Trapdoor Hash Functions and Their Applications

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TPMPC 2019, Bar Ilan University

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Setting: Sender-Receiver Computation

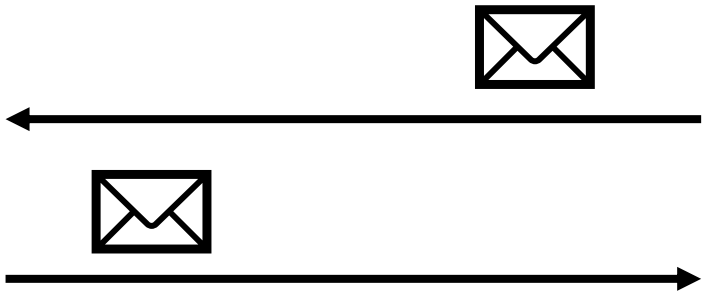
$f: X \times Y \to Z$

Sender

Receiver

$y \in Y$

$x \in X$



communication

Output:

$f(x, y)$

Setting: Sender-Receiver Computation

Focus: **two-message** protocols with **minimum communication**.

Sender

Input: $y \in Y$

receiver message

sender message

Receiver

Input: $x \in X$

Output: $f(x, y)$

Ideal World:

Simple and **optimal** solutions.

Real World (semi-honest security):

Sender and receiver do not trust each other, and want to keep their inputs **private**.

Main question in this work:

Can Secure Protocols be as Efficient as Ideal-World Solutions?

Can Secure Protocols be as Efficient as Ideal-World Solutions?

Setting I.

Sender input is larger than receiver input

$$|y| \gg |x|$$

Example:

String Oblivious Transfer (OT)

$y_0, y_1 \in \{0,1\}^n$ ⟵ $x \in \{0,1\}$

⟶ Output: y_x

In Ideal World, communication dominated by length of **second message** $|f(x,y)| = n$.

Goal

Optimize length of **second message**, i.e.

$$\textbf{download rate} = \frac{|f(x,y)|}{|Second\ Message|}$$

Ideal-World Solution:

download rate = 1

Negative Answer

Secure protocols with **exact** rate 1 **do not exist.**
(even with correlated randomness)

For 2^{λ} security,

$$|second\ message| > n + 2\lambda$$

Best we can hope for:

$$\textbf{download rate} \to 1$$

when $n \to \infty$.

Positive Answer

Protocols with rate approaching 1 using **Trapdoor Hash**.

Results in Setting I: Rate-1 Oblivious Transfer and More

Life before Trapdoor Hash

- Rate - ½ OT is easy to achieve.

- Only known solution for higher rate uses **high-rate homomorphic encryption**.
- Only rate-1 homomorphic encryption scheme: **Damgård-Jurik** scheme based on **DCR**.

Exception: [Gentry-Halevi'19]

	DCR	LWE	QR	DDH
Oblivious Transfer (OT)	**rate-1 [DJ00]**	**rate-½**	**rate-½**	**rate-½**

Life after Trapdoor Hash

	DCR	LWE	QR	DDH
Oblivious Transfer (OT)	**rate-1**	**rate-1**	**rate-1**	**rate-1**
Oblivious Linear-Function Evaluation (OLE)	**rate-1**	**rate-1**	**rate-1**	**rate-1**
Oblivious Matrix-Vector Product (OMV)	**-**	**rate-1**	**rate-1**	**-**

Results in Setting I: Application of Rate-1 OT

1. Single-Server Private Information Retrieval [Kushilevitz-Ostrovsky'97]
First **rate-1** PIR protocols with **polylogarithmic communication** [Ishai-Paskin'07].

	DCR	LWE	QR	DDH
Before TDH	$O(\log^c n)$, **rate-1**	$O(\log^c n)$	$O(2^{\sqrt{\log n}})$ **[KO97]**	$O(2^{\sqrt{\log n}})$ **[KO97]**
After TDH	$O(\log^c n)$, **rate-1**	$O(\log^c n)$, **rate-1**	$O(\log^c n)$, **rate-1**	$O(\log^c n)$, **rate-1**

2. Homomorphic Encryption for Branching Programs
First **semi-compact** homomorphic encryption from **DDH,QR** [Ishai-Paskin'07].

3. Lossy Trapdoor Functions [Peikert-Waters'07]
First **rate-1** constructions.

	DCR	LWE	QR	DDH
Before TDH	**rate-1**	-	-	**constant rate [GGH18]**
After TDH	**rate-1**	**rate-1**	**rate-1**	**rate-1**

Can Secure Protocols be as Efficient as Ideal-World Solutions?

Setting I.

Sender input is larger than receiver input

$$|y| \gg |x|$$

Example:

String Oblivious Transfer (OT)

In Ideal World, communication dominated by length of **second message** $|f(x,y)| = n$.

