Efficient Secure Ridge Regression from Randomized Gaussian Elimination

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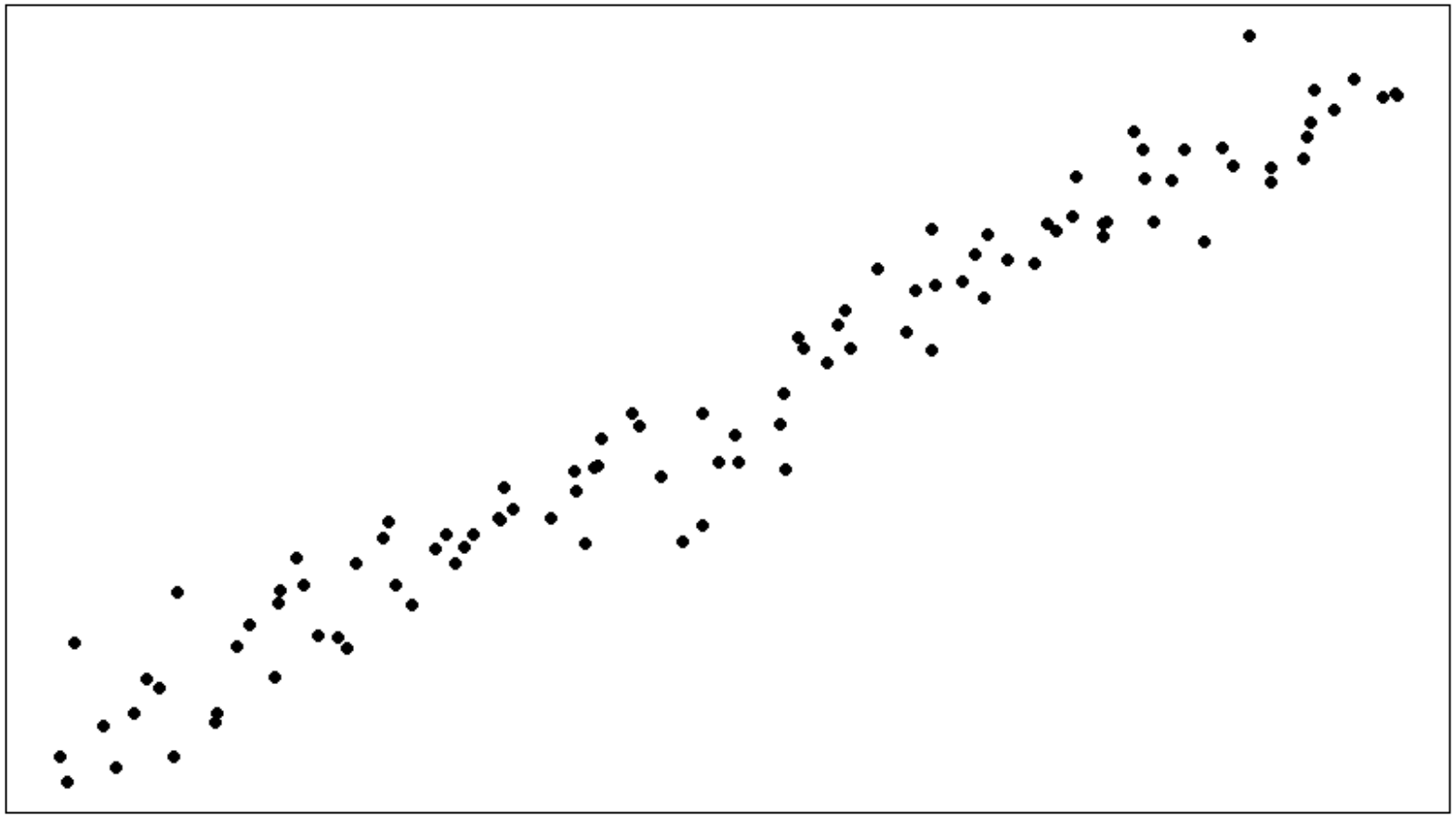
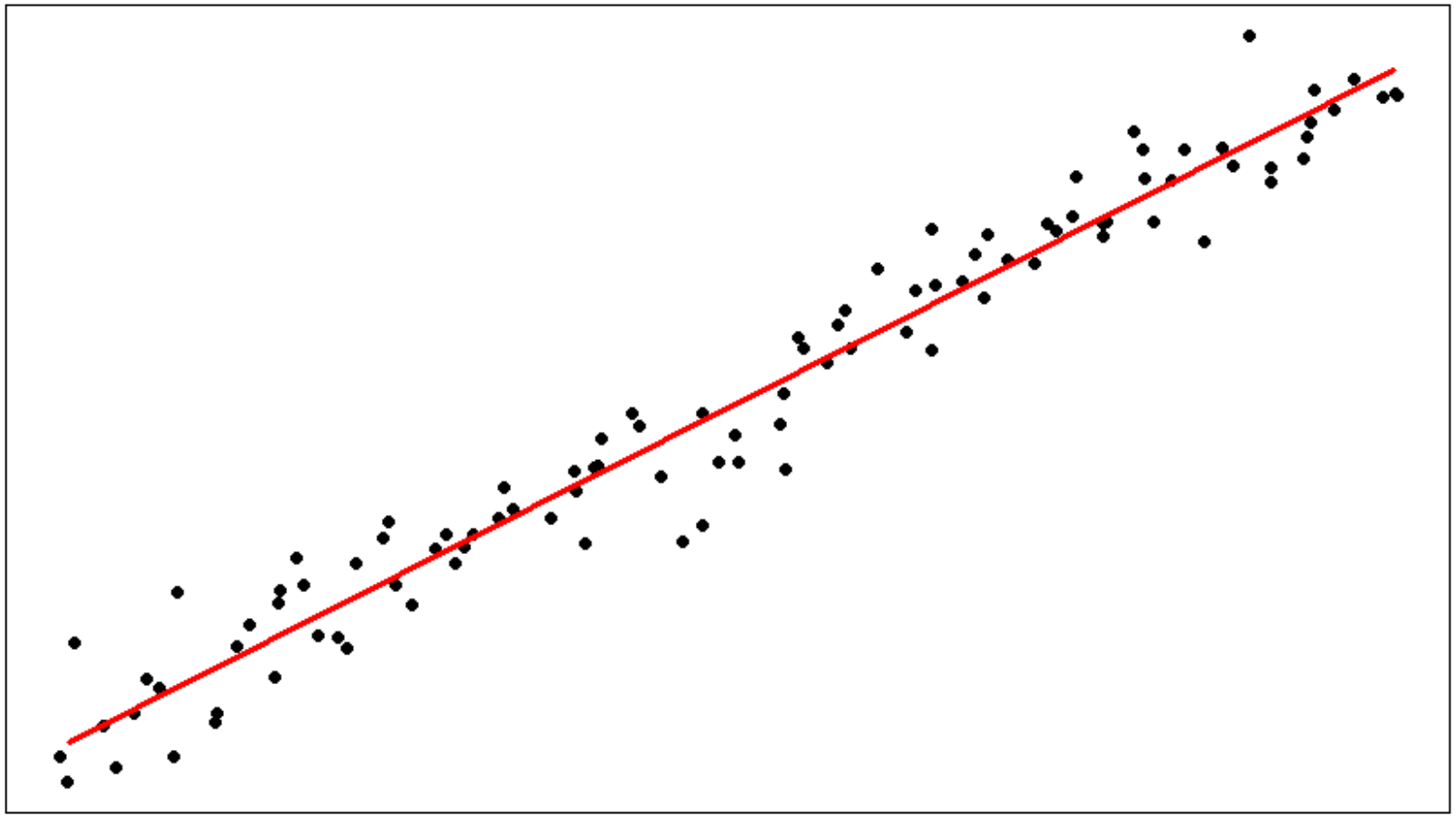
Linear regression



