \subseteq \bar{B} \theta \bar{B} \).

Proof. If \( \sigma \in \mathbb{R}^+ \theta \mathbb{R}^+ \) then \( \sigma = r_1 \theta r_2 \) where \( r_1, r_2 \in \mathbb{R}^+ \). Thus,

\[
B\sigma B = Br_1 \theta r_2 B \\
\subseteq Br_1 \theta r_2 B \\
\subseteq \bar{B} \theta \bar{B} \quad \text{since } Br_1, r_2 B \subseteq \bar{B}.
\]

Conversely, if \( B\sigma B \subseteq \bar{B} \theta \bar{B} \) then there exist \( r_1, r_2 \in \mathbb{R}^+ \) such that \( B\sigma B \subseteq Br_1 \theta Br_2 B \). On the other hand, by Theorem 2.6, \( r_1 \theta B \subseteq \bigcup_{r \leq r_1} Br_2 \theta B \), while by Theorem 2.5, \( r_3 \theta Br_2 \subseteq \bigcup_{r \leq r_3} Br_3 \theta r_4 B \). Hence \( B\sigma B \subseteq Br_1 \theta Br_2 B \subseteq \bigcup_{r \leq r_3} Br_3 \theta r_4 B \), from which it follows that \( \sigma = r_3 \theta r_4 \) for some \( r_3 \leq r_1 \) and \( r_4 \leq r_2 \). But \( r_1, r_2 \in \mathbb{R}^+ \) and so \( r_3, r_4 \in \mathbb{R}^+ \).

Theorem 2.7 above gives us one of the major ingredients in the decomposing the Adherence Order on \( \mathbb{R} \). It turns out that any relation \( \sigma \leq \tau \) can be obtained as a chain of elementary relations \( \sigma = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_n = \tau \), where, for each \( i \), either \( \theta_i \in \mathbb{R}^+ \theta \mathbb{R}^+ \), or else \( \theta_i+1 \) is obtained from \( \theta_i \) by a “Bruhat Interchange” (see Definition 2.9 below).

The next theorem will get us closer to the major result of this section.

2.8. Theorem. Let \( \beta \in \text{Ref}(W) = \bigcup_{w \in W} wSw^{-1} \) and \( x, y \in W \). Assume that \( x < x\beta \). Then the following are equivalent.

(a) \( Bxey^{-1}B \subseteq \bar{B}xey^{-1}B \) and \( xey^{-1} \neq x\beta ey^{-1} \)

(b) \( \beta \not\in W(e) = \{ x \in W \mid xe = ex \} \).

Proof. Assume (b). So \( \beta \in \text{Ref}(w) \setminus W(e) \). Then \( x\beta \) can be written in reduced form as \( x\beta = s_1 \cdots s_i \beta \), where \( s_i \in S \) for all \( i \), \( s_1, \ldots, s_m \in D(e) \) and \( s_{m+1}, \ldots, s_i \in W(e) \). Clearly, \( (x\beta) \beta = x \) and \( (x\beta) \beta < x\beta \). Thus, by the strong exchange condition, there exists a unique \( i \) such that

\[
x = (x\beta) \beta = s_1 \cdots s_{i-1}s_{i+1} \cdots s_i.
\]

Hence \( \beta = s_1 \cdots s_{i+1}s_i(s_{i+1} \cdots s_i) \). Also,

\[
x e \beta x^{-1} = s_1 \cdots s_{i-1}s_{i+1} \cdots s_i ex \cdots s_i.
\]
But now \( \beta \in \text{Ref}(W) \setminus W(e) \), so we must have \( i \leq m \). Expanding further, we obtain

\[
x e \beta x^{-1} = s_1 \ldots s_{i-1}s_{i+1} \ldots s_m e s_m \ldots s_1
\]

with

\[
s_1 \ldots s_{i-1}s_{i+1} \ldots s_m \prec s_1 \ldots s_m
\]

and

\[
s_1 \ldots s_m \in D(e).
\]

But from Lemma 2.3 we obtain that \( xe \beta x^{-1} \in \mathbb{R}^+ \). Hence,

\[
B x e y^{-1} B \subseteq \overline{B x e y^{-1} B}
\]

\[
= \overline{B (xe \beta x^{-1}) (x e y^{-1}) B}
\]

\[
\subseteq \overline{B x e y^{-1} B}.
\]

Finally, if \( \beta \notin W(e) \) then \( \beta \notin W(e) \). Hence, \( e \neq \beta e \) and it follows that \( x e y^{-1} \neq x e y^{-1} \).

Conversely, assume \( \beta \in W(e) \). Thus, \( xe \beta y^{-1} = xe y^{-1} \). Now if \( xe \beta x^{-1} \in \overline{B} \) then \( e \beta = e = \beta e \) and so \( xe y^{-1} = x e y^{-1} \). Hence, either \( xe \beta x^{-1} \notin \overline{B} \) or else \( xe y^{-1} = x e y^{-1} \). If \( xe \beta x^{-1} \notin \overline{B} \) then \( B x e y^{-1} B \nsubseteq B x e y^{-1} B \). (For if \( B x e y^{-1} B \subseteq B x e y^{-1} B \) then by Theorem 2.7, there exists \( u, v \in \mathbb{R}^+ \) such that \( u x e y^{-1} = x e y^{-1} \). Replacing \( u \) and \( v \) by \( u x e y^{-1} \) and \( y e y^{-1} \), if necessary, we can assume \( u \in x e x^{-1} C_p(x e x^{-1}) \) and \( v \in y e y^{-1} C_p(y e y^{-1}) \). But also, \( u, v \in \mathbb{R}^+ \) and hence \( u = x e x^{-1} \) and \( v = y e y^{-1} \). But then \( xe y^{-1} = x e y^{-1} \). This completes the proof.

2.9. Definition. Let \( x e y^{-1} \) and \( s e t^{-1} \) be in standard form, \( x e y^{-1} \neq s e t^{-1} \). A \textit{Bruhat Interchange} occurs between \( x e y^{-1} \) and \( s e t^{-1} \) if there exists \( \alpha \in \text{Ref}(W) \cap W(e) \) such that \( x \preceq x \alpha \) and \( x \alpha e y^{-1} = s e t^{-1} \).

Notice that for \( x e y^{-1} \neq x \alpha e y^{-1} \), \( x \preceq x \alpha \) if and only if \( x e y^{-1} \preceq x \alpha e y^{-1} \).