Goal

Optimize length of **second message**, i.e.

$$\textbf{download rate} = \frac{|f(x,y)|}{|Second\ Message|}$$

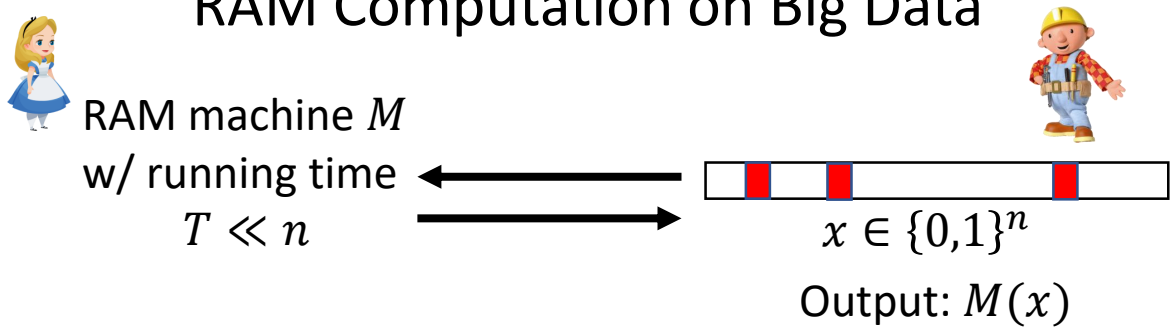
Setting II.

Receiver input is larger than sender input

$$|y| \ll |x|$$

Example:

RAM Computation on Big Data



(DNA analysis, suspects lookup, etc.)

Can Secure Protocols be as Efficient as Ideal-World Solutions?

Setting I.

Sender input is larger than receiver input

$$|y| \gg |x|$$

Example:

String Oblivious Transfer (OT)

In Ideal World, communication dominated by length of **second message** $|f(x,y)| = n$.

Goal

Optimize length of **second message**, i.e.

$$\textbf{download rate} = \frac{|f(x,y)|}{|Second\ Message|}$$

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Receiver input is larger than sender input

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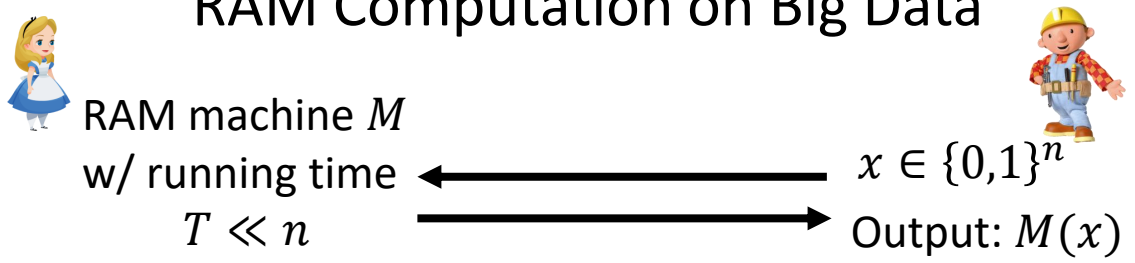
Example:

RAM Computation on Big Data



In Ideal World, communication is $|y| \ll n$:

"much smaller" than n

Goal

Communication **smaller than $|x| = n$.**

Non-trivial for **two-message** protocols.

Results in Setting II: Sublinear Secure RAM Computation

Life before Trapdoor Hash

Only **fully secure** solution is based on **lattice assumptions**.

	Overall Com.	Assumption	Security
Laconic Function Evaluation [QWW18]	$\tilde{O}(T)$	**LWE**	**full**
Laconic OT [CDGGMP17]	$\tilde{O}(T)$	**DDH**	**UMA**

Life after Trapdoor Hash

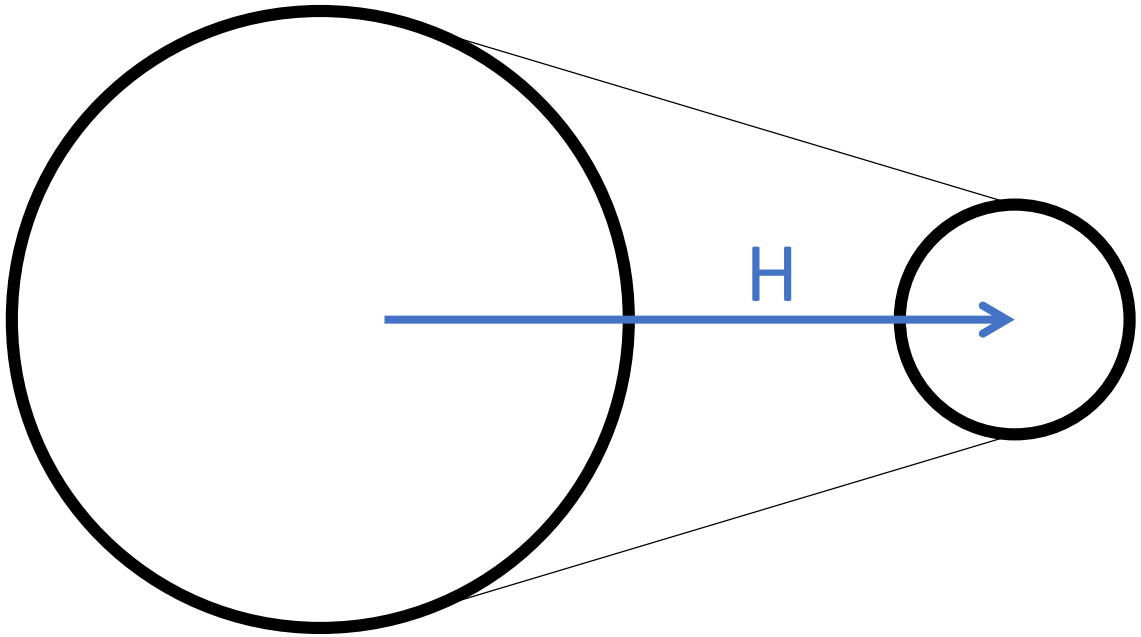
First sublinear two-message protocols under **number-theoretic assumptions**.