Linear regression



Multi-linear regression

UCI Machine Learning Repository
Center for Machine Learning and Intelligent Systems

residual sugar	chlorides	pH	alcohol	wine quality
1.50	0.091	3.25	9.6	7
3.50	0.029	3.53	11.7	3
12.30	0.044	3.11	9.9	5
10.60	0.035	3.20	10.4	9

Data Set Characteristics:	Multivariate	**Number of Instances:**	4898	**Area:**	Business
Attribute Characteristics:	Real	**Number of Attributes:**	12	**Date Donated**	2009-10-07
Associated Tasks:	Classification, Regression	**Missing Values?**	N/A	**Number of Web Hits:**	943674

Multi-linear regression

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$X \in \mathbb{R}^{n \times d}$

$y \in \mathbb{R}^{n}$

Multi-linear regression

$X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^n$ represent an n x d over-determined system:

$$Xw = y \ (n \gg d)$$

Least-squares solution:

$$\arg\min_w \{\|Xw - y\|_2\}$$

Least-squares solution is an unbiased estimator, but might have high variance

Ridge regression: reduce variance (at the cost of some bias)

Ridge regression

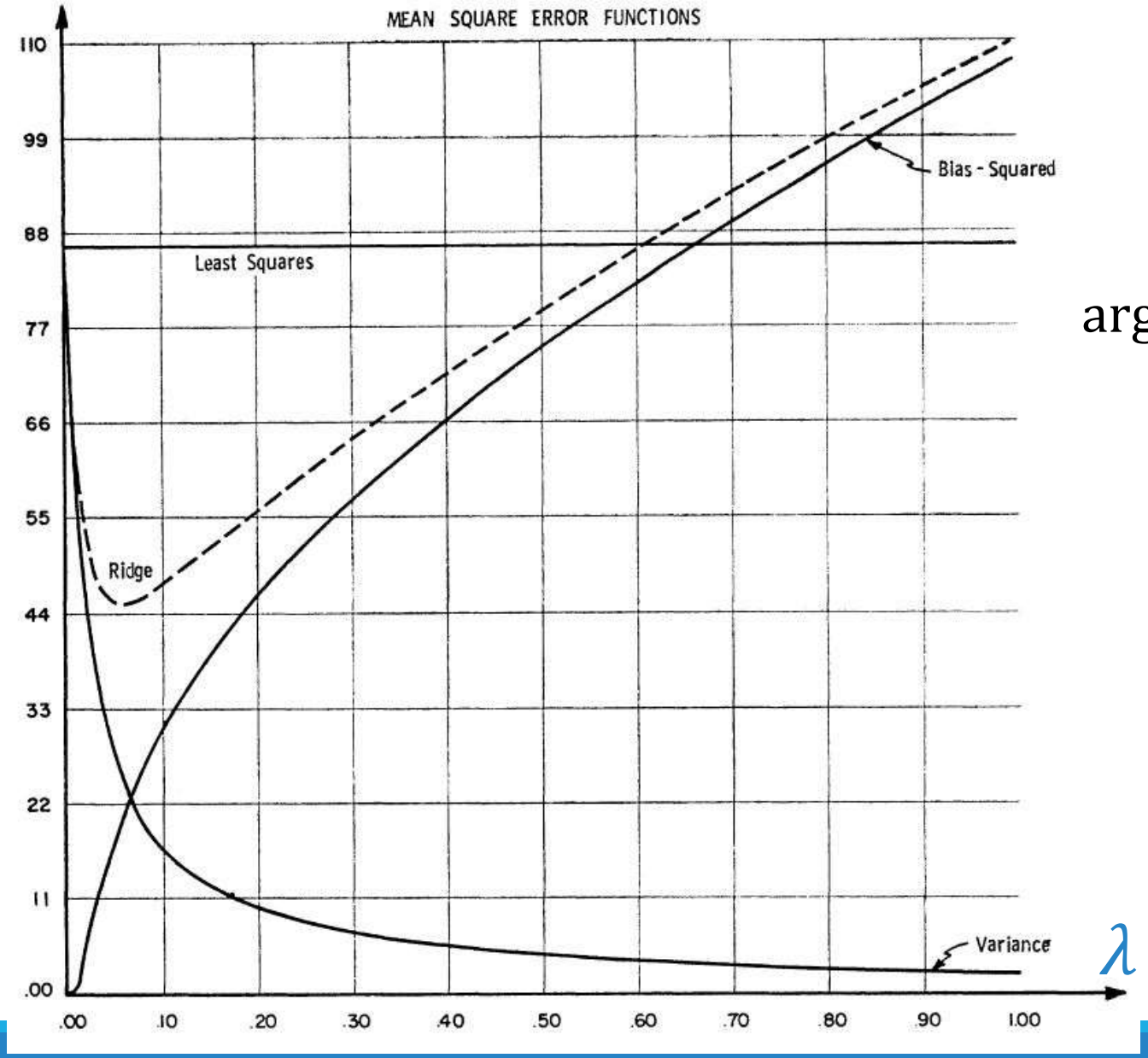
$X \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^n$ represent an n x d over-determined system:

$$Xw = y \ (n \gg d)$$

Regularized least-squares solution:

$$\arg\min_w \{\|Xw - y\|_2 + \lambda \|w\|_2\}_{\lambda > 0}$$

Bias-variance trade-off



$$\arg\min_w \{\|Xw - y\|_2 + \lambda\|w\|_2\}_{\lambda > 0}$$

MSE = Bias^2 + variance + noise

How to solve Ridge Regression?