2.10. Example. Consider, in \( M_3(k) \),

\[
\sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = xe y^{-1}
\]

\[
\tau = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = x \alpha e y^{-1}.
\]
Here
\[ e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W(e). \]

Clearly,
\[ x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_0. \]

2.11. **Theorem.** Let \( \sigma = xey^{-1} \) and \( \theta = sft^{-1} \) be in standard form. Then \( \sigma \leq \theta \) if and only if there exist \( \theta_0, \theta_1, \ldots, \theta_r \in \mathbb{R} \) such that \( \sigma = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_r = \theta \), and for each \( l \), either \( \theta_{l-1} \leq \theta_l \) or else \( \theta_{l-1} \) and \( \theta_l \) differ by a Bruhat Interchange.

**Proof.** Obviously, the condition is sufficient. So assume \( \sigma \leq \theta \). Then \( Bxey^{-1}B \subseteq Bsft^{-1}B \), and there exists \( w \in W(f)W_e \) such that \( x \leq sw \) and \( tw \leq y \). Write \( w = w_1w_2 \) with \( w_1 \in W(f) \) and \( w_2 \in W_e \). Using this fact, we obtain
\[
sw^{-1} = sw_1w_2ey^{-1} = sw_1few_2y^{-1} = sw_1ew_2y^{-1} = swey^{-1}.
\]

Using this, we also obtain
\[
Bswey^{-1}B \subseteq Bsft^{-1}twey^{-1}B \subseteq Bsft^{-1}B, \quad \text{since } twey^{-1} \in \mathbb{R}^+.
\]

But also \( Bxey^{-1}B \subseteq Bswey^{-1}B \), since \( x \leq sw \). In any case, we obtain
\[
xey^{-1} \leq sway^{-1} \leq sft^{-1}.
\]

So we let \( \theta_0 = sft^{-1} \) and \( \theta_{r-1} = sway^{-1} \). Then we only need to find the chain of \( \theta \)'s from \( xey^{-1} \) to \( sway^{-1} \). Since \( x \leq sw \) we can find \([4, \text{Proposition 5.11}] \gamma_1, \ldots, \gamma_k \in Rf(W) \) such that
\[
x < x\gamma_1 < x\gamma_1 \gamma_2 < \cdots < x\gamma_1 \cdots \gamma_r = sw
\]
and for every \( i \in \{1, 2, \ldots, r\} \) there exists \( \delta_i \in W_e \) such that \( x\gamma_1 \cdots \gamma_i \delta_i \in \)
Since $x \in D(e)$,

$$x \leq x\gamma_1\delta_1 \leq x\gamma_1\gamma_2\delta_1 \leq \cdots \leq x\gamma_1\gamma_2\cdots\gamma_r\delta_r.$$  

If $x\gamma_1\gamma_2\cdots\gamma_r\delta_r = x$ then

$$swey^{-1} = sw\delta^{-1}ey^{-1} = x\gamma_1\cdots\gamma_rey^{-1} = xey^{-1}.$$  

In this case, we can just let $\theta_0 = swey^{-1}$ and $\theta_1 = sft^{-1}$ and we are done.  

If $x\gamma_1\gamma_2\cdots\gamma_r \neq x$ then let

$$\theta_i = x\gamma_1\cdots\gamma_i\delta_i ey^{-1}$$

$$= x(\gamma_1\cdots\gamma_{i-1})\gamma_i ey^{-1}.$$  

If $\gamma_i \in \text{Ref}(W) \setminus W(e)$ then by Theorem 2.8, $B\theta_{i-1}B \subseteq \overline{B}\theta_i\overline{B}$, while if $\gamma_i \in \text{Ref}(W) \cap W(e)$ then $\theta_{i-1}$ and $\theta_i$ differ by a Bruhat Interchange.  

2.12. Example. Let $M = M_n(k)$. Then

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \theta.$$  

We can illustrate Theorem 2.11 as

$$\theta_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \theta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\leq \theta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \theta_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

One could invoke Theorem 2.8 to prove $\theta_0 \leq_i \theta_1 \leq_i \theta_2$. However, we can see this directly (using Theorem 2.7) by observing

$$\theta_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \theta_2.$$  

Definition 2.9 easily yields that $\theta_2 \rightarrow \theta_3$ is a Bruhat Interchange. (The Bruhat Interchange for $M_n(k)$ will be discussed systematically in Section 3.)
3. THE BRUHAT–CHEVALLEY ORDER ON $M_n(k)$

In this section we illustrate our theory for the reductive monoid $M_n(k)$ of $n \times n$ matrices. In this example, the symmetric inverse semigroup $R = R_n$ can be identified with the set of $n \times n$ zero-one monomial matrices. Our approach in this section is to identify and interpret the main results of Section 2 (especially Theorem 2.11) as they apply to $M_n(k)$. To help focus the reader, we consider the following question: Is there a purely combinatorial description for the Adherence Order on $R_n$?

3.1. Theorem. Let $E_{ij}$ be the $n \times n$ matrix $(a_{st})$ such that

$$a_{st} = \begin{cases} 1, & (s, t) = (i, j) \\ 0, & (s, t) \neq (i, j). \end{cases}$$

Then $E_{ij} \leq E_{km}$ if and only if $i \leq k$ and $m \leq j$. ($(i, j)$ is “up” and “to the right” of $(k, m)$.)

**Proof.** If $E_{ij} \leq E_{km}$ then there exists upper triangular matrices $X$ and $Y$ such that $XE_{km}Y = E_{ij}$. However, this implies that $(XE_{km}Y)_{st} = 0$ if $s > k$ or $t < m$. Thus, $i \leq k$ and $m \leq j$.

Conversely, suppose $i \leq k$ and $m \leq j$. Then $X = E_{ik}$ and $Y = E_{mj}$ are both upper triangular. But one calculates $XE_{km}Y = E_{ij}$. \[\square\]

**Remark.** It is easy to check that $E_{ij} \leq E_{km}$ if and only if $E_{ij} \leq E_{km}$. This leads to a number of special properties in the case of $M = M_n(k)$.

We refer to $E_{ij}$ as an elementary matrix.

3.2. Theorem. Let $A, C \in R_n$ and write

$$A = \sum_{l=1}^{s} A_l \quad \text{and} \quad C = \sum_{l=1}^{t} C_l,$$

where $\{A_l\}$ and $\{C_l\}$ are elementary matrices. Define $S_A = \{A_1, \ldots, A_s\}$ and $S_C = \{C_1, \ldots, C_t\}$. Then the following are equivalent:

(a) $A <_c C$

(b) There exists an injection $\theta: S_A \to S_C$ such that $A_l \leq \theta(A_l)$ for all $l = 1, 2, \ldots, s$. 