Private Laconic OT (through TDH)	$O(T \cdot \sqrt{n})$	**DDH**	**full**
	$O(T \cdot \sqrt[3]{n})$	**SXDH+**	**full**

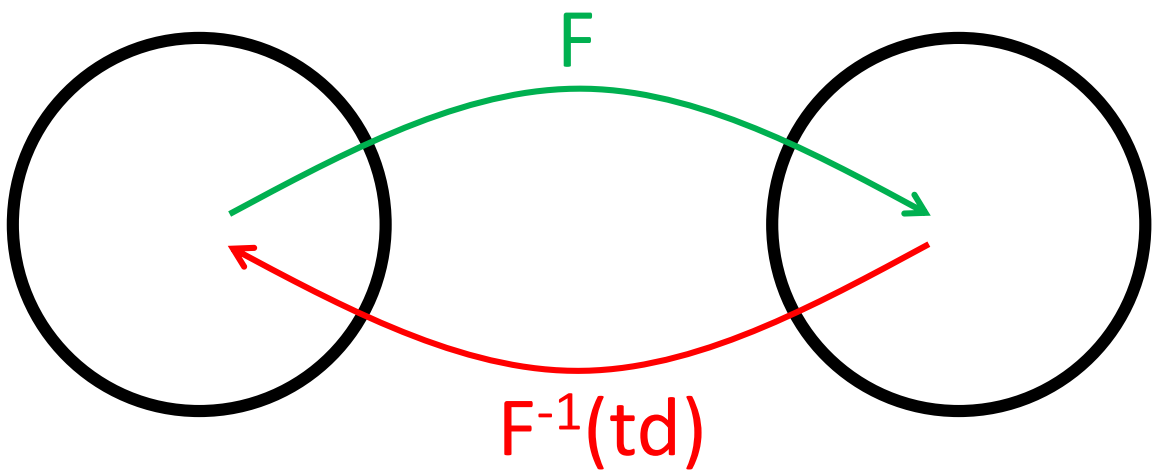
Open Question (already): can we close the efficiency gap?