Iterative method:

- possibly: numerical instability
- stopping condition

Direct method:

$$\arg\min_w \{\|Xw - y\|_2 + \lambda\|w\|_2\}_{\lambda > 0}$$

$$w = (X^T X + \lambda I)^{-1} X^T y$$

$$w = A^{-1} b$$

$$A = X^T X + \lambda I,$$
$$b = X^T y$$

MPC roadmap

- Design choices
- Converting input: real numbers to field elements
- Sketch of main ideas
- Choosing the modulus
- Experimental results

Ridge Regression as a Secure Multiparty Computation

- Arithmetic Black Box over a finite field with “cheap” inner product
- “Cheap” inner product: same round complexity as single secure multiplication
- Example: Shamir’s scheme

Notation: $[\![x]\!]$ represents a sharing of x

$[\![x]\!]_p$ represents a sharing of x with specific prime modulus p

Design choices

- Direct solution (as opposed to iterative solution)
- Avoid rational reconstruction
- No assumption on data partitioning

Horizontal Partitioning

Party i

residual sugar	chlorides	pH	alcohol	wine quality
3.50	0.029	3.53	11.7	3
12.30	0.044	3.11	9.9	5

Design choices

- Direct solution (as opposed to iterative solution)
- Avoid rational reconstruction
- No assumption on data partitioning

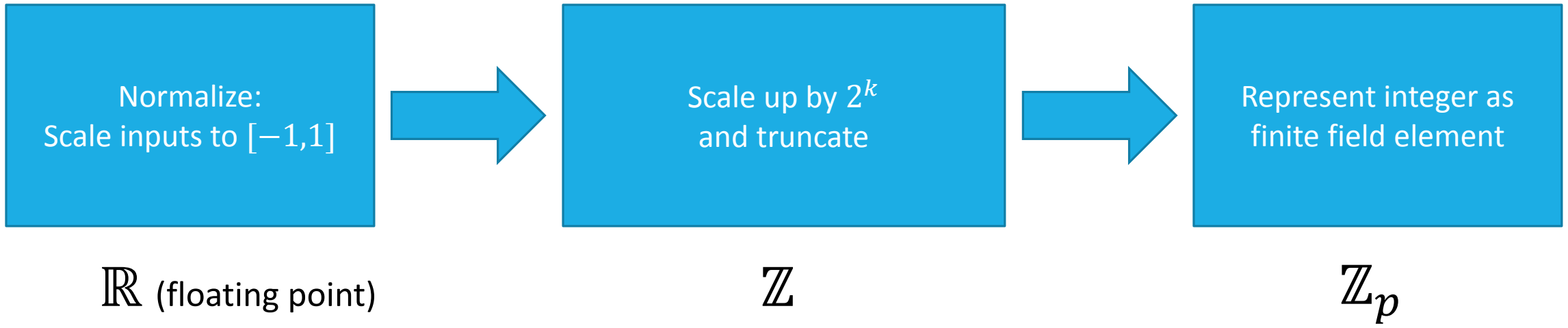
Vertical Partitioning

Party i

chlorides	pH
0.091	3.25
0.029	3.53
0.044	3.11
0.035	3.20

Input scaling

Plaintext precomputation:



Representing the output

$$w = A^{-1}b = \frac{1}{\det(A)} \operatorname{adj}(A)b$$

$$A \in \mathbb{Z}^{d\times d} \implies \operatorname{adj}(A) \in \mathbb{Z}^{d\times d}$$

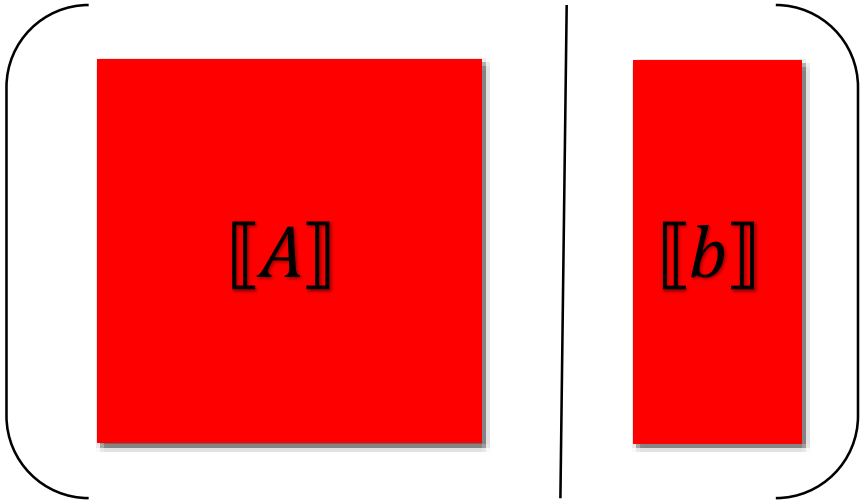
Represent w in two parts

Numerator: $\det(A)\, w\ (= \operatorname{adj}(A)b)$

Denominator: $\det(A)$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

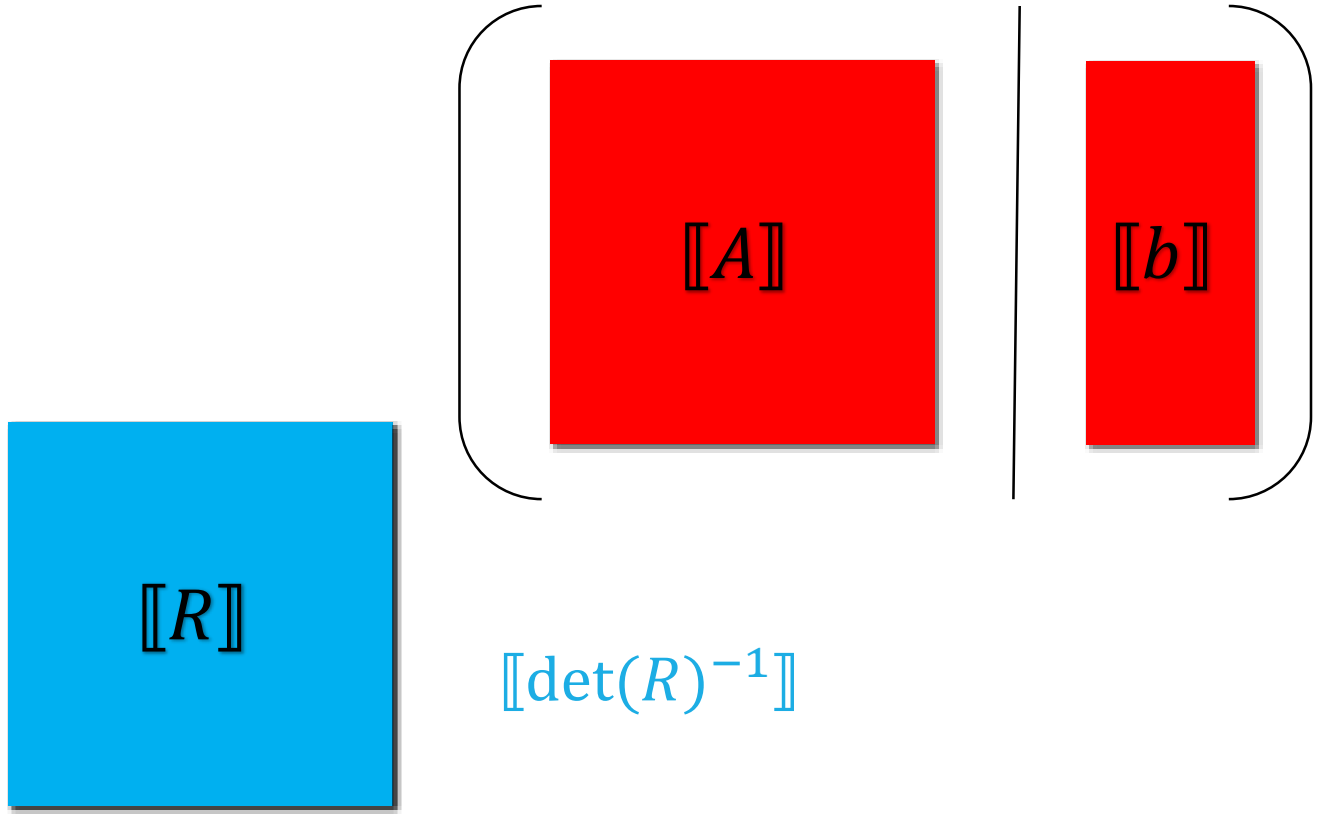


Return:([[adj$(A)b$]], [[det(A)]])