Proof. Assume $A \leq_j C$. Notice that $s = \text{rank}(A)$ and $t = \text{rank}(C)$. Write

$$A = w_1 \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} w_2$$

$$C = w_3 \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} w_4,$$

where $w_1, w_2, w_3, w_4 \in S_n$, the unit group of $\mathbb{R}_n$. Let

$$X = w_1 w_3^{-1}$$

$$Y = w_4^{-1} \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} w_2.$$ 

Then we have

$$XCY = A.$$ 

Furthermore, one can check that, for all $k \leq t$,

$$XC_k Y \leq C_k,$$

and

$$\text{rank}(XC_k Y) \leq 1.$$ 

Hence, for each $k = 1, 2, \ldots, t$

- either $XC_k Y \in S_A$
- or else $XC_k Y = 0$.

But for any $A_i \in S_A$ there exists $C_k$ such that $XC_k Y = A_i$ (since $XCY = A$). So define $\theta(A_i) = C_k$, where $XC_k Y = A_i$.

Conversely, if there exists an injection $\theta: S_A \to S_C$ such that $A_i \leq_j \theta(A_i)$ for all $i = 1, 2, \ldots, s$, then we can find elementary upper triangular matrices $X_i$ and $Y_i$ such that $X_i \theta(A_i) Y_i = A_i$. Let

$$X = \sum_{i=1}^{s} X_i$$

and

$$Y = \sum_{i=1}^{s} Y_i.$$
Then $X$ and $Y$ are both upper triangular and (one checks) $XCY = A$. Thus $A \preceq_j C$.

3.3. Example.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \preceq_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C$$

since $S_A = \{E_{11}, E_{23}\}$, $S_C = \{E_{13}, E_{22}, E_{33}\}$, and $\theta; S_A \rightarrow S_C$ defined by $\theta(E_{11}) = E_{12}$, $\theta(E_{23}) = E_{22}$ satisfies the criterion of Theorem 3.2.

To properly illustrate Theorem 2.11 in the situation of $R_n$ it remains to describe concretely the notion of a Bruhat Interchange (Definition 2.9).

We begin our description with the rank two case.

3.4. Theorem. Let $A, C \in R_n$ be such that $A = E_{ij} + E_{kl}$ and $C = E_{kj} + E_{li}$. Then $A < C$ if and only if $i < k$, $j < l$ or $i > k$, $j > l$. (Notice that $A < \overline{a}$ if and only if a Bruhat Interchange has occurred (see Definition 2.9).)

Proof. Suppose $i < k$ and $j < l$ (the other case is similar). If $E = E_{11} + E_{22}$ then $A = (i - 1)i \ldots (k - 1k) \ldots (34)(23)E(23) \ldots (l - 1l)(12) \ldots (j - 1j)$ where $(ab)$ is the permutation matrix which interchanges the $a$th row and the $b$th row. This is the standard form for $A$. One checks that

$$C = [(i - 1i) \ldots (12)(k - 1k) \ldots (34)(23)]$$

$$(12)E(23) \ldots (l - 1l)(12) \ldots (j - 1j).$$

But $[(i - 1i) \ldots (23)] < [(i - 1i) \ldots (23)][12]$. Therefore $A < C$.

Conversely, assume $i < k$ and $j > l$ or $i > k$ and $j < l$. But then the same argument, as above, shows that $C < A$.

3.5. Theorem. Let $A = E_{i,j} + \ldots + E_{i,j} \in R$ (where, as usual, $E_{ij}$ is an elementary matrix) and suppose $C = E_{i,j} + \ldots + E_{i,j} + E_{i,j} + E_{j,i} + \ldots + E_{i,j}$ (so $C$ is obtained from $A$ by interchanging two nonzero rows). Relabel the two interchanged matrices $E_{ij}$ and $E_{kl}$. Then $A < C$ if and only if $i < k$, $j < l$ or $i > k$, $j > l$.

Proof. If $i < k$ and $j < l$, then by Theorem 3.4, $(E_{ii} + E_{kk})A \prec (E_{ii} + E_{kk})C$. A little more calculation then shows that $A < C$.

Conversely, if $i > k$ and $j < l$, or $i < k$ and $j > l$ then the above argument shows that $C < A$.

Notice that the situation above describes exactly the case where a Bruhat Interchange occurs between $A$ and $C$. 

Combining Definition 2.9 and Theorems 2.11 and 3.5 we obtain the following description of the Adherence Order on \( R_n \).

3.6. Theorem (Theorem 2.11 for \( R_n \)). Let \( \sigma, \tau \in R_n \). Then \( \sigma \leq \tau \) if and only if \( \sigma = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_r = \tau \) where, for each \( l \), either

(a) \( \gamma_{l-1} \in R_n^+ \cap R_n^+ \), or else
(b) \( \gamma_l \) is obtained from \( \gamma_{l-1} \) via a Bruhat Interchange.

To calculate in case (a), we use Theorem 3.2, and in case (b) we use Theorem 3.5.

Recall that for \( \sigma, t \in R_n \) we write \( S_\sigma = \{ E_1, \ldots, E_r \} \) if \( \sigma = \sum_{i=1}^r E_i \) and each \( E_i \) is an elementary matrix.

3.7. Theorem. Suppose \( \sigma \leq \tau \) and \( l(\sigma) = l(\tau) - 1 \). (So \( \sigma \leq \gamma \leq \tau \) implies \( \sigma = \gamma \) or \( \gamma = \tau \).) Then one of the following holds:

(a) \( |S_\sigma| = |S_\tau| - 1 \) and \( S_\sigma \subseteq S_\tau \).
(b) \( |S_\sigma| = |S_\tau| \) and \( |S_\sigma \cap S_\tau| = |S_\tau| - 1 \) with \( E_\sigma < E_\tau \) where \( S_\sigma \setminus (S_\sigma \cap S_\tau) = \{ E_\tau \} \) and \( S_\tau \setminus (S_\sigma \cap S_\tau) = \{ E_\sigma \} \). Furthermore, \( \sigma \) and \( \tau \) are either row equivalent or column equivalent.
(c) \( \tau \) is obtained from \( \sigma \) via a Bruhat Interchange.

Proof. Assume (c) is not the case, and rank(\( \sigma \)) < rank(\( \tau \)). By Theorems 3.2 and 3.6 there exists an injection \( \theta: S_\sigma \rightarrow S_\tau \) such that \( \sigma_i \leq \theta(\sigma_i) \) for each \( \sigma_i \in S_\sigma \). Define \( \theta(\sigma) \in R_n \) by \( S_{\theta(\sigma)} = \theta(S_\sigma) \). Then \( \sigma \leq \theta(\sigma) < \tau \). Thus \( \sigma = \theta(\sigma) \). Therefore, (a) holds.