Trapdoor Hash Functions

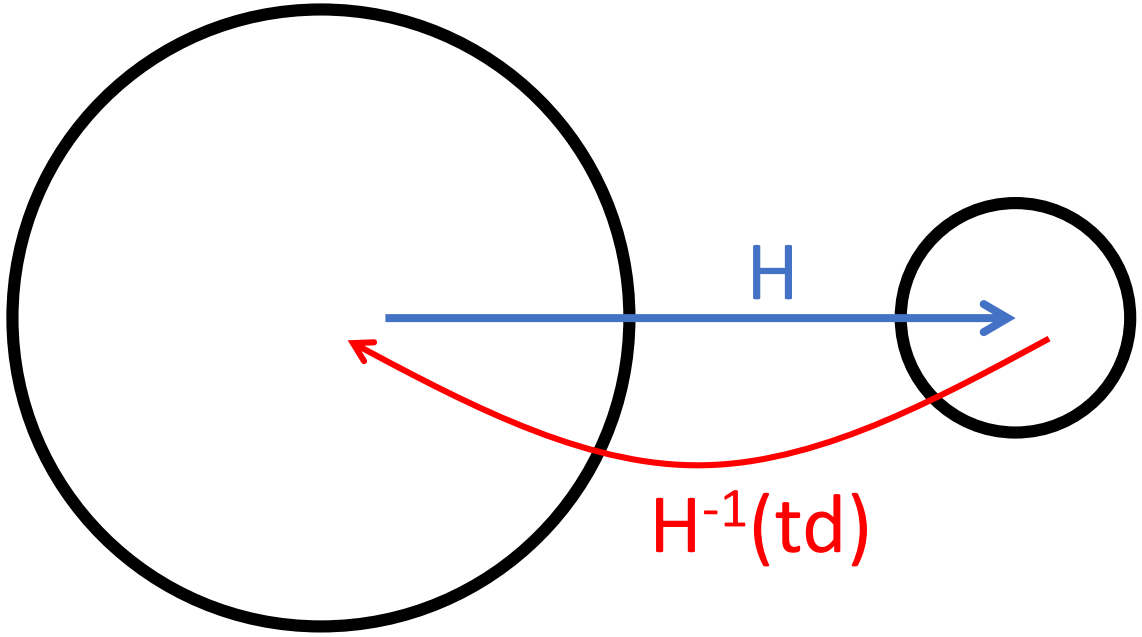
Hash Functions



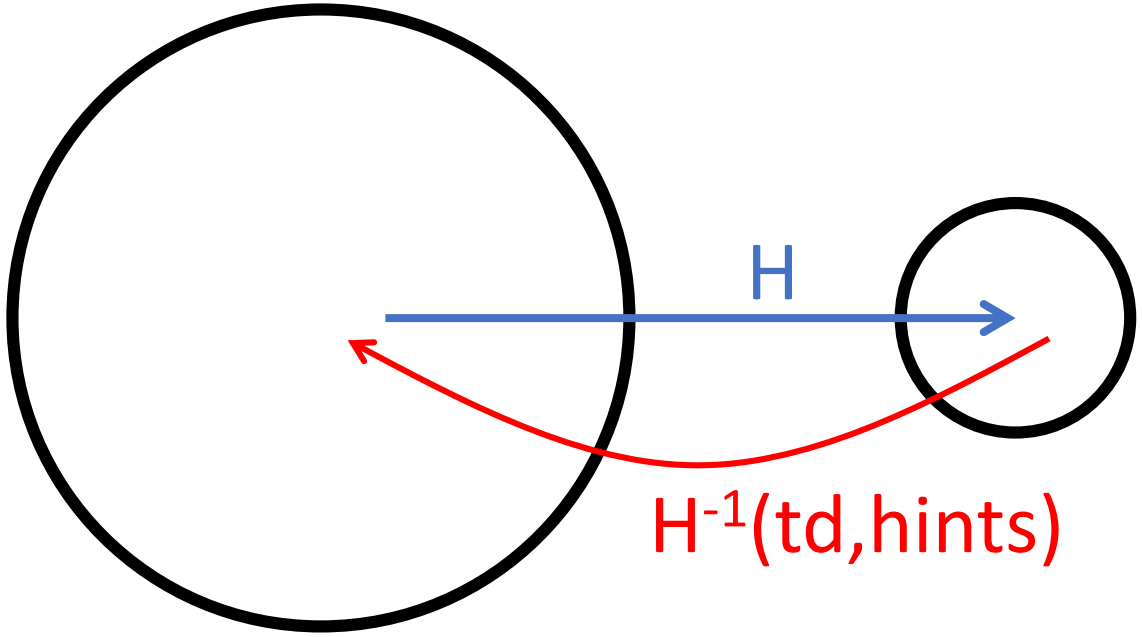
Trapdoor Functions



Trapdoor Hash Functions



Trapdoor Hash Functions



Defining Trapdoor Hash

Alice

Bob

The Hash Function:

Input Privacy: h hides x.

h

h

"I want to learn $x[i]$, give me some hints"

key

Key Generation:

(key, trapdoor)← **G**(i)

Index Privacy: key hides i.

The "Hinting" Function:

e

Decoding:

$x[i] \leftarrow$ **D**(trapdoor, h, e)

Efficiency: small hints, i.e. **high rate**

Defining Trapdoor Hash

Alice

Bob

The Hash Function:

Input Privacy: h hides x.

h

h

"I want to learn $x[i_1, \ldots, i_m]$, give me some hints"

$\text{key}_1, \ldots, \text{key}_m$

Key Generation:

$(\text{key}_j, \text{trapdoor}_j) \leftarrow \mathbf{G}(i_j)$

Index Privacy: key hides i.

The "Hinting" Function:

$e_1, \ldots, e_m$

Decoding:

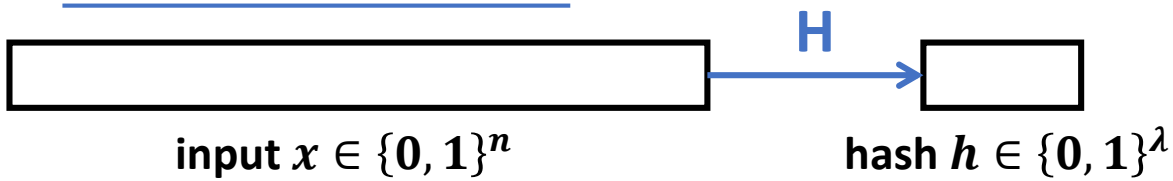
$x[i_j] \leftarrow \mathbf{D}(\text{trapdoor}_j, h, e_j)$

