1. Introduction

My research lies in mathematical logic, particularly set theory, and, within set theory, it focuses primarily on forcing, large cardinals, and combinatorial set theory.

Much of my work involves the investigation of compactness and incompactness phenomena in set theory. Roughly speaking, a statement of compactness (or, dually, a statement of reflection) takes a form similar to: “Whenever [some property] holds for all (or many) small substructures of a given structure $\mathfrak{A}$, then [same property] holds for $\mathfrak{A}$ as well.” By varying the type of structure and property under consideration and the definition of “small,” one can obtain an incredibly diverse class of interesting statements.

A well-known instance of compactness is the De Bruijn-Erdős theorem [11], which states that, given a natural number $k$ and a graph $G$, $\chi(G) \leq k$ if and only if $\chi(H) \leq k$ for every finite subgraph $H$ of $G$, where $\chi(G)$ denotes the chromatic number of $G$. Another is König’s infinity lemma [23], which states that, if $T$ is an infinite, finitely-branching tree, then there is an infinite path through $T$. Both of these statements are finitary in nature and illustrate the fact that $\aleph_0$, the smallest infinite cardinal, exhibits a great deal of compactness.

One can also consider infinitary versions of these and related compactness assertions. When doing so, one quickly runs into statements that are independent of the axioms of ZFC (the usual axioms of set theory) and is led to the consideration of a hierarchy of large cardinals, which are cardinal numbers whose existence cannot be proven by the axioms of ZFC. Generally speaking, compactness principles tend to be implied by the existence of large cardinals and by forcing axioms (such as the Proper Forcing Axiom (PFA), a generalization of the Baire Category Theorem which goes beyond the axioms of ZFC), while incompactness is prevalent in canonical inner models. For example, Jensen [21] showed that, in Gödel’s constructible universe $L$, the combinatorial principle $\square_\kappa$, which is in some sense a canonical witness to incompactness at $\kappa^+$, holds for all infinite cardinals $\kappa$. On the other hand, a result of Burke and Kanamori (see [41]), implies that, if $\kappa$ is a strongly compact cardinal, then $\square_\lambda$ fails for every cardinal $\lambda \geq \kappa$.

This sets up a fundamental tension in set theory between inner models and incompactness on one hand and large cardinals, forcing axioms, and compactness on the other. Many instances of compactness or incompactness can be formulated as infinitary combinatorial statements, and part of my work has focused on calibrating their strength by proving implications or independences between them. Another fruitful line of investigation in this area involves the extent to which compactness properties of large cardinals can hold at smaller, more accessible cardinals. Some of my recent research has been in this direction, in particular focusing on the extent to which these properties can be made to hold robustly.

Infinite cardinals can be divided into two natural classes: singular and regular. A cardinal $\kappa$ is singular if it is the union of fewer than $\kappa$ sets, each of which has size less than $\kappa$. $\kappa$ is regular if it is not singular. Questions of compactness and incompactness, as well as questions of cardinal arithmetic, are of particular interest at singular cardinals and their successors. This is partly because, due to various covering lemmas, compactness principles at successors of singular cardinals have much higher consistency strength than the same principles at successors of regular cardinals and are generally more difficult to obtain. Also, as a result of a celebrated theorem of Silver and later work of Shelah in PCF theory, there are interesting ZFC restrictions on the values of the continuum function at singular cardinals, in sharp contrast to the situation at regular cardinals. Therefore, questions about singular cardinal combinatorics touch on many areas of set theory, including inner model theory, cardinal arithmetic, large cardinals, and forcing.

The primary method for obtaining independence results in set theory is through forcing, a technique first developed by Paul Cohen, who used it to prove that the Continuum Hypothesis is independent of the axioms.
of ZFC [6]. The technique of forcing is roughly as follows: Given a transitive model $M$ of ZFC and a poset $\mathbb{P} \in M$, one adjoins an object called a $\mathbb{P}$-generic filter, $G$, to $M$ to create a model of ZFC, $M[G]$, that is, in a sense, the smallest model of ZFC such that $M \subseteq M[G]$ and $G \in M[G]$ (in non-trivial cases, $G \notin M$). By altering the properties of the poset $\mathbb{P}$, one can determine to an extent what statements are true or false in the generic extension, $M[G]$.