Now assume (c) is not the case, and rank(\( \sigma \)) = rank(\( \tau \)). By Theorem 3.6 we must have \( \sigma = x_\tau y \) for some \( x, y \in R_n \). Then \( \sigma = x_\tau y \leq x_\tau y \leq \tau \). Hence, either \( \sigma = x_\tau y \) or else \( x_\tau y = \tau \). In the first case \( \sigma = x_\tau y \), and in the second case \( \sigma = x_\tau \). Assume, without loss of generality that \( \sigma = x_\tau \) (\( \sigma \) and \( \tau \) have the same nonzero rows). Now by Theorem 3.2, the map \( \theta: S_\tau \rightarrow S_\tau \) has \( \sigma_i \leq \theta(\sigma_i) \) for all \( \sigma_i \in S_\sigma \). But by Theorem 3.1, this means that if \( \sigma_i = E_{uv} \) and \( \theta(\sigma_i) = E_{ij} \) then \( r \geq u \) and \( s \leq v \) (\( E_{uv} \) is “up” an “to the right” of \( E_{ij} \)). But \( \sigma \) and \( \tau \) have the same nonzero rows. Since \( \theta \) is row increasing it follows easily that \( \theta \) is row preserving. So write

\[
S_\sigma = \{ E_{i_1}, \ldots, E_{i_r} \}
\]

and

\[
S_\tau = \{ E_{j_1}, \ldots, E_{j_r} \},
\]

where \( E_{i_k} \) and \( E_{j_k} \) are nonzero in the same row (so that \( \theta(E_{i_k}) = E_{j_k} \)). Now let \( E_{ik} \in S_\sigma \) be the elementary matrix with the maximum column
value. Define \( \tau' \in \mathbb{R}^n \) via \( S_{\tau'} = (S_\tau \setminus \{E_{ik}\}) \cup \{E_{jk}\} \) where \( E_{ik} \) is nonzero in the same row as \( E_{jk} \) (so \( \theta(E_{jk}) = E_{ik} \)). Then \( \tau' = \sum_{E \in S_{\tau'}} E \in \mathbb{R}^n \) since \( \theta \) is column decreasing. Also \( \sigma \leq_j \tau' \leq_j \tau \) since \( E_{ik} \) has larger column value than \( E_{jk} \). If \( \tau' \neq \tau \) then we are done, since we obtain \( \sigma = \tau' \), and so (b) holds. If \( \tau' = \tau \) (i.e., \( E_{ik} = E_{jk} \)) we move on; i.e., define \( \tau^{(2)} \) using the \( E_{ik} \in S' \) with the next largest column value, thereby substituting it for the corresponding \( E_{jk} \in S \). Again one checks that \( \tau^{(2)} \in \mathbb{R}^n \), and \( \sigma \leq \tau^{(2)} \leq \tau \). If \( \tau^{(2)} = \tau \) then define \( \tau^{(3)} \), and so on. Eventually, one obtains \( \tau^{(r)} \neq \tau \). But then \( \sigma = \tau^{(r)} \) since \( \sigma \leq \tau^{(r)} < \tau \) and \( l(\tau) = l(\sigma) + 1 \). So \( \tau \) is obtained from \( \sigma \) by "moving" one elementary matrix of \( \sigma \) "to the left." So (b) holds.

Theorem 3.7 allows us to give a combinatorial description of the Adherence Order on \( \mathbb{R}^n \). First we represent the elements of \( \mathbb{R}^n \) by sequences of nonnegative integers. Let \( \sigma \in \mathbb{R}^n \). We associate to \( \sigma \) a sequence \( (e_1, \ldots, e_n) \) where, for all \( i, 1 \leq i \leq n, e_i \) is defined as

\[
e_i = \begin{cases} 0, & \text{if } \sigma \text{ is zero in the } i \text{th column} \\ k, & \text{if } [\sigma]_{ki} = 1. \end{cases}
\]

**Example.**

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} \leftrightarrow (3042).
\]

Notice that a sequence \( (e_1, \ldots, e_n) \) can occur this way for some \( r \in \mathbb{R}^n \) if and only if \( 0 \leq e_i \leq n \) for all \( i \); and whenever \( e_i = e_j \), either \( i = j \) or else \( e_i = e_j = 0 \). For convenience we simply write \( \sigma = (e_1, \ldots, e_n) \in \mathbb{R}^n \). We can now interpret Theorem 3.7 combinatorially.

**3.8. Theorem.** The Adherence Order on \( \mathbb{R}^n \) is the smallest partial order generated by declaring

\[\sigma = (\delta_1, \ldots, \delta_n) < \tau = (\epsilon_1, \ldots, \epsilon_n)\]

if either

(a) \( \delta_j = \epsilon_j \) for \( j \neq i \) and \( \delta_i < \epsilon_i \) (\( \sigma \) is obtained from \( \tau \) by replacing some \( \epsilon_i \) by a smaller number)

(b) \( \delta_k = \epsilon_k \) if \( k \neq i \) or \( j \), \( i < j \), \( \delta_i = \epsilon_i = \delta_j \), and \( \epsilon_j > \epsilon_i \) (\( \sigma \) is obtained from \( \tau \) by interchanging \( \epsilon_i \) and \( \epsilon_j \) (\( i < j \)) where \( \epsilon_i > \epsilon_j \)).
Proof. We simply interpret each case of Theorem 3.7.

Case 3.7(a). This says that \( \sigma \) is obtained from \( \tau \) by setting some nonzero entry in \( \tau \) (as a matrix) to zero. This is included in Theorem 3.8(a).

Case 3.7(b). First assume that \( \sigma \) and \( \tau \) have the same nonzero rows. So, as we saw in the proof of Theorem 3.7, the matrix \( \sigma \) is obtained from the matrix \( \tau \) by moving a “one” to the right. This is included in Theorem 3.8(b). If \( \sigma \) and \( \tau \) have the same columns then \( \sigma \) is obtained from \( \tau \) by moving a “one” upward. This is included in Theorem 3.8(a).

Case 3.7(c). This is easily seen as the special case of Theorem 3.8(b) where both \( e_i \) and \( e_j \) are nonzero.