Efficiency: small hints, i.e. **high rate**

Defining Trapdoor Hash

TDH = (H,G,E,D)

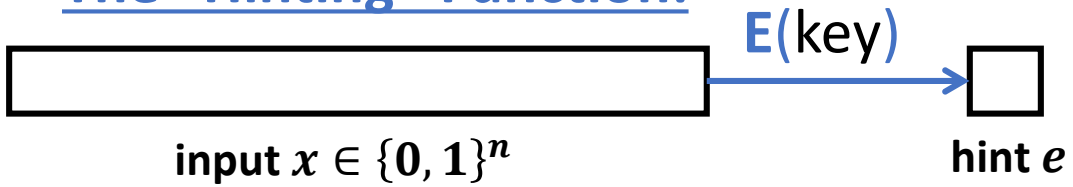
The Hash Function:



Key Generation:

(key, trapdoor)← **G**(i)

The "Hinting" Function:



Decoding:

$x[i] \leftarrow$ **D**(trapdoor, h, e)

Input Privacy: h hides x.

Index Privacy: key hides i.

Rate:

$$\frac{\mathbf{1}}{|\boldsymbol{e}|}$$

Optimally, **rate = 1**.

Main technical contribution:

Rate-1 TDH from **DDH,QR,LWE,DCR**

We also consider TDH for general functions of x.

Trapdoor Hash from DDH

Similar technique used to construct **IBE [DG17], laconic OT [CDGGMP17]** and **Trapdoor Functions [GH18,GGH19]**.

- **Multiplicative abelian group $\mathbb{G}$ of prime order p, with a public generator $g \in \mathbb{G}$.**
  For all $g_1, g_2 \in \mathbb{G}$,

$$\boldsymbol{g_1 \cdot g_2 = g_2 \cdot g_1 \in \mathbb{G}}$$

- **The DDH (Decisional Diffie-Hellman) assumption**
  For **uniform** $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{Z}_p$ and an element $g \in \mathbb{G}$,

$$\left(\boldsymbol{g^a, g^b, g^{ab}}\right) \equiv \left(\boldsymbol{g^a, g^b, g^c}\right)$$

Trapdoor Hash from DDH

Alice

public parameters

$2 \times n$ uniform group elements

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix} \in \mathbb{G}^{2\times n}$$

Bob

input $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{n}}$

Trapdoor Hash from DDH

Alice

public parameters

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix}$$

Bob

Hash Function:

$$H(x) = \prod_j g_{j,x[j]}$$

$h = H(x)$ →

$$\begin{pmatrix} g_{1,0} & \boldsymbol{g_{2,0}} & \boldsymbol{g_{3,0}} & \cdots & g_{n,0} \\ \boldsymbol{g_{1,1}} & g_{2,1} & g_{3,1} & \cdots & \boldsymbol{g_{n,1}} \end{pmatrix}$$

1	0	0	⋯	1

H → $h \in \mathbb{G}$

input $\boldsymbol{x \in \{0,1\}^n}$

Input Privacy: statistical*.

Trapdoor Hash from DDH

Alice

public parameters

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix}$$

Bob

Hash Function:

$$H(x) = \prod_j g_{j,x[j]}$$

$$h = H(x) \longrightarrow$$

Key Generation:

"I want to learn $x[i]$"

$$key = \begin{pmatrix} \tilde{g}_{1,0} & \tilde{g}_{2,0} & \tilde{g}_{3,0} & \cdots & \tilde{g}_{n,0} \\ \tilde{g}_{1,1} & \tilde{g}_{2,1} & \tilde{g}_{3,1} & \cdots & \tilde{g}_{n,1} \end{pmatrix} \longleftarrow$$

trapdoor: uniform $t \in \mathbb{Z}_p$

$$key = \begin{pmatrix} g_{1,0}^t & \cdots & g_{i,0}^t & \cdots & g_{n,0}^t \\ g_{1,1}^t & \cdots & g_{i,1}^t \cdot g & \cdots & g_{n,1}^t \end{pmatrix}$$

Index Privacy: assuming DDH,

$$\begin{pmatrix} \tilde{g}_{1,0} & \tilde{g}_{2,0} & \tilde{g}_{3,0} & \cdots & \tilde{g}_{n,0} \\ \tilde{g}_{1,1} & \tilde{g}_{2,1} & \tilde{g}_{3,1} & \cdots & \tilde{g}_{n,1} \end{pmatrix} \equiv \text{uniform matrix in } \mathbb{G}^{2\times n}$$

Trapdoor Hash from DDH

Alice

public parameters

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix}$$

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$$key = \begin{pmatrix} g_{1,0}^t & \cdots & g_{i,0}^t & \cdots & g_{n,0}^t \\ g_{1,1}^t & \cdots & g_{i,1}^t \cdot g & \cdots & g_{n,1}^t \end{pmatrix}$$