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$

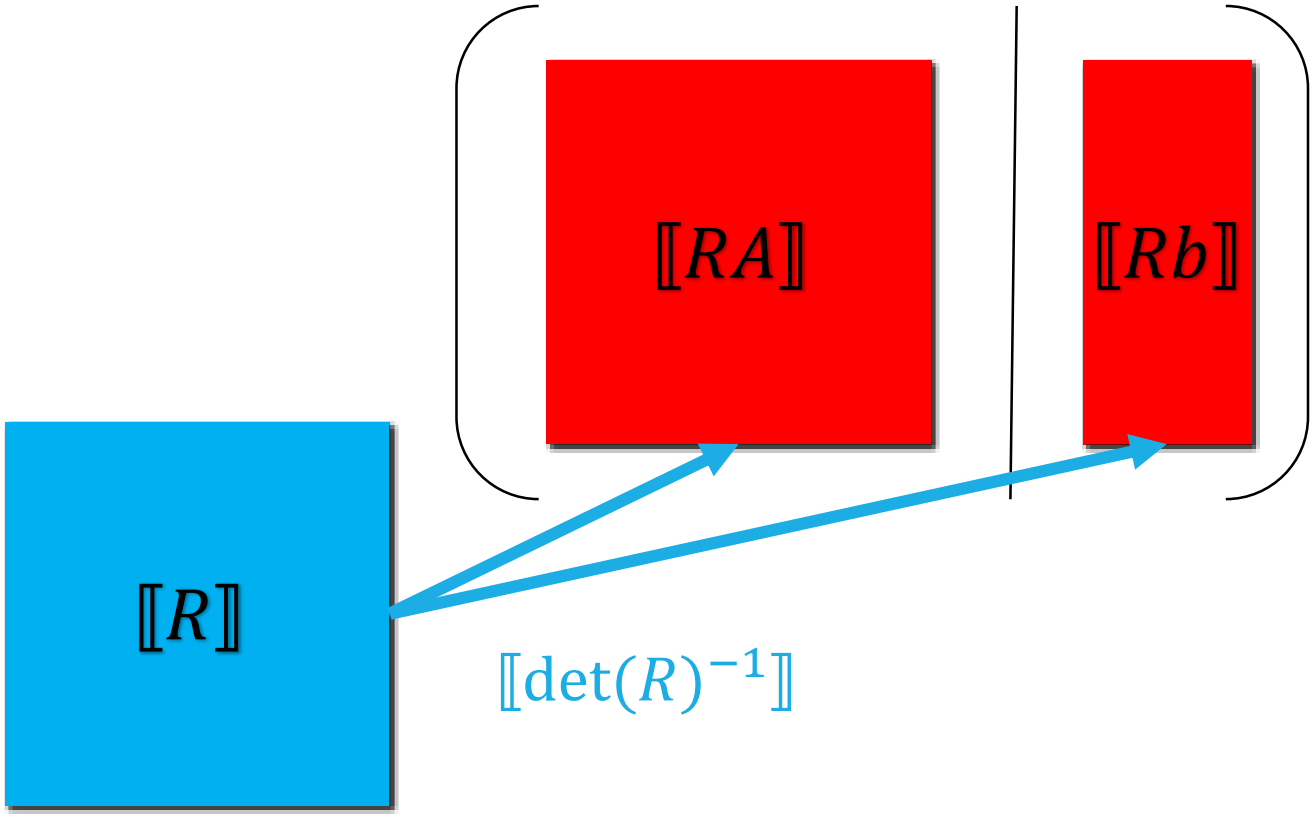


Return:$([\![\text{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$

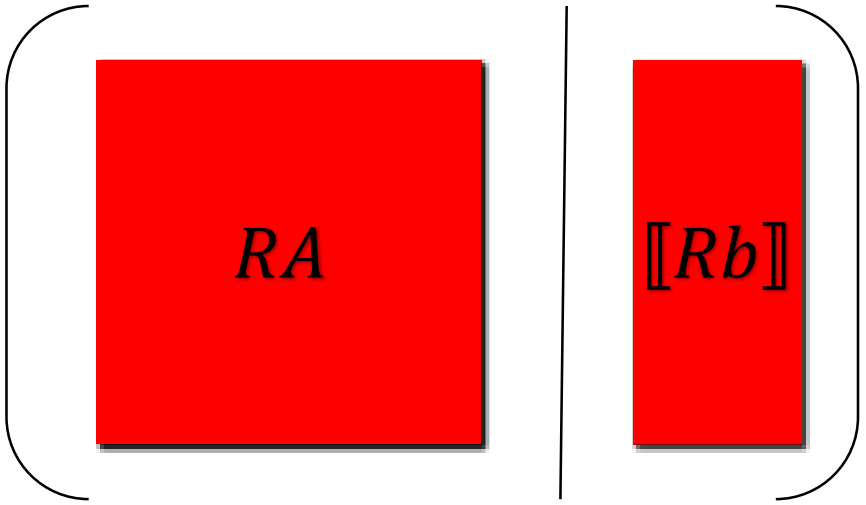


Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$
3. Open RA