3.9. Example. Let \( r = (21403) \) and \( s = (35201) \) in \( \mathbb{R}_5 \). Then \( r < s \), since

\[
(21403) < (31402), \quad \text{by Theorem 3.8(b)}
\]
\[
< (34102), \quad \text{by Theorem 3.8(b)}
\]
\[
< (35102), \quad \text{by Theorem 3.8(a)}
\]
\[
< (35201), \quad \text{by Theorem 3.8(b)}.
\]

4. Subadditivity of Length

In this section we consider a natural length on \( \mathbb{R} \), extending the much studied length function on \( W \), the Weyl group. In keeping with our point of view in this paper, we shall define this length function in terms of the length function on \( W \) and a notion of length for the elements of \( \Lambda \). This appears to be a natural generalization of length function considered by Solomon [12] for \( M = M_n(\mathbb{F}_q) \), where \( \mathbb{R} \) can be identified with the symmetric inverse semigroup on \( n \) letters. There is also a topological/geometric definition for this length function. Indeed, one can define

\[
l(x) = \dim(B_xB) - \dim(B_vB),
\]

where \( v \in W \times W \) is the unique element such that \( Bv = vB \). This is discussed in some detail by the third named author in [11]. Our main result in this section proves that the length function is subadditive.
Let $W = \langle S \rangle$ be the Weyl group, where $S \subseteq W$ is the set of simple involutions relative to $B$ and $T \subseteq B$. Define
\[ \text{Ref}(W) = \bigcup_{x \in W} xSx^{-1}. \]

For $I \subseteq S$ we have
\[ \text{Ref}(W_I) = W_I \cap \text{Ref}(W). \]

Let
\[ l(I) = |\text{Ref}(W) \setminus \text{Ref}(W_I)| \]
\[ D(I) = \{x \in W | x \text{ has minimal length in } xW_I\}. \]

Thus,
\[ l(I) = \max\{l(x) | x \in D_I\}. \]

For $I, J \subseteq S$ let
\[ l(I, J) = |\text{Ref}(W_I) \setminus W_J|. \]

So
\[ l(I) = l(S, I) \]
and
\[ l(I, J) = l(I, I \cap J) \]
\[ = \max\{l(x) | x \in W_I \cap D_I\}. \quad (1) \]

Now let $R$ be as in section one with unit group $W$ and cross section lattice $\Lambda$. For $e \in \Lambda$, recall that
\[ W(e) = \{x \in W | xe = ex\} \]
\[ W_e = \{x \in W | xe = e\} \triangleleft W(e). \]

Notice that there exists a unique subgroup $\tilde{W}(e)$ of $W$ such that $\tilde{W}(e)$ is generated by a subset of $S$ and
\[ W(e) = W_e \times \tilde{W}(e) \quad (2) \]
\[ D(e) = D_I, \quad \text{where } W(e) = \langle I \rangle, \]
\[ D_e = D_J, \quad \text{where } W_e = \langle J \rangle, \]
and
\[ l(e) = l(I). \]

If \( e, f \in \Lambda \), let
\[ l(f, e) = l(J, I), \quad \text{where } W(f) = \langle J \rangle \text{ and } W(e) = \langle I \rangle. \]

If \( e, f \in \Lambda \), \( x \in W \), then
\[
\begin{align*}
& e \geq f \quad \text{iff } e \geq f \text{ and } x \in W(e)W(f) \\
& f \leq x \quad \text{iff } e \geq f \text{ and } x \in W(f)W(e).
\end{align*}
\]

If \( e, f \in \Lambda \) then
\[ e \geq f \implies W(f) \subseteq W(e). \]

Thus, by (2) and (4),
\[ e \geq f \implies W(f) = W(e, f)W_f, \]
where \( W(e, f) = W(e) \cap W(f) \). Recall from the end of Section 1 that if \( \sigma \in WeW, e \in \Lambda \), then
\[ \sigma = xey^{-1} \quad \text{for some } x \in D_e \text{ and } y \in D(e). \]

Furthermore, \( x \) and \( y \) are unique (by an easy counting argument).

4.1. Definition. Let \( \sigma = xey^{-1} \) be in standard form. Define
\[ l(\sigma) = l(x) + l(e) - l(y), \]
where \( l(x), l(y) \) are the lengths of \( x \) and \( y \) as elements of \( W \), and \( l(e) \) is as above.

It is easy to see that \( l(z\sigma) \leq l(z) + l(\sigma) \) for any \( z \in W \). However, much more is true. The main result of this section (Theorem 4.5 below) says that for any \( \sigma, \tau \in R \), \( l(\sigma\tau) \leq l(\sigma) + l(\tau) \). For example, let
\[
\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
so that
\[ \sigma\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]
Then
\[
\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = xy^{-1},
\]
\[
\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = xu^{-1}
\]
and
\[
\sigma \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = rsf^{-1}
\]
determine the standard form of \( \sigma \), \( \tau \), and \( \sigma \tau \). Thus, by Definition 4.1
\[
\begin{align*}
l(\sigma) &= l(x) + l(e) - l(y) = 3 + 2 - 2 = 3 \\
l(\tau) &= l(u) + l(e) - l(v) = 1 + 2 - 0 = 3 \\
l(\sigma \tau) &= l(r) + l(f) - l(s) = 2 + 2 - 0 = 4.
\end{align*}
\]
So \( l(\sigma \tau) \leq l(\sigma) + l(\tau) \) as required.

4.2. Lemma. Let \( e, f, h \in E \) be such that \( eWf \cap WhW \neq \emptyset \). Then
\[
l(h) \leq l(e) + l(f) - l(h, e) - l(h, t).
\]

Proof. Let \( x \in W \) be such that \( exf = ahb^{-1} \) is in standard form. Then \( eah \in WhW \). So by (3) and (5) above \( a \in W(e)W(h) = W(e)W_h \). Since \( a \in D_h \), we see that \( a \in W(e) \). Also, since \( hh^{-1}f \in WhW \), we see by (3) that \( b^{-1} \in W(h)W(f) \). Finally, since \( b \in D(h) \), we see that \( b \in W(f) \). We conclude that
\[
ea^{-1}xbf = a^{-1}exfb = h.
\]
Thus, without loss of generality we may assume that \( exf = h \). So we obtain \( hxf = x \) and, by (3), \( x = W(h) \). Since \( x^{-1}exf \) is an idempotent we obtain that \( x^{-1}exf = h \). Thus, \( W(x^{-1}ex) \cap W(f) \subseteq W(h) \), and it follows that
\[
x^{-1}R \text{ef}(W(e))x \cap R \text{ef}(W(f)) \subseteq R \text{ef}(W(h)).
\]
Thus, in \( R \text{ef}(W) \),
\[
R \text{ef}(W(h))^c \subseteq x^{-1}R \text{ef}(W(e))^c x \cup R \text{ef}(W(f))^c.
\]
and so
\[ \text{Ref}(W(h))^c \subseteq [x^{-1}\text{Ref}(W(e))^c x \setminus \text{Ref}(W(h)) \cap x^{-1}\text{Ref}(W(e))^c x] \]
\[ \cup [\text{Ref}(W(f))^c \setminus \text{Ref}(W(h)) \cap \text{Ref}(W(f))^c]. \]