One of the most prominent methods for studying questions of singular cardinal combinatorics and cardinal arithmetic is to start in a model $P$ in a sense, the smallest model of ZFC such that $2^\kappa > \kappa^+$, led to the first proof of the consistency of the failure of the Singular Cardinals Hypothesis (SCH). Since then, a number of ‘Prikry-type’ forcings have been developed to answer questions about singular cardinal combinatorics (see [18] for a survey of many of these forcing notions). Some of my work has centered around Prikry-type forcings, in particular around a diagonal supercompact Prikry-type forcing used by Gitik and Sharon in [19] and generalizations thereof.

I have also done some work in infinite Ramsey theory. Together with Kolesnikov, we made partial progress toward a model-theoretic conjecture of Grossberg regarding the Hanf number for amalgamation in abstract elementary classes. Our result relies on a detailed combinatorial analysis of a class of colorings of certain scattered linear orders. In work with Weinert, we obtained results about partition relations involving generalizations of the notion of a scattered linear order. We investigated the behavior of $\kappa$-scattered linear orders and weakly $\kappa$-scattered linear orders for regular cardinals $\kappa$, showing that these two classes of orders can have quite different Ramsey-theoretic behaviors.

2. Compactness and incompactness in set theory

2.1. Square sequences and simultaneous stationary reflection. In joint work [20] with Hayut, we investigate the relationship between certain weak square principles and simultaneous stationary reflection. As stationary reflection is an instance of compactness and square principles are instances of incompactness, the two are naturally at odds with one another. In [8], Cummings, Foreman, and Magidor prove a number of results about the compatibility and incompatibility of square principles of the form $\square_{\mu, \kappa}$ with various instances of stationary reflection. In [20] we address similar questions about square principles of the form $\square(\lambda, \kappa)$. If $\lambda = \mu^+$, then $\square(\lambda, < \kappa)$ is a weakening of $\square_{\mu, \kappa}$ with many of the same incompactness consequences. In our work, we show that some genuinely new behavior arises when moving from $\square_{\mu, \kappa}$ to $\square(\lambda, < \kappa)$. In what follows, if $S$ is a stationary subset of $\lambda$ and $\kappa$ is a cardinal, then $\text{Refl}(< \kappa, S)$ (resp. $\text{Refl}(\kappa, S)$) is the assertion that any collection of fewer than $\kappa$ (resp. at most $\kappa$) stationary subsets of $S$ reflects simultaneously. As $\kappa$ increases, then, the strength of $\text{Refl}(< \kappa, S)$ increases, while the strength of $\square(\lambda, < \kappa)$ decreases. We start by obtaining the following new ZFC results.

**Theorem 2.1** (Hayut-LH, [20]). Suppose $\lambda$ is a regular, uncountable cardinal and $\square(\lambda, < \omega)$ holds. Then, for all stationary $S \subseteq \lambda$, $\text{Refl}(2, S)$ fails.

**Theorem 2.2** (Hayut-LH, [20]). Suppose $\kappa < \lambda$ are uncountable cardinals, with $\lambda$ regular, and suppose $\square(\lambda, < \kappa)$ holds. Then, for all stationary $S \subseteq \lambda \cap \text{cof}(\geq \kappa)$, $\text{Refl}(< \kappa, S)$ fails.

We also obtain consistency results showing that the ZFC results are almost sharp.

**Theorem 2.3** (Hayut-LH, [20]). Assume the consistency of certain large cardinals. Then the following are individually consistent, where $\kappa < \lambda$ are infinite, regular cardinals.