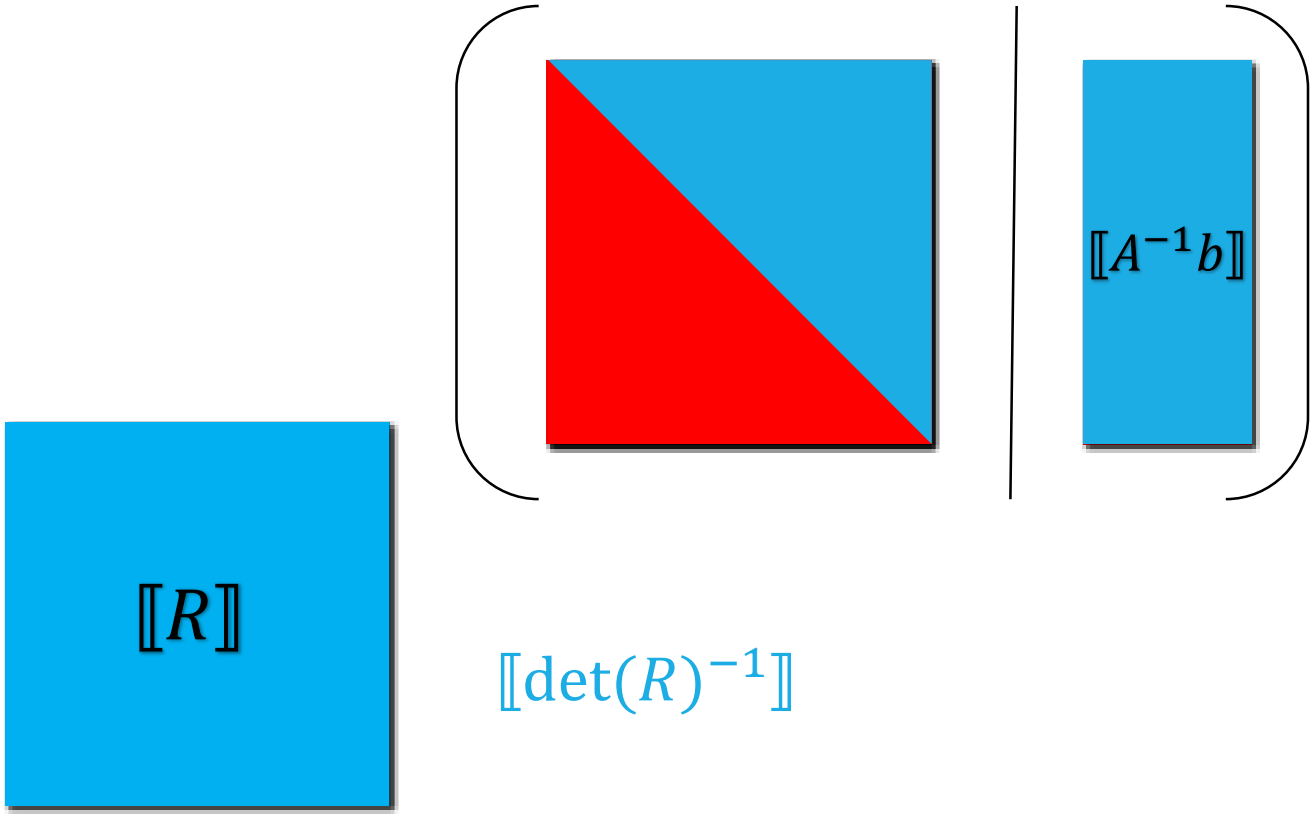
$[\![R]\!]$

$[\![\det(R)^{-1}]\!]$

Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

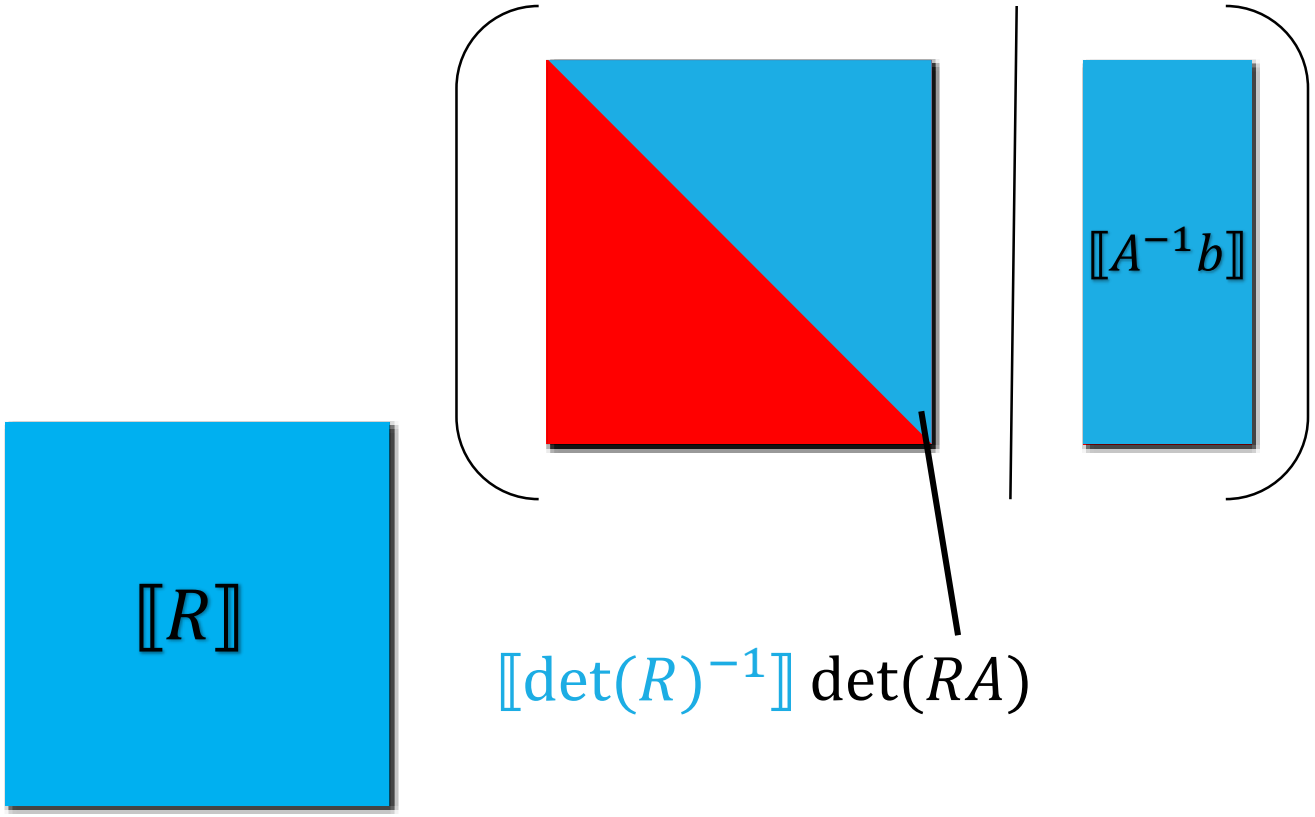


1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$
3. Open RA
4. Gaussian Elimination over $(RA|[\![Rb]\!])$

Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

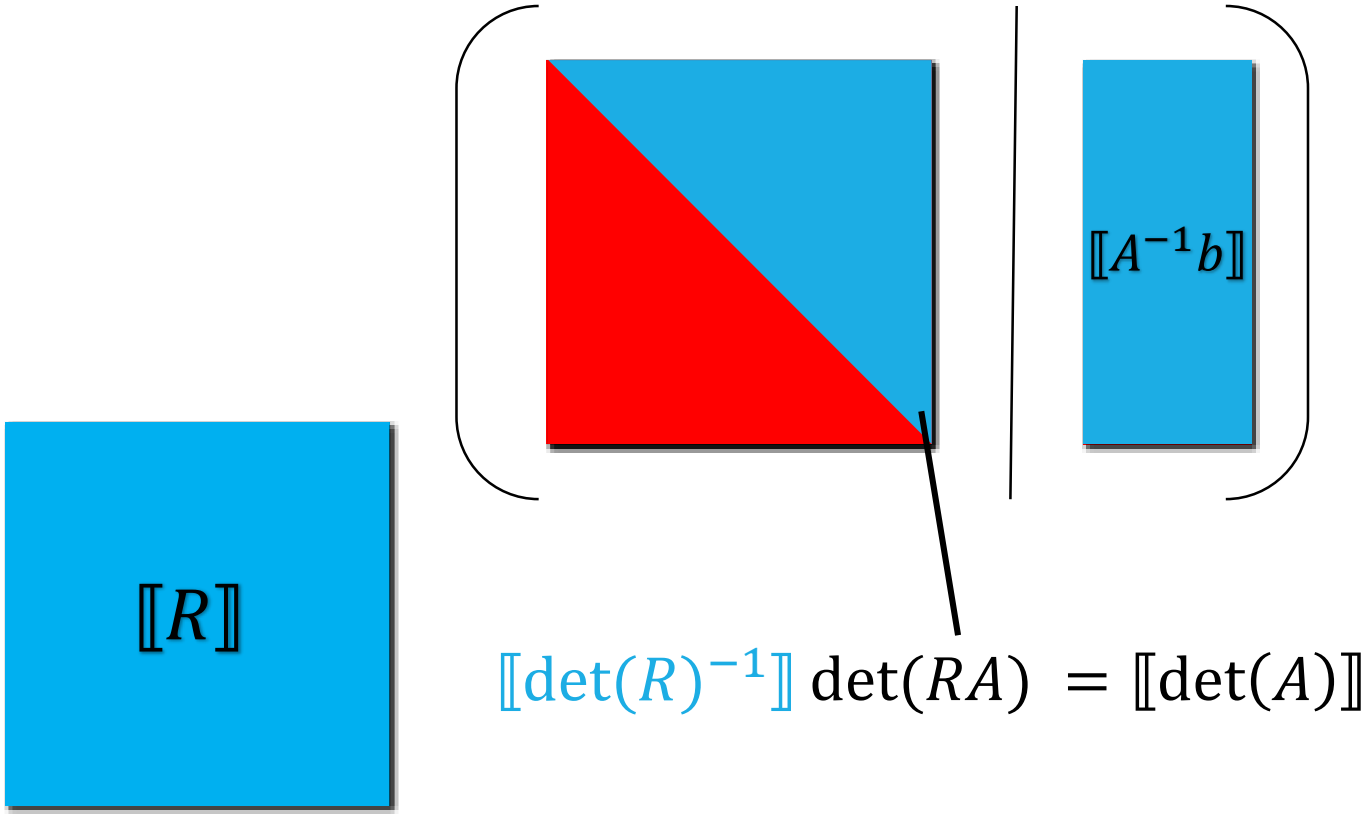


1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$
3. Open RA
4. Gaussian Elimination over $(RA|[\![Rb]\!])$
5. Extract $\det(RA)$

Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$



1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$
3. Open RA
4. Gaussian Elimination over $(RA|[\![Rb]\!])$
5. Extract $\det(RA)$
6. Compute $[\![\det(A)]\!]$

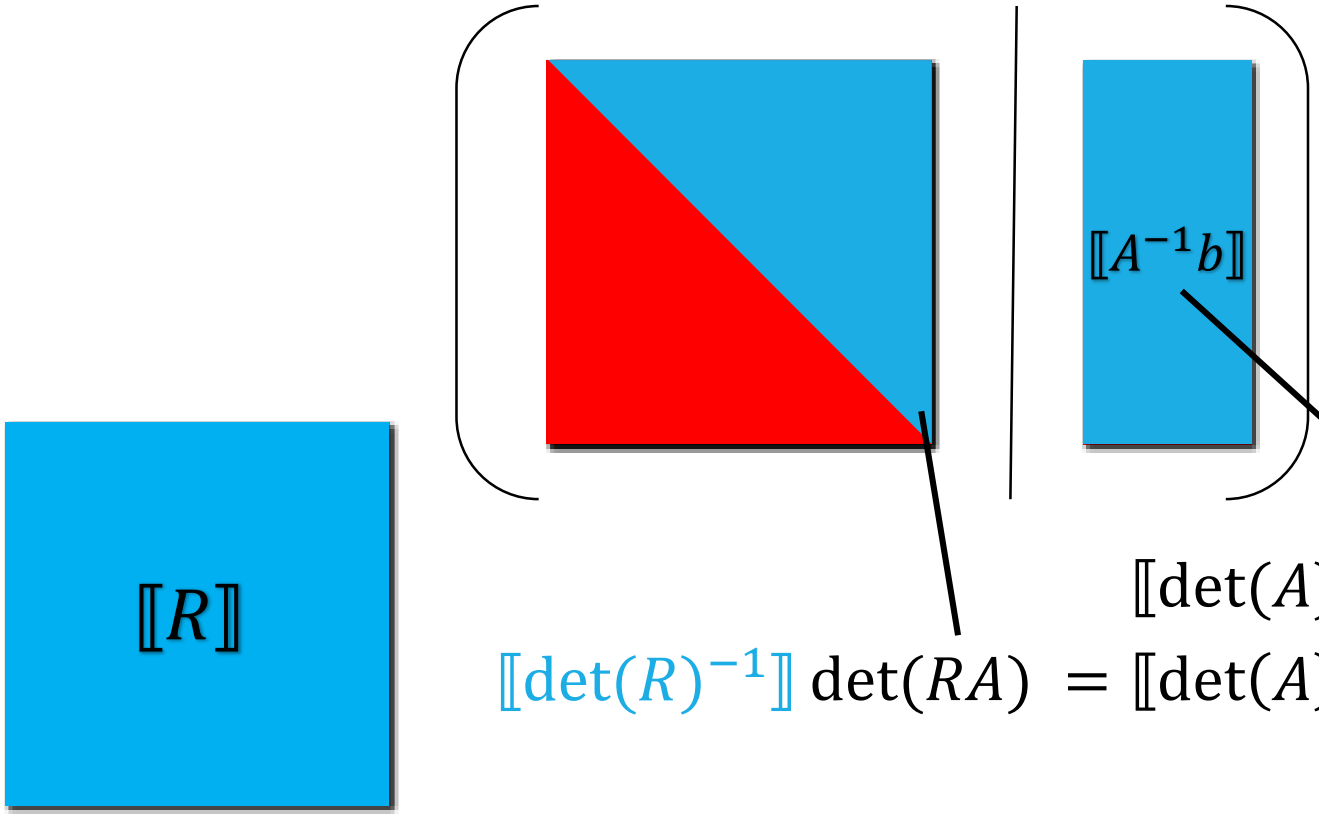
Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$

Randomized Gaussian Elimination

$A \in GL_d(\mathbb{Z}_p)$

1. Generate random mask $[\![R]\!] \in_R GL_d(\mathbb{Z}_p)$
2. Randomize system $(A|b)$
3. Open RA
4. Gaussian Elimination over $(RA|[\![Rb]\!])$
5. Extract $\det(RA)$
6. Compute $[\![\det(A)]\!]$
7. Compute $[\![\mathrm{adj}(A)b]\!]$