But
\[ |\text{Ref}(W(h))^c| = l(h) \]
\[ |x^{-1}\text{Ref}(W(e))^c x| = |\text{Ref}(W(e))^c| = l(e) \]
and
\[ |\text{Ref}(W(h)) \setminus x^{-1}\text{Ref}(W(e))x| = |x^{-1}[\text{Ref}(W(h)) \setminus \text{Ref}(W(e))]x| \]
\[ = l(h, e) \]
since \( x \in W(h) \). Furthermore,
\[ |\text{Ref}(W(f))^c| = l(f) \]
\[ |\text{Ref}(W(h)) \setminus \text{Ref}(W(f))| = l(h, f). \]

Thus, we conclude that \( l(h) \leq [l(e) - l(h, e)] + [l(f) - l(h, f)]. \]

**4.3. Corollary.** Let \( e, f, h \in \Lambda \) and \( x \in D(e) \) be such that \( ex^{-1}f = ahb^{-1} \) is in standard form. Then
\[ l(a) + l(h) - l(b) \leq l(e) + l(f) - l(x) - l(h, f). \]

**Proof.** Now \( a \in D_h, b \in D(h), \) and \( eah, hba^{-1}f \in WhW \). Thus (3) and (5), \( a \in W(e) \) and \( b \in W(f) \). Hence \( ea^{-1}x^{-1}bf = a^{-1}ex^{-1}fb = h \). So \( ha^{-1}x^{-1}bh = hae^{-1}x^{-1}bf = h. \) But then by (3), \( a^{-1}x^{-1}b \in W(h) \), and so \( a^{-1}x^{-1}bh = h \). Hence, \( a^{-1}xb \in W_h \). Let \( \alpha = a^{-1}x^{-1}b \in W_h \). So
\[ a^{-1}x^{-1} = \alpha b^{-1}. \] (6)

Since \( a \in W(e), x \in D(e), \alpha \in W_h, \) and \( b \in D(h), \)
\[ l(a) + l(h) - l(b) = l(e) + l(f) - l(x) - l(h, f). \] (7)

But \( \alpha = \alpha_1\alpha_2 \) with \( \alpha_1 \in W_h \cap W(e), \alpha_2 \in W_h \cap D(e)^{-1}, \) and \( l(\alpha) = l(\alpha_1) + l(\alpha_2). \) So by (6), \( \alpha^{-1}a^{-1}x^{-1} = \alpha_2b^{-1}. \) Then
\[ l(\alpha_2^{-1}a) = l(\alpha_2) + l(\alpha) \] since \( \alpha_2 \in W_h, a \in D_h \)
and
\[ l(\alpha_1^{-1}a^{-1}x^{-1}) = l(\alpha_1^{-1}a^{-1}) + l(x) \]
\[ = l(\alpha_1) + l(a) + l(x) \quad \text{since } \alpha_1^{-1}a^{-1} \in W(e) \text{ and } x \in D(e). \]

Furthermore,
\[ l(\alpha_2 b^{-1}) = l(\alpha_2) + l(b) \quad \text{since } b \in D(h). \]

Thus,
\[ l(\alpha_1) + l(a) + l(x) = l(\alpha_2) + l(b). \]

But \( l(\alpha) = l(\alpha_1) + l(\alpha_2) \), so by (7),
\[ l(a) + l(x) = l(\alpha_1) + l(\alpha_2) + l(b). \]

Conclude that \( l(\alpha_1) = 0 \), and so \( \alpha = \alpha_2 \in W_h \cap D(e) \subseteq W(h) \cap D(e). \) By (1), \( l(\alpha) \leq l(h, e). \) Hence, \( l(a) + l(x) \leq l(h, e) + l(b). \) Combining this with the lemma we obtain \( l(a) + l(\alpha_1) - l(b) \leq l(e) + l(f) - l(x) - l(h, f). \]

4.4. Corollary. Let \( e, f, h \in \Lambda, \ x \in D(e), \ t \in D(f). \) Then
\[ l(ex^{-1}ft^{-1}) \leq l(ex^{-1}) + l(ft^{-1}). \]

Proof. Let \( ex^{-1}f = ahb^{-1}, \ h \in \Lambda, \) be in standard form. Then as in Corollary 4.3,
\[ a \in D_h \cap W(e), \ b \in D(h) \cap W(f) \quad (8) \]

and
\[ l(a) + l(h) - l(b) \leq l(e) + l(f) - l(x) - l(h, f). \quad (9) \]

Let
\[ b^{-1}t^{-1} = \alpha v^{-1}, \quad \alpha \in W(f), \, v \in D(h). \quad (10) \]

By (8) and (10)
\[ l(b) + l(t) = l(b^{-1}t^{-1}) = l(\alpha v^{-1}) = l(\alpha) + l(v). \]

But \( \alpha = \alpha_1 \alpha_2 \) with \( \alpha_1 \cap (f) \cap W(h) \) and \( \alpha_2 \in D(f)^{-1} \cap W(h). \) So \( l(\alpha) = l(\alpha_1) + l(\alpha_2). \) By (10), \( \alpha_1^{-1}b^{-1}t^{-1} = \alpha_2 v^{-1} \) and (8), \( l(\alpha_1) + l(b) = l(\alpha_1^{-1}b^{-1}). \) Since \( \alpha_1^{-1}b^{-1} \in W(h) \) and \( t \in D(h), \) we obtain
\[ l(\alpha_1) + l(b) + l(t) = l(\alpha_1^{-1}b^{-1}) + l(t) \]
\[ = l(\alpha_1^{-1}b^{-1}t^{-1}) \]
\[ = l(\alpha_2) + l(v). \]
But by (11), \( l(b) + l(t) = l(\alpha_2) + l(\alpha_2) + l(v) \). Hence \( l(\alpha_2) = 0 \) and so \( \alpha = \alpha_2 \in D(f)^{-1} \cap W(h) \). By (1)
\[
l(\alpha) \leq l(h, f).
\]
Equation (12) follows.

Since \( f \geq h \) it follows from (5) that \( \alpha \in W(h) \cap D(f) \subseteq W_h \cap D(f) \).