1. $\square(\lambda) + \text{Refl}(\lambda)$.
2. $\square(\lambda, 2) + \text{"for all stationary } S, T \text{ such that } S \cup T = \lambda, S \text{ and } T \text{ reflect simultaneously."}$
3. $\square(\lambda, \kappa) + \text{Refl}(< \kappa, S_{<\mu})$ for every regular $\mu < \lambda$.

In this theorem, $\lambda$ can be a successor of a regular cardinal, a successor of a singular cardinal, or an inaccessible cardinal.
2.2. Aronszajn trees, square principles, and stationary reflection. In recent work [3, 4, 5], Brodsky and Rinot introduce strengthenings of □(λ, < κ) that incorporate some guessing mechanisms and are used, together with ◊(λ), to construct a variety of A-Souslin trees. Rinot [39] asked whether these principles are consistent with stationary reflection. In [24], we obtain a positive answer in many cases, indicating that the existence of higher Souslin trees is compatible with a high degree of simultaneous stationary reflection.

In the same paper, we answer a question of Lücke [35] regarding the existence of special trees with narrow ascent paths. In particular, we prove the following theorem.

Theorem 2.4 (LH, [24]). Suppose µ is a singular cardinal.

1. If □µ holds, then there is a special µ⁺-tree with a cf(µ)-ascent path.
2. It is consistent that □µ, 2 holds and, for every regular λ < µ, if T is a tree of height µ⁺ with a λ-ascent path, then T has a cofinal branch.

Clause (2) of Theorem 2.4 is due, independently and by a different proof, to Shani [42].

2.3. Square principles and covering matrices. Two of my papers deal with relationships between square principles and combinatorial objects known as covering matrices. Covering matrices were introduced by Viale in his proof that SCH follows from PFA [45]. In this work and in subsequent work with Sharon [43], he isolated two important reflection principles, CP and S, which can hold of covering matrices. In what follows, CP(λ, θ) is the assertion that CP(D) holds whenever D is a θ-covering matrix for λ. CP(λ, < µ) is the statement that CP(λ, θ) holds for every regular θ < µ.

The heart of Viale’s argument that PFA implies SCH consists of the following.

Theorem 2.5 (Viale, [45]). Let λ > ℵ₂ be a regular cardinal. PFA implies that CP(λ, ω) holds.

In [29] and [25], I investigate the possible failure of CP and S, connecting the existence of covering matrices for which CP and S fail with square principles. In [29], I introduce certain strengthenings of □(λ) and prove the following.

Theorem 2.6 (LH, [29]). Suppose θ < λ are infinite, regular cardinals, with λ > ω₁. Suppose also that □₀(λ) holds. Then there is a θ-covering matrix for λ, D, for which CP(D) and S(D) fail.

Also in [29], I investigate the implications and non-implications that exist between various strengthenings of □(λ), including those appearing in Theorem 2.6. I am able to obtain a complete picture in the case λ = ℵ₂, as illustrated in this diagram.

In [46], Viale shows that, if λ > ω₁ is a regular cardinal, then CP(λ, ω) implies the failure of □(λ). In the following result, I show that this is sharp. This indicates that CP(λ, ω) is significantly weaker than PFA (or even the weaker statements MRP and PID, which Viale uses to derive CP(λ, ω)), which entails the failure of □(λ, ω₁) for all regular λ > ω₁.
Theorem 2.7 (LH, [25]). If there are infinitely many supercompact cardinals, then there is a forcing extension in which \( \text{CP}(\aleph_{\omega+1}, < \aleph_{\omega+1}) \) and \( \square_{\aleph_{\omega+2}} \) both hold.

Similar techniques yield analogous results at other successors of singular cardinals as well as at inaccessible cardinals or successors of regular cardinals.

2.4. Narrow systems. One of the most prominent examples of compactness in set theory is given by the tree property.

**Definition 2.8.** Let \( \kappa \) be a regular cardinal. \( \kappa \) has the tree property if every \( \kappa \)-tree (i.e. tree of height \( \kappa \) with levels of size less than \( \kappa \)) has a cofinal branch.