Return:$([\![\mathrm{adj}(A)b]\!], [\![\det(A)]\!])$



Generate random mask

Slight twist on Cramer, Damgård Protocol for secure computation of $\det(A)$ over a finite field [CD01]

$$R = LU = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ [\![l_{2,1}]\!] & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ [\![l_{d,1}]\!] & \cdots & [\![l_{d,d-1}]\!] & 1 \end{pmatrix} \begin{pmatrix} [\![u_{1,1}]\!] & [\![u_{1,2}]\!] & \cdots & [\![u_{1,d}]\!] \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & [\![u_{d-1,d}]\!] \\ 0 & \cdots & 0 & [\![u_{d,d}]\!] \end{pmatrix}$$

$$[\![\det(R)^{-1}]\!] = \left(\prod_{i=1}^{d} [\![u_{i,i}]\!]\right)^{-1}$$

Secure Ridge Regression

$$\text{Ridge}([\![X]\!], [\![y]\!], \lambda)$$

1. $[\![A]\!] = [\![X^T]\!][\![X]\!] + \lambda I$, and $[\![b]\!] = [\![X^T]\!][\![y]\!]$
2. $([\![\text{adj}(A)b]\!], [\![\det(A)]\!]) = \text{RandGaussElim}([\![A]\!], [\![b]\!])$
3. Set $[\![\det(A)w]\!] = [\![\text{adj}(A)b]\!]$ // $w = A^{-1}b$
4. Return $[\![\det(A)w]\!], [\![\det(A)]\!]$

Choosing the modulus size

- Return:$(\llbracket \mathrm{adj}(A)b \rrbracket, \llbracket \det(A) \rrbracket)$

  Requirement: $\mathrm{p}/2 > \max\{\|\mathrm{adj}(A)b\|_{max}, |\det(A)|\}$

- Hadamard's Bound: $|\det(M)| \leq \prod_{i=1}^{d} \|m_i\|_2 \leq d^{\frac{d}{2}} \|M\|_{\max}^{d}$

  For $A = X^T X + \lambda I, b = X^T y$ with $X \in [-2^k, 2^k]^{n \times d}, y \in [-2^k, 2^k]^n$

  Bit-length p dominated by: $d \cdot \log(n)$ $n \gg d$

Choosing the modulus size

Requirement: $\mathrm{p}/2 > \max\{\|\mathrm{adj}(A)b\|_{max}, |\det(A)|\}$

n	d	bit–length p
4.208.261	16	744
11.000.000	7	278

Bit-length of p:
we save a factor of two by avoiding rational reconstruction!

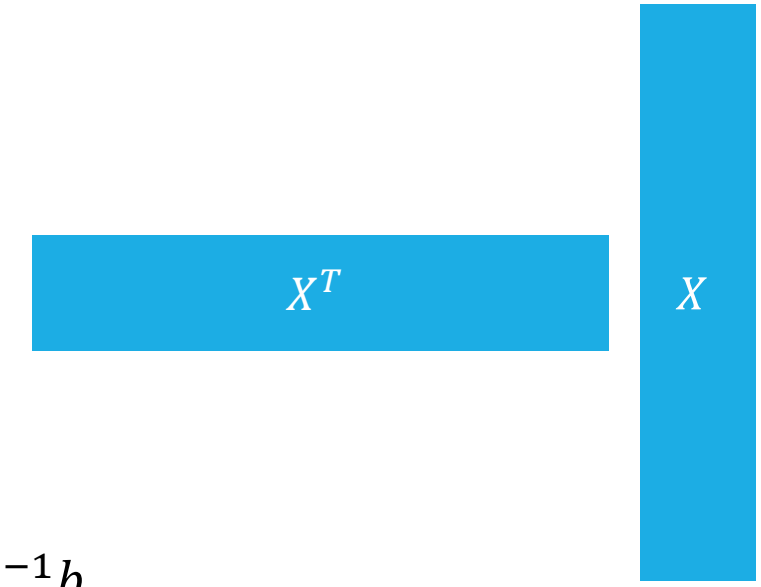
Avoiding rational reconstruction

- Represent fractions $a/b \in \mathbb{Q}$ (uniquely) as $ab^{-1} \in \mathbb{Z}_{\mathrm{p}}$
- Requirement: $p/2 > \|\mathrm{adj}(A)b\|_{max} \cdot |\det(A)|$

Large prime

$$\text{Ridge}([\![X]\!], [\![y]\!], \lambda)$$

1. $[\![A]\!] = [\![X^T]\!][\![X]\!] + \lambda I$, and $[\![b]\!] = [\![X^T]\!][\![y]\!]$
2. $([\![\text{adj}(A)b]\!], [\![\det(A)]\!]) = \text{RandGaussElim}([\![A]\!], [\![b]\!])$
3. Set $[\![\det(A)w]\!] = [\![\text{adj}(A)b]\!]$ // $w = A^{-1}b$
4. Return $[\![\det(A)w]\!], [\![\det(A)]\!]$



n	d	bit–length p
4.208.261	16	744
11.000.000	7	278

✓ Cheap inner product

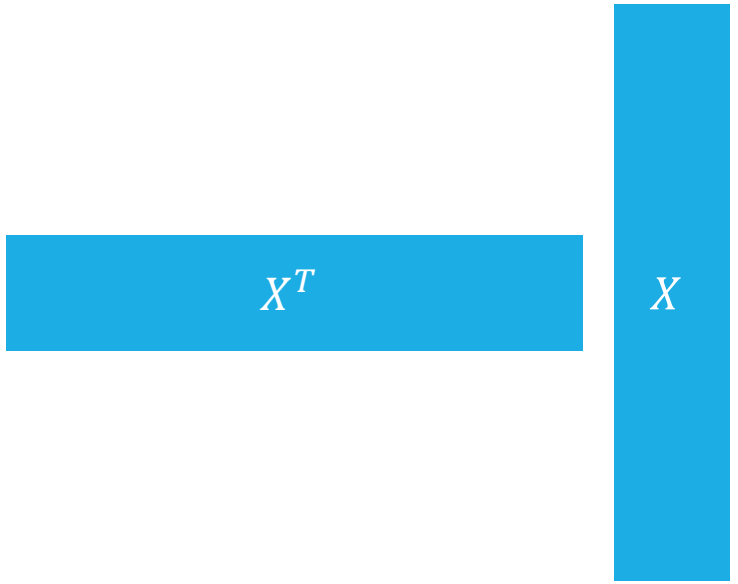
× Many field elements

High memory usage

Long computation time

Two prime moduli: $q < p$

$$\text{Ridge}(\llbracket X \rrbracket_q, \llbracket y \rrbracket_q, \lambda)$$



1. $\llbracket A \rrbracket_q = \llbracket X^T \rrbracket_q \llbracket X \rrbracket_q + \lambda I$, and $\llbracket b \rrbracket_q = \llbracket X^T \rrbracket_q \llbracket y \rrbracket_q$
2. Convert $(\llbracket A \rrbracket_q, \llbracket b \rrbracket_q)$ to $(\llbracket A \rrbracket_p, \llbracket b \rrbracket_p)$
3. $\left(\llbracket \text{adj}(A)b \rrbracket_p, \llbracket \det(A) \rrbracket_p\right) = \text{RandGaussElim}(\llbracket A \rrbracket_p, \llbracket b \rrbracket_p)$
4. Set $\llbracket \det(A)w \rrbracket_p = \llbracket \text{adj}(A)b \rrbracket_p$ // $w = A^{-1}b$
5. Return $\llbracket \det(A)w \rrbracket_p, \llbracket \det(A) \rrbracket_p$