Hinting:

$$E(key, x) = \prod_j \tilde{g}_{j,x[j]}$$

$$\begin{pmatrix} \tilde{g}_{1,0} & \tilde{g}_{2,0} & \tilde{g}_{3,0} & \cdots & \tilde{g}_{n,0} \\ \tilde{g}_{1,1} & \tilde{g}_{2,1} & \tilde{g}_{3,1} & \cdots & \tilde{g}_{n,1} \end{pmatrix}$$

1	0	0	⋯	1

input $x \in \{0,1\}^n$

E

$$e = E(key, x)$$

$$e \in \mathbb{G}$$

Rate:

$$\frac{1}{|e|} = \frac{1}{\lambda}$$

Trapdoor Hash from DDH

Alice

public parameters

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix}$$

Bob

Hash Function:

$$H(x) = \prod_j g_{j,x[j]}$$

$$h = H(x)$$

Key Generation:

"I want to learn $x[i]$"

$$key = \begin{pmatrix} \tilde{g}_{1,0} & \tilde{g}_{2,0} & \tilde{g}_{3,0} & \cdots & \tilde{g}_{n,0} \\ \tilde{g}_{1,1} & \tilde{g}_{2,1} & \tilde{g}_{3,1} & \cdots & \tilde{g}_{n,1} \end{pmatrix}$$

trapdoor: uniform $t \in \mathbb{Z}_p$

$$key = \begin{pmatrix} g_{1,0}^t & \cdots & g_{i,0}^t & \cdots & g_{n,0}^t \\ g_{1,1}^t & \cdots & g_{i,1}^t \cdot g & \cdots & g_{n,1}^t \end{pmatrix}$$

Hinting:

$$E(key, x) = \prod_j \tilde{g}_{j,x[j]}$$

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1	0	0	⋯	1

input $x \in \{0,1\}^n$

E

$$e = E(key, x)$$

$$e \in \mathbb{G}$$

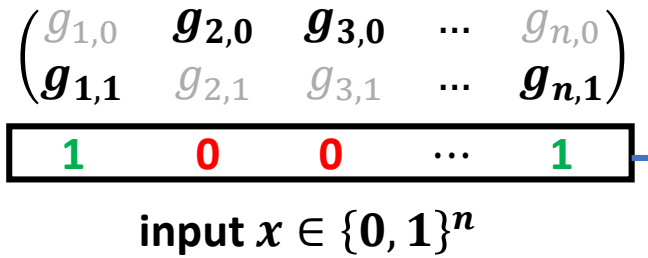
How to recover $x[i]$ given

- The **hash** h
- The **hint** e
- The **trapdoor** t ?

Trapdoor Hash at The Bar: Decoding

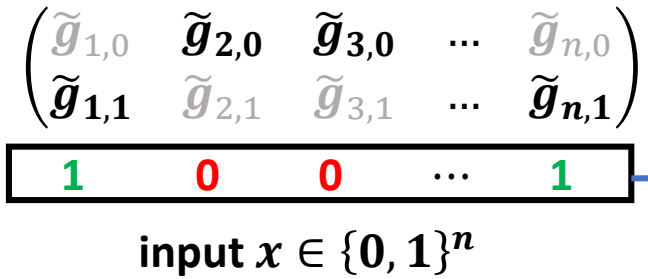
Observation: given that

$$\begin{pmatrix} \tilde{g}_{1,0} & \tilde{g}_{2,0} & \tilde{g}_{3,0} & \cdots & \tilde{g}_{n,0} \\ \tilde{g}_{1,1} & \tilde{g}_{2,1} & \tilde{g}_{3,1} & \cdots & \tilde{g}_{n,1} \end{pmatrix} = \begin{pmatrix} g_{1,0}^t & \cdots & g_{i,0}^t & \cdots & g_{n,0}^t \\ g_{1,1}^t & \cdots & g_{i,1}^t \cdot g & \cdots & g_{n,1}^t \end{pmatrix}$$



H

$$h = \prod_j g_{j,x[j]}$$



E

$$e = \prod_j \tilde{g}_{j,x[j]} = \prod_j g_{j,x[j]}^t \cdot g^{x[i]} = h^t \cdot g^{x[i]}$$

Trapdoor Hash from DDH

Alice

public parameters

$$\begin{pmatrix} g_{1,0} & g_{2,0} & g_{3,0} & \cdots & g_{n,0} \\ g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{n,1} \end{pmatrix}$$

Bob

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trapdoor: uniform $t \in \mathbb{Z}_p$

$$\boldsymbol{key} = \begin{pmatrix} g_{1,0}^t & \cdots & g_{i,0}^t & \cdots & g_{n,0}^t \\ g_{1,1}^t & \cdots & \boldsymbol{g_{i,1}^t \cdot g} & \cdots & g_{n,1}^t \end{pmatrix}$$

Hinting:

$$E(key, x) = \prod_j \tilde{\boldsymbol{g}}_{j,x[j]}$$

$$e = E(key, x) \longrightarrow$$

Decoding: using hash h and trapdoor t

$$\boldsymbol{e = h^t} \quad \rightarrow \boldsymbol{x[i] = 0}$$

$$\boldsymbol{e = h^t \cdot g} \quad \rightarrow \boldsymbol{x[i] = 1}$$

Rate-$1/\lambda$ Trapdoor Hash

- For applications in **Setting II (Sublinear Secure RAM Computation)**:
  Rate-$1/\lambda$ TDH **is sufficient***.