König’s infinity lemma states that \( \aleph_0 \) has the tree property, whereas Aronszajn showed that \( \aleph_1 \) does not have the tree property. At larger regular cardinals, the tree property is independent of ZFC. A counterexample to the tree property at a cardinal \( \kappa \) is known as a \( \kappa \)-Aronszajn tree. A related combinatorial notion is that of a system, introduced by Magidor and Shelah in [37]. Systems, and their strengthenings, strong systems, generalize trees by allowing multiple order relations to replace the single order relation of a tree, and they are useful in the study of the tree property. A system \( S \) is called narrow if \( \text{width}(S) < \text{height}(S) \). My motivation for studying narrow systems lies in the fact that, in all known cases in which the tree property holds at the successor of a singular cardinal, \( \mu \), the proof of this fact roughly divides into two steps. In the first, it is shown that every \( \mu^+ \)-tree has a narrow subsystem. In the second, it is shown that every narrow \( \mu^+ \)-system has a cofinal branch.

**Definition 2.9.** Let \( \lambda \) be a regular cardinal. \( \lambda \) satisfies the narrow system property (denoted NSP(\( \lambda \)) if every narrow \( \lambda \)-system has a cofinal branch.

In [27], I show that, unlike the tree property, the narrow system property is compatible with some weak square principles. I also prove that the narrow system can hold globally and that PFA yields branches through narrow systems with countable width but not through wider systems.

Theorem 2.10 (LH, [27]). Suppose there are infinitely many supercompact cardinals. Then there is a forcing extension in which NSP(\( \aleph_{\omega+1} \)) and \( \square_{\aleph_{\omega}, < \aleph_{\omega}} \) both hold.

Theorem 2.11 (LH, [27]). Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which every narrow system has a cofinal branch.

Theorem 2.12 (LH, [27]). Suppose PFA holds. Then every narrow system of countable width has a cofinal branch. However, PFA is consistent with the statement that, for all regular \( \lambda > \omega_2 \), there is a narrow \( \lambda \)-system with width \( \aleph_1 \) having no cofinal branch.

2.5. Robust reflection principles.

**Definition 2.13.** Suppose \( P \) is a property that can hold of a cardinal (e.g. being measurable, having the tree property, etc.). We say \( \kappa \) has the robust property \( P \) if, whenever \( Q \) is a forcing poset and \( |Q| + < \kappa \), \( \kappa \) satisfies \( P \) after forcing with \( Q \).

By an argument of Lévy and Solovay [34], most large cardinal notions are robust. Thus, if reflection properties hold because of the existence of large cardinals, then they themselves are robust. In recent work, I have considered reflection principles following from large cardinals that can fail to be robust when holding at small cardinals, identified natural strengthenings of these principles which are always robust, and investigated the extent to which these strengthenings can hold at small cardinals. I have focused in particular on stationary reflection and the tree property.

Shelah, in [44], constructs a model in which stationary reflection holds at the successor of every singular cardinal. It is easily verified that, in Shelah’s model, this reflection is robust everywhere. Moreover, stationary reflection is necessarily robust when it holds at \( \aleph_{\omega+1} \). In [30], building on the main result of [9], which answered a question of Eisworth [14], I obtain the following result.

Theorem 2.14 (LH, [30]). Suppose there is consistently a proper class of supercompact cardinals. Then it is consistent that, for every singular cardinal \( \mu > \aleph_\omega \), stationary reflection holds non-robustly at \( \mu^+ \).

In [26], I introduce a natural strengthening of the tree property.
**Definition 2.15.** Let $\lambda$ be a regular cardinal. $\lambda$ satisfies the **strong system property** if, whenever $S$ is a strong $\lambda$-system and $|R_S|^+ < \lambda$, $S$ has a cofinal branch, where $R_S$ denotes the set of relations of $S$.

A strong $\lambda$-system with one relation is precisely a $\lambda$-tree, so the strong system property implies the tree property. If $\lambda$ is inaccessible, then the tree property is equivalent to the strong system property (both are equivalent to weak compactness). Also, the strong system property is always robust.