Hence, by (10)
\[
ex^{-1}f^{-1} = ahb^{-1}t^{-1} = ah\alpha v^{-1} = ahv^{-1}.
\]
Finally, by (9), (11), and (12)
\[
l(ex^{-1}f^{-1}) = l(a) + l(h) - l(b) - l(t) + l(\alpha)
\leq l(e) + l(f) - l(x) - l(h, f) - l(t) + l(\alpha),
\]
by Corollary 4.3
\[
\leq l(e) + l(f) - l(x) - l(t)
= l(ex^{-1}) + l(ft^{-1}).
\]

4.5. Theorem. Let \( \sigma_1, \sigma_2 \in R \). Then
\[
l(\sigma_1 \sigma_2) \leq l(\sigma_1) + l(\sigma_2).
\]
Proof. Write \( \sigma_1 = xey^{-1} \) and \( \sigma_2 = vft^{-1} \) in standard form with \( e, f \in \Lambda \).
Let \( y^{-1}v = wy_1^{-1} \) where \( w \in W(e) \) and \( y_1 \in D(e) \). Since \( y \in D(e) \) and \( w \in W(e) \) we obtain that \( w^{-1}y^{-1} = y_1^{-1}v^{-1} \) and \( l(w^{-1}y^{-1}) = l(w) + l(y) \).
Hence,
\[
l(w) + l(y) = l(y^{-1}v^{-1}) \leq l(y_1) + l(v). 
\]
Finally,
\[
l(\sigma_1 \sigma_2) + l(xey^{-1}vft^{-1})
= l(xwy_1^{-1}ft^{-1})
= l(xwey_1^{-1}ft^{-1})
\leq l(xw) + l(\sigma_1 y_1^{-1}ft^{-1})
\leq l(xw) + l(e) - l(y_1) + l(f) - l(t), \quad \text{by Corollary 4.4}
\leq l(x) + l(w) + l(e) - l(y_1) + l(f) - l(t)
\leq l(x) + l(v) - l(y) + l(e) + l(f) - l(t), \quad \text{by (13)}
= l(\sigma_1) + l(\sigma_2), \quad \text{by definition.}
Remark. The question of subadditivity of length was raised by L. Solomon. It turns out that the issue of integral structure constants for monoid Hecke algebras hinges on Theorem 4.5. See the proof of [12, Theorem 4.12] for an illustration of this issue in the special case \( M = M_n(\mathbb{F}_q) \).

5. RELATED RESULTS

In this section we complete our current study by establishing a number of supplementary results concerning the Adherence Order. We focus on the relationship between the Adherence Order and the H-relations on \( R \). It is clear to the authors that there are an intriguing number of uncharted possibilities that could be pursued here.

Let \( M \) be a reductive monoid with maximal torus \( T \). If \( \sigma, \tau \in R \) we say that \( \sigma \) and \( \tau \) are H-related (in \( R \)) if \( \sigma R = \tau R \) and \( R \sigma = R \tau \). We write \( \sigma \ll \tau \). Similarly, elements \( x \) and \( y \) of \( M \) can be H-related. We denote the H-class in \( M \) of \( x \in M \) by \( H_x \) and the H-class of \( x \in R \) in \( R \) by \( H_x \). So \( H_x = \{ y \in M \mid yM = xM \} \) and \( H_x = \{ y \in M \mid xy = yx \} \). Two elements \( x \) and \( y \) of \( M \) are \( l \)-related \((x \ll y)\) is \( MxM = MyM \). We write \( x \gg y \) if \( MyM \subseteq MxM \). For \( n \times n \) matrices, \( x \gg y \) in the \( l \)-order if \( \text{rank}(x) \geq \text{rank}(y) \).

Let \( e, f \in E(T) \).

5.1. Lemma. Suppose \( x \in eMf \cap eG \). Then there exists \( e' \in E(T) \) such that \( e' \in Cl_w(e), e' \leq f \), and \( xe' \in eG \).

Proof. Write \( x = eg \) and let \( b_1, b_2 \in B \) be such that \( g = b_1wb_2 \). Here \( B \) is a Borel subgroup containing \( T \) such that \( B \subseteq \{ h \in G \mid eh = ehe \} \). Thus \( eb_1 = eb_1c = ce \) for some \( c \in C_u(e) \). Hence, \( eg = cewb_2 = cew^{-1}b_2 = ge'b \), where \( e' = w^{-1}ew \) and \( b = b_2 \). So \( xe' = ege' = ye'be' \in Ge' \) since \( e'be' \in Be' \). But also \( eg = x = xf \). So if \( ege' \in Ge' \) then \( f' \geq e' \). Otherwise, \( f \not\prec e' \) and so \( xe' \prec e' \) which implies that \( xfe' < e' \) in the \( l \)-order on \( M \). But this is absurd since \( xfe' = ege' \in Ge' \).

5.2. Corollary. Suppose \( eBf \cap eG \neq \emptyset \). Then there exists \( e' \in E(T) \) such that \( e' \ll e, e' \leq f \), and \( eBe' \cap eG \neq \emptyset \).

Proof. Let \( x \in eBf \cap eG \) and choose \( e' \) as in Lemma 5.1. Then \( xe' \in eBe' \cap eG \) and the conclusion follows.

For \( s \in R \) let \( H_s \) denote the H-class of \( s \) in \( M \).

5.3. Corollary. Let \( r, s \in R \) and let \( e, f \in E(T) \) be such that \( eR = sR \) and \( Rf = R \). Suppose \( eBrf \cap H_s \neq \emptyset \). Then there exist \( e', f' \in E(T) \) with \( e'f' \ll s \) such that \( eBe'rf'Bf \cap H_s \neq \emptyset \). In particular, \( e'r = rf's \) and if we write \( r = e_0s = sf_0 \) with \( \sigma \in W \) then \( e' \leq e_0 \) and \( f' \leq f_0 \).
Proof. Since $eBf \cap H_s \neq \emptyset$ it follows easily that $eB_0 \cap eG \neq \emptyset$ and $f_0 Bf \cap Gf \neq \emptyset$. So by Corollary 5.2 and its left-right dual we obtain $e' \leq e_0$ and $f' \leq f_0$ such that $eB'e \cap eG \neq \emptyset$ and $f'Bf \cap Gf \neq \emptyset$.

If we let $\sigma H_f$ denote the $H$-class of $e_1 \sigma$ where $\sigma^{-1} e_1 \sigma = e_2$ then it is easy to see that $e_1 H_0 \subseteq e_1 H_f$. But we have $\emptyset \neq eBe' \cap e_1 H_f$ (since $e'r = rf'$) and $\emptyset \neq f'Bf \cap e_1 H_f$. Thus, recalling that $H_f = e_1 H_f$, we obtain $eBe' \cap e_1 H_f \neq \emptyset$.