n	d	bit–length p	bit–length q
4.208.261	16	744	75
11.000.000	7	278	66

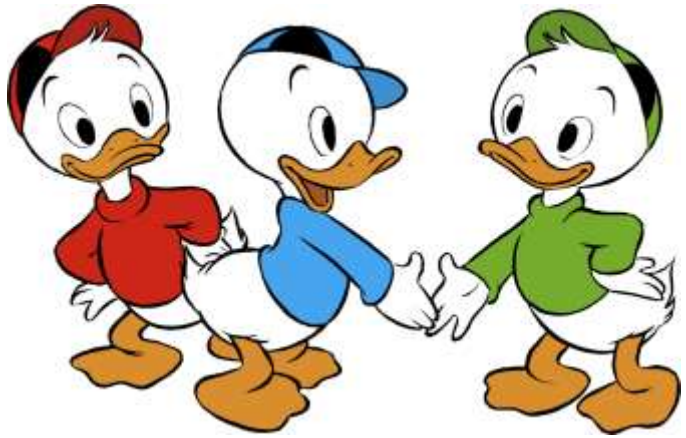
Implementation

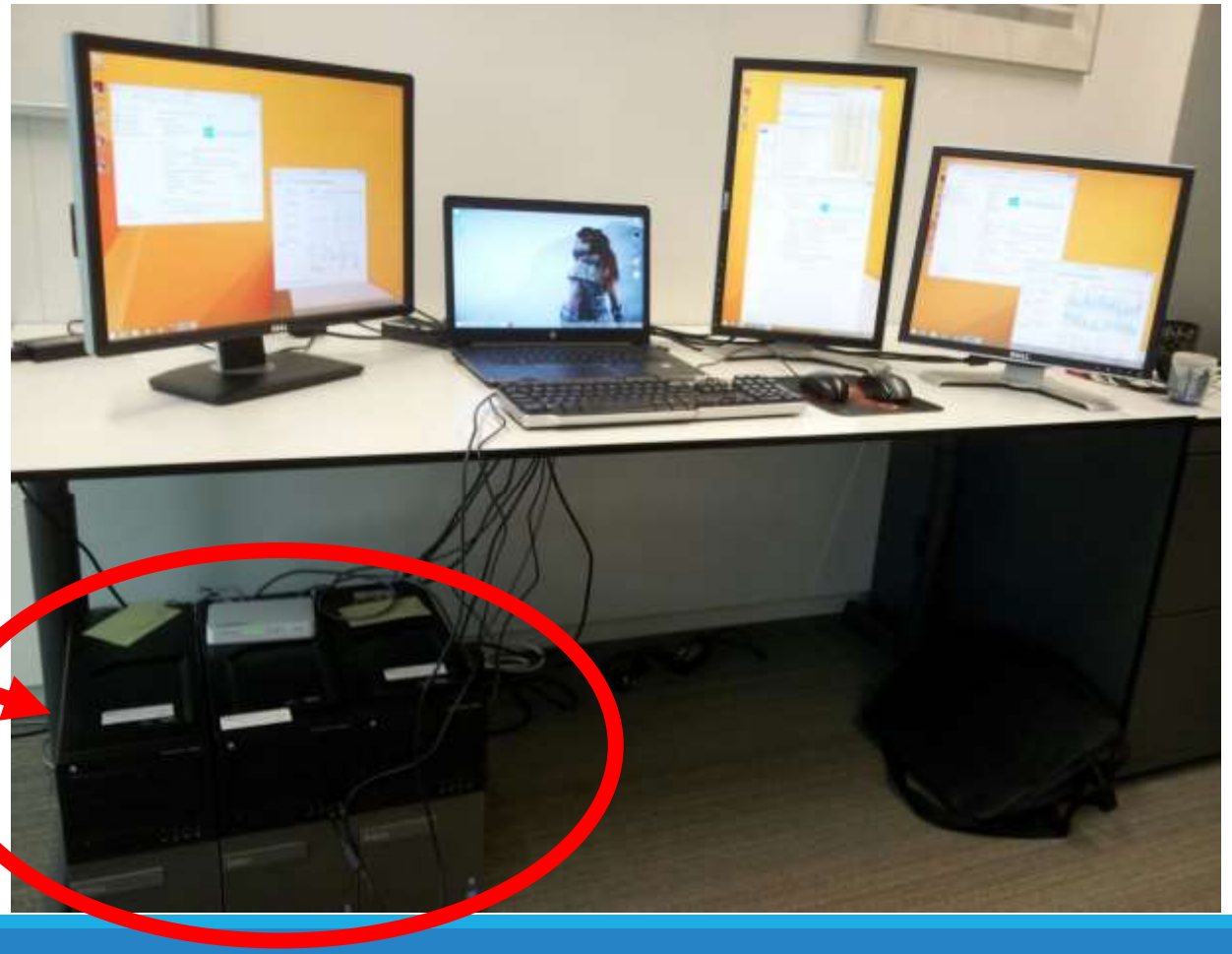
Implementation in MPyC (Shamir Secret Sharing)

MPyC **Secure Multiparty Computation in Python**

github.com/lschoe/mpyc

Multiparty computation (3 parties)





Implementation results UCI Datasets

- Nikolaenko etal. [NWI+13]
  - Horizontal data-partitioning,
  - Fixed-point arithmetic
- Giacomelli etal. [GJJ+13]
  - Horizontal & Vertical data-partitioning
  - Rational Reconstruction
- Gascón etal. [GSB+17]
  - Vertical data-partitioning
  - Fixed-point arithmetic

Dataset	n	d	This Work Time (s)	[NWI+13] HP	[GJJ+13] HP	[GJJ+13] VP	[GSB+17] VP 32-bit	[GSB+17] VP 64-bit
Student	395	30	1,61	-	39,76	328,06	5 (-0,0%)	35 (-0,0%)
Wine (white)	4898	11	0,16	45	4,09	-	0 (4,2%)	4 (-0,0%)
Year Prediction MSD	515.345	90	255	-	-	-	230 (0,0%)	808 (0,0%)
Gas Sensor Array (CO)	4.208.261	16	63,2	-	-	-	42 (5.2%)	69 (0,0%)
HIGGS	11.000.000	7	35,0	-	-	-	-	-

$(< 0.1\%)$

Conclusion

- Secure Integer valued linear system solving
- Efficient ridge regression without data-partitioning assumptions
- Competitive with existing solutions

win.tue.nl/~berry/mpyc/bareiss.pdf

Final remarks

- Slight changes to the protocol to allow underdetermined systems $(n \ll d)$
- Avoiding rational reconstruction also useful for secure computation of Moore-Penrose pseudoinverse [BdV19]

References

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A practical approach to the secure computation of the moore-penrose pseudoinverse over the rationals.
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