By a modification of an argument of Fontanella and Magidor from [16], I show that the strong system property can be achieved at certain small cardinals.

**Theorem 2.16 (LH, [26]).** Suppose there are infinitely many supercompact cardinals. Then there is a forcing extension in which $\aleph_{\omega^2+1}$ has the strong system property.

I can also, to a degree, fine-tune the extent to which the strong system property holds.

**Theorem 2.17 (LH, [26]).** Suppose there are infinitely many supercompact cardinals, and let $\alpha < \omega^2$. Then there is a forcing extension in which, whenever $S$ is a strong $\aleph_{\omega^2+1}$-system and $|R_S| < \aleph_\alpha$, $S$ has a cofinal branch, but in which $\aleph_{\omega^2+1}$ fails to have the strong system property.

3. **Prikry-type forcings and singularizing cardinals**

My work concerning Prikry-type forcings and singularizing cardinals appears primarily in three papers. In [28], I undertake a careful analysis of an important model first studied by Gitik and Sharon in [19], in which they prove the consistency of the simultaneous failure of SCH and the approachability property (AP) at a singular cardinal $\mu$. In [28], I address three questions of Cummings and Foreman from [7] regarding the behavior of scales in the Gitik-Sharon model and the ideal of sets carrying good scales.

In [2], which is joint with Ben-Neria and Unger, we introduce a variant on the diagonal supercompact Prikry forcing used in [19] which preserves the inaccessibility of the cardinal at which the forcing is based. We use this forcing to obtain the following result.

**Theorem 3.1 (Ben Neria-LH-Unger, [2]).** If there are a supercompact cardinal $\kappa$ and a weakly inaccessible cardinal $\theta > \kappa$, then there is a forcing extension in which $\kappa$ is inaccessible and there is a club $E \subseteq \kappa$ of singular cardinals $\nu$ at which SCH and AP both fail.

In particular, $\square^*_\nu$ fails at each $\nu \in E$. Moreover, as $\kappa$ is inaccessible in the forcing extension, by moving to $V_\kappa$ of the extension, we obtain a model of ZFC with a class club of singular cardinals at which SCH and weak square both fail. This is progress toward a desired model of ZFC in which weak square fails at every uncountable cardinal, which would in turn be progress toward a model of ZFC in which the tree property holds at every regular cardinal $\lambda \geq \aleph_2$, which would answer a long-standing open question of Magidor that has been the focus of a great deal of work in recent years.

Given the importance that techniques for singularizing regular cardinals have in the study of singular cardinal combinatorics, it is often of interest to study what is true in general in situations in which a regular cardinal becomes singular in some outer model. In such situations, it often turns out to be the case that, in the outer model, there is a sequence, known as a *pseudo-Prikry sequence*, which is in some way close to a generic sequence over the inner model for some Prikry-type forcing. Theorems to this effect have been obtained by Gitik [17], Džamonja and Shelah [12], and Magidor and Sinapova [38].

In [32], we revisit and extend these results. We prove a very general theorem that yields as corollaries the results on the existence of pseudo-Prikry sequences from [17], [12], and [38] and applies in a greater number of situations. Our proof is quite different from the proofs of the earlier results, using structures defined in the inner model to do a PCF-theoretic analysis in the outer model. We also obtain results about the existence of *diagonal pseudo-Prikry sequences*, which approximate the behavior of generic sequences introduced by diagonal Prikry-type forcings such as that from [19].

4. **Infinite Ramsey theory**

Two of my papers deal with infinite Ramsey theory, in a model-theoretic and order-theoretic context, respectively. In [22], a collaboration with Kolesnikov, we make progress towards a conjecture of Grossberg that, for an infinite cardinal $\kappa$, the Hanf number for amalgamation for $L_{\kappa^+, \omega}$ is $\beth_1$. In [22], we show that
Grossberg’s conjecture is optimal. In particular, we introduce a type of abstract elementary class called a coloring class and prove the following, improving results of Baldwin, Kolesnikov, and Shelah [1].