We recall now some results from [11]. Define the set of order preserving elements of $R$ by

$$O = \{ r \in R \mid rB_r^* \subseteq B_r^* \}.$$

Then by [11, Sect. 2]

$$O \subseteq R$$

is an inverse subsemigroup,

$$E( R ) \subseteq O,$$

and

$$|O \cap H| = 1$$

for each $H$-class of $R$.

For each $s \in R$ we can write uniquely

$$s = s_+ s_0 s_-,$$

where $s_+, s_- \in O$, $s_+ R = s R$, $R s_- = R s$, $s_0 H \nu$, and $\nu I s$ is the unique such element that satisfies $B_\nu = \nu B$.

To illustrate these notions, consider our familiar example of $R_n$. One can easily calculate that, in this example,

$$O = O_n = \{ \sigma \in R_n \mid \text{if } \sigma_{ij}, \sigma_{kl} \neq 0 \text{ and } i < k, \text{then } j < l \};$$

i.e., for nonzero entries, the column value is an increasing function of the row value, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in O_4, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \notin O_4.$$

It is interesting to notice that for each $\sigma \in R_n$, there exists a unique $\bar{\sigma} \in O_n$ so that $\sigma$ be obtained form $\bar{\sigma}$ via a sequence of Bruhat interchanges. $O_n$ is an inverse semigroup because it is closed under matrix multiplication and transpose.
Another easy calculation yields that

\[
\{ \nu \in \mathbb{R}_n | B \nu = \nu B \} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \right\},
\]

The decomposition

\[ s = s_+ s_0 s_- \quad \text{(for any} \ s \in \mathbb{R}_n) \]

can be thought of as follows:

- \( s_+ \) indicates the nonzero rows of \( s \).
- \( s_- \) indicates the nonzero columns of \( s \).
- \( s_0 \) indicates the nonzero columns as a function of the nonzero rows.

One should think of \( s_0 \) as the “order reversing” part of \( s \). For example, \( s_0 = \nu \) if and only if \( s \in 0 \). As an illustration, consider the example

\[
s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = s_+ s_0 s_-.
\]

As promised, \( s_+ \) says “rows one and three are involved in \( s \),” \( s_- \) says “columns one and two are involved in \( s \),” \( s_0 \) represents the function

\[
R_1 \mapsto C_2 \\
R_3 \mapsto C_1.
\]

See [11, Sect. 3] for more details concerning \( 0 \) and the \( +0 - \) decomposition.

5.4. Definition. For \( s \in \mathbb{R} \) let \( \bar{s} \in 0 \) denote the unique element of \( 0 \cap H \). So if \( s = s_+ s_0 s_- \) as above then \( \bar{s} = s_+ \nu s_- \).

For the elements of \( \mathbb{R}_n \), it is easy to find \( \bar{s} \) from \( s \) without finding \( s_+, s_0, \) and \( s_- \). \( \bar{s} \) is the unique element of \( 0_n \) with the same nonzero rows and
columns as \( s \). For example, if

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

then

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\].

5.5. Corollary. The following are equivalent for \( r, s \in \mathbb{R} \) with \( e \mathcal{R} = s \mathcal{R} \) and \( s \mathcal{R} = r \mathcal{F} \).

(a) \( e \mathcal{R} \mathcal{F} \cap H_s \neq \emptyset \)

(b) \( r \geq s \)

(c) \( r \geq s \).

Proof. By [11, Corollary 4.4], \( H_s \subseteq \bigcup_{H_s} BtB = BH_sB \), yet by continuity of multiplication \( e \mathcal{R} \mathcal{F} \subseteq BrB \) since \( e, f \in T \) and \( TBrBT \subseteq BrB \). Hence, \( e \mathcal{R} \mathcal{F} \cap H_s \neq \emptyset \) iff there exists \( i \mathcal{H} \) such that \( r \geq t \). On the other hand \( s \in H_s \) is the unique smallest element in \( H_s \). Thus, (a) and (b) are equivalent. But also \( r \geq s \). So (c) implies (b). So we may assume \( r \geq s \).

By Corollary 5.3 there exist \( e', f' \in \mathbb{R} \), \( l \)-equivalent to \( s \), such that \( r \geq e'f' \) and \( e'f' \geq s \). Thus,

\[
\bar{r} \geq e'f' = e'f'.
\]

The equality here results from the fact that \( e'f' \mathcal{H} e'f' \) and \( e'f', e'f' \in \mathcal{H} \). But also \( e'f' \mathcal{H} s \) (and \( e'f' \geq s \)). So by [11, Proposition 3.11], \( e'f' \geq s \) implies that \( (e'f') \geq s \). Thus

\[
\bar{e'f'} \geq s
\]

by [11, Proposition 3.2] since \( \bar{(e'f')} = (s) \). Putting things together we obtain

\[
r \geq s \Rightarrow r \geq e'f' \geq s \Rightarrow r \geq e'f'
\]

(by (1)).

But \( e'f' \geq s \), by (2).

Remarks. (1) From the above proof we see that for \( r \in \mathbb{R} \), \( s \in \mathcal{O} \), and \( r \geq s \) there exists \( t \mathcal{H} s \) such that

\[
t \geq s, \quad \text{and} \quad t = e'f', \text{where } e' \mathcal{R} = t \mathcal{R} \text{ and } \mathcal{R} t = R f'.
\]

This should be useful in getting a better picture of the Adherence Order generally.

(2) Assume \( e \mathcal{R} \mathcal{F} \cap H_s \neq \emptyset \) where \( e \mathcal{R} = s \mathcal{R} \) and \( s \mathcal{R} = r \mathcal{F} \). From [11, Corollary 4.4] we know that \( H_s \subseteq \bigcup_{H_s} BtB \). Thus, there exists a
unique \( t \in H \) such that \( eBBrf \cap BtB \subseteq eBBrf \) is dense. It would be nice to describe this \( r \) somehow. In any case we obtain the following corollary.

5.6. COROLLARY. If \( r \geq \bar{s} \) then there exists a maximum \( t \in H \) such that \( r \geq t \).

5.7. EXAMPLE. We can use Theorem 3.8 to illustrate Corollary 5.6. Recall from Section 3 that the elements of \( R_n \) can be thought of as sequences of nonnegative integers \( r = (e_1, \ldots, e_n) \). So let \( n = 5 \), and consider \( r = (24105), s = (31020) \in R_5 \). Then \( \bar{s} = (12030) \) and one can easily check, using Theorem 3.8, that \( r \geq \bar{s} \). Again using 3.8, we see that \( r \geq (23010) = t \). Furthermore, \( t \) is maximum in the Adherence Order with the properties \( r \in H \) and \( r \geq t \).

REFERENCES