- For applications in **Setting I (Rate-1 OT)**:
  We need **Rate-1 TDH**, i.e. TDH where the hint is a single bit.

Rate-1 Trapdoor Hash from DDH

Alice

Bob

Hinting:

$$e = \prod_i \tilde{\boldsymbol{g}}_{i,x[i]}$$

$x[i] = 0$ → $\boldsymbol{h^t} \in \mathbb{G}$

$x[i] = 1$ → $\boldsymbol{h^t \cdot g} \in \mathbb{G}$

$$e = E(key, x)$$

Decoding: using hash h and trapdoor t

e

$= h^t$? → $\boldsymbol{x[i] = 0}$

$= h^t \cdot g$? → $\boldsymbol{x[i] = 1}$

Rate-1 Trapdoor Hash from DDH

Alice

Bob

Hinting:

$$e = \prod_i \tilde{\boldsymbol{g}}_{i,x[i]}$$

$x[i] = 0$ ↙ $\boldsymbol{h^t} \in \mathbb{G}$

$x[i] = 1$ ↘ $\boldsymbol{h^t \cdot g} \in \mathbb{G}$

$LSB(e)$ →

Decoding: using hash h and trapdoor t

$LSB(e)$

$= LSB(h^t)\,?$ ↙ $\boldsymbol{x[i] = 0}$

$= LSB(h^t g)?$ ↘ $\boldsymbol{x[i] = 1}$

Attempt I:

Send **LSB** of e.

Fails when $LSB(h^t) = LSB(h^t g)$ is **equal**: happens with probability $1/2$.

Rate-1 Trapdoor Hash from DDH

Alice

Bob

Hinting:

$$e = \prod_i \tilde{\boldsymbol{g}}_{i,x[i]}$$

$x[i] = 0$ → $\boldsymbol{h^t} \in \mathbb{G}$

$x[i] = 1$ → $\boldsymbol{h^t \cdot g} \in \mathbb{G}$

$\Phi(e)$ →

Decoding: using hash h and trapdoor t

$\Phi(e)$

$= \Phi(h^t)$? → $\boldsymbol{x[i] = 0}$

$= \Phi(h^t g)$? → $\boldsymbol{x[i] = 1}$

Attempt I:

Send **LSB** of e.

Fails when $LSB(h^t) = LSB(h^t g)$ is **equal**: happens with probability $1/2$.

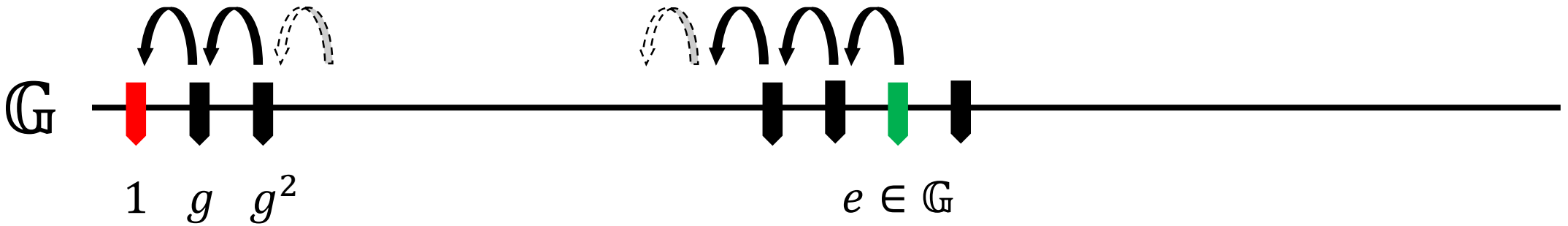
Goal: an encoding $\Phi: \mathbb{G} \to \{0,1\}$ such that with high probability

$$\Phi(h^t) \neq \Phi(h^t g)$$

Rate-1 Trapdoor Hash from DDH

Goal: an encoding $\Phi: \mathbb{G} \to \{0,1\}$ such that with high probability

$$\Phi(h^t) \neq \Phi(h^t g)$$



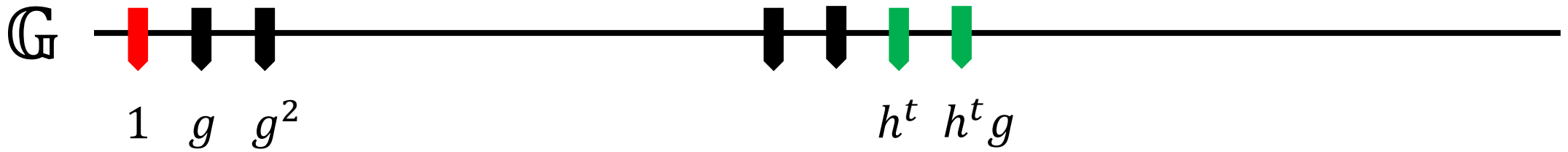
Attempt II:

$\Phi(e) =$ LSB of **number of steps to reach 1**

Rate-1 Trapdoor Hash from DDH

Goal: an encoding $\Phi: \mathbb{G} \to \{0,1\}$ such that with high probability

$$\Phi(h^t) \neq \Phi(h^t g)$$



Attempt II:

$$\Phi(e) = \text{LSB of } \textbf{number of steps to reach 1}$$

Clearly, $\Phi(h^t) \neq \Phi(h^t g)$ for all $t \in \mathbb{Z}_p$.

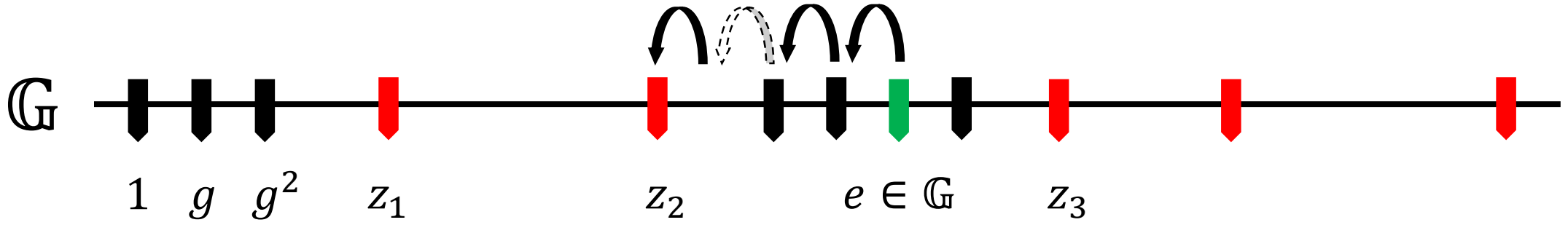
Problem: **not efficient** (this is DLOG).

Rate-1 Trapdoor Hash from DDH

Goal: an encoding $\Phi: \mathbb{G} \to \{0,1\}$ such that with high probability

$$\Phi(h^t) \neq \Phi(h^t g)$$

Idea: distributed discrete log [BGI16,DKK18]



Attempt III: using a PRF, define many random "**reference points**": $\boldsymbol{z}$ **s.t.** $\boldsymbol{PRF(z) = 0}$.

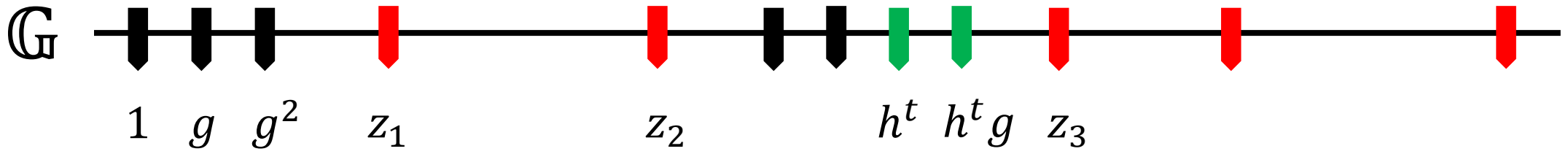
$\Phi(e) =$ LSB of **number of steps to reach the closest reference point**

Rate-1 Trapdoor Hash from DDH

Goal: an encoding $\Phi: \mathbb{G} \to \{0,1\}$ such that with high probability

$$\Phi(h^t) \neq \Phi(h^t g)$$

Idea: distributed discrete log [BGI16,DKK18]



Attempt III: using a PRF, define many random "**reference points**": $\boldsymbol{z}$ **s.t.** $\boldsymbol{PRF(z) = 0}$.

$\Phi(e) =$ LSB of **number of steps to reach the closest reference point**

If the reference points are **dense enough**, this is **efficient**.
If the reference points are **sparse enough**, $\Phi(h^t) \neq \Phi(h^t g)$ **with high probability**.

Conclusion and Further Research

We introduce **Trapdoor Hash**:

- Simple primitive with **constructions from various standard assumption**.
- Powerful primitive with **significant applications** in communication-efficient protocols.

Applications in **Setting I: Rate-Optimal Protocols for OT and Other Functions**:

- Constructions from new standard assumptions.
- More **general functionalities**?
- Constructions from **general assumptions**?

Applications in **Setting II: Sublinear Protocols for Secure RAM Computation**:

- First constructions from number-theoretic assumptions.
- Can we get constructions **as efficient as lattice-based schemes**?

Other applications of trapdoor hash or similar techniques?

Thanks for Listening!