**Theorem 4.1** (Kolesnikov-LH, [22]). Suppose $\kappa$ is an infinite cardinal.

1. For coloring classes in a language of size $\kappa$, disjoint amalgamation is equivalent to amalgamation.
2. The Hanf number for amalgamation for coloring classes in a language of size $\kappa$ is precisely $\beth_{\kappa^+}$.
3. Given a limit ordinal $\beta < \kappa^+$ and a natural number $k$, there is a coloring class in a language of size $\kappa$ such that amalgamation first fails somewhere between $\beth_{\beta+k}$ and $\beth_{\beta+\langle k+3 \rangle}$.

A linear order is scattered if it does not contain a copy of the rationals. In [13], Džamonja and Thompson generalize the notion of a scattered order by introducing the notions of weakly $\kappa$-scattered and $\kappa$-scattered orders. In [33], which is joint work with Weinert, we investigate partition relations involving these generalizations of scattered orders, obtaining the following partition results, the first positive and the second negative.

**Theorem 4.2.** Suppose $\kappa$ is an infinite cardinal, $\kappa^\kappa = \kappa$, and $\varphi$ is a weakly $\kappa$-scattered linear order of size $\leq \kappa$. Then there is a weakly $\kappa$-scattered linear order $\tau$ of size $\leq \kappa$ such that, for all $n < \omega$, $\tau \to (\varphi, n)^2$.

**Theorem 4.3.** Suppose $\kappa$ is an uncountable regular cardinal.

1. If $\kappa = \omega_1$ and CH holds, then there is a $\kappa$-scattered linear order $\tau$ of size $\kappa$ such that, for all $\kappa$-scattered linear orders $\varphi$ of size $\kappa$, $\varphi \not\equiv (\tau, 3)$.
2. If $\kappa > \omega_1$, then there is a ccc forcing extension in which there is a $\kappa$-scattered linear order $\tau$ of size $\kappa$ such that, for all $\kappa$-scattered linear orders $\varphi$ of size $\kappa$, $\varphi \not\equiv (\tau, 3)^2$.

5. **Current and future work**

There are a number of directions in which I plan to take my future research and in which I am currently working. I continue to work on questions surrounding the strong system property and the tree property, including the following two central open questions about the strong system property.

**Question 5.1.** Is the strong system property equivalent to the robust tree property?

A negative answer to this question would likely require a fundamentally new way of achieving the tree property as, in all known models of the tree property, it is actually the case that all strong systems with countably many relations have a cofinal branch.

**Question 5.2.** Can $\aleph_{\omega_1}$ consistently have the strong system property?

Some questions about the tree property that originally brought me to the study of systems also remain open and very much of interest to me.

**Question 5.3** (Woodin). Is it consistent that $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$, and $\square^{\aleph_\omega}_{\aleph_\omega}$ fails?

**Question 5.4.** Is it consistent that there is a singular cardinal $\mu$ such that $\mu^+$ has the tree property and $AP_\mu$ holds?

**Question 5.5.** Is it consistent that $\text{Refl}(\aleph_{\omega+1})$ and the tree property at $\aleph_{\omega+1}$ both hold?

In [16], Fontanella and Magidor prove that the tree property at $\aleph_{\omega+1}$ is consistent with $\text{Refl}(\aleph_{\omega+1})$, but the question remains completely open at $\aleph_{\omega+1}$. All of these questions will likely require new techniques to solve, and their solutions will likely shed much light on a number of currently mysterious aspects of singular cardinal combinatorics and have implications for the following, perhaps the most important open question in the area.

**Question 5.6** (Magidor). Can the tree property consistently hold simultaneously for all regular, uncountable $\lambda > \aleph_1$?

Also, depending on the solutions, Questions 5.2 and 5.5 may provide new and interesting instances of ways in which $\aleph_{\omega+1}$ differs from $\aleph_{\omega+1}$, a phenomenon brought to light in [36], in which Magidor and Shelah prove that a certain compactness principle known as $\Delta$-reflection can consistently hold at $\aleph_{\omega+1}$ but necessarily fails at all smaller uncountable, regular cardinals.
In ongoing joint work with Rinot [31], we are investigating incompactness phenomena relating to chromatic numbers for graphs. We are particularly interested in uncovering relations between chromatic incompactness, square principles, and reflection principles. We have thus far obtained the following result.

**Theorem 5.7 (LH-Rinot, [31]).** Suppose $\lambda$ is a regular, uncountable cardinal. If $\mathbb{Z}^- (\lambda)$ holds, then there is a graph $G$ of size $\lambda$ such that $\chi(H) = \aleph_0$ for all subgraphs $H$ such that $|H| < \lambda$ but $\chi(G) = \lambda$.

By results from [24], this shows that high amounts of chromatic incompactness are compatible with stationary reflection. Further, by combining results of Fontanella-Hayut [15] and Rinot [40], this shows that $\Delta$-reflection does not imply compactness for the chromatic number, which is in sharp contrast with the situation for compactness for freeness of Abelian groups or the coloring number for infinite graphs. We plan to continue this work and investigate the extent to which chromatic incompactness is compatible with stronger reflection principles.

I am also interested in the following suite of related questions, which has partially motivated my work on covering matrices.

**Question 5.8.** Are any of the following consistent?

1. The Chang’s Conjecture variation, $(\mathbb{R}_{\omega+1}, \mathbb{R}_\omega) \rightarrow (\mathbb{R}_2, \mathbb{R}_1)$.
2. The existence of an infinite $A \subseteq \omega$ and a scale $\vec{f}$ in $\prod_{n \in A} \mathbb{R}_n$ of length $\mathbb{R}_{\omega+1}$ such that $\vec{f}$ has stationarily many bad points of cofinality $\omega_2$.
3. Transitive models of set theory $V \subseteq W$ such that $(\mathbb{R}_{\omega+1})^V = (\mathbb{R}_2)^W$.

Some of my results from [25] give an alternate proof that, under an appropriate stationary reflection assumption, (2) must fail (this also follows from results of [43]). It seems likely that further investigation into the interaction between PCF-theoretic scales and covering matrices could shed more light on these questions.

One of the applications of the pseudo-Prikry results discussed in Section 3 is in the proof of the following theorem, due in the special case of Prikry forcing to Cummings and Schimmerling [10].

**Theorem 5.9** (by combined arguments of Džamonja-Shelah and Cummings-Schimmerling). Suppose $W$ is an outer model of $V$, $\kappa$ is a regular cardinal in $V$, $\kappa$ is a singular cardinal of countable cofinality in $W$, and $(\kappa^+)^W = (\kappa^+)^V$. Then $\square_{n,\omega}$ holds in $W$.

Viale asked whether a similar situation holds when $\kappa$ becomes singular of uncountable cofinality. This question remains open and of considerable interest in my continuing research.

**Question 5.10** (Viale). Suppose $W$ is an outer model of $V$, $\kappa$ is a regular cardinal in $V$, $\kappa$ is a singular cardinal of cofinality $\aleph_1$ in $W$, and $(\kappa^+)^W = (\kappa^+)^V$. Must $\square_{\kappa,\aleph_1}$ hold in $W$?

It seems plausible that a solution to this question will involve a careful analysis of quotients of Prikry-type forcing notions, a highly technical topic which could have widespread application to other questions of singular cardinal combinatorics.

I am also interested in looking into questions regarding Jónsson cardinals. In particular, I plan on looking into matters surrounding the following two important open questions.

**Question 5.11.** Is it consistent that $\mathbb{R}_\omega$ is Jónsson?

**Question 5.12.** Is it consistent that there is a singular cardinal $\mu$ such that $\mu^+$ is Jónsson?

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