RESEARCH STATEMENT

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My research lies mostly in logic and set theory, and in applications of set-theoretic tools to other areas of mathematics, such as graph theory, algebra, and topology.

My set-theoretic work is largely combinatorial in nature and comes in one of two flavors: ZFC results and independence results. ZFC stands for *Zermelo-Fraenkel axioms with choice* and is the standard set of axioms in which set theory is done. Many interesting set-theoretic statements can be proven outright from the axioms of ZFC. However, it has been known since the 1930s [12] that, given any sufficiently strong, effectively axiomatizable formal system (such as ZFC), there are statements that can be neither proven nor disproven within the system. In 1963, Cohen [4] introduced the method of *forcing*, which, together with earlier work of Gödel [13], allowed him to show that the Continuum Hypothesis, the most famous open problem of set theory at the time, is undecidable by the axioms of ZFC. Since then, forcing, which allows one to construct new models of ZFC by introducing certain "generic" sets, has become a central technique in set theory, and I make use of it extensively in my work.

Another central concept in modern set theory, and in my work on independence results, is the notion of a *large cardinal*. Roughly speaking, a large cardinal is a type of infinite cardinal number whose consistent existence is not implied by ZFC. For example, a *weakly compact cardinal* is an uncountable cardinal κ for which the following generalization of Ramsey's theorem holds:

> Whenever the edges of the complete graph on κ vertices are colored with two colors, there is a complete subgraph of size κ, all of whose edges are the same color.

The existence of a weakly compact cardinal turns out to imply the consistency of ZFC, so, by Gödel's Second Incompleteness Theorem, ZFC cannot imply the consistency of the existence of a weakly compact cardinal. There is a great variety of large cardinal notions, and, rather strikingly, they form a largely linear hierarchy when ordered in terms of consistency strength. Assuming the existence of large cardinals can allow set theorists to prove certain consistency results that could not otherwise be obtained and, indeed, one can often show a natural infinitary combinatorial statement to be equiconsistent with the existence of a certain type of large cardinal. For example, the *tree property at* $\aleph_2$ ($\mathrm{TP}(\aleph_2)$), which is a generalization of König's infinity lemma, is equiconsistent over ZFC with the existence of a weakly compact cardinal [25]. In other words, if there is a model of ZFC in which $\mathrm{TP}(\aleph_2)$ holds, then there is a model in which there is a weakly compact cardinal, and vice versa.

We recall here some basic notions and notations. An *ordinal* is the order-type of a well-ordered set; in practice, an ordinal is identified with the set of all ordinals less than it. A *cardinal* is an equivalence class of sets under the equivalence relation of "having a bijection between." Cardinals are identified with the least ordinal of their cardinality. $\aleph_0$ is the smallest infinite cardinal number, $\aleph_1$ is the next smallest cardinal number, and so on. If κ is a cardinal, then κ^+ denotes the smallest cardinal greater than κ. If X is a set, then $|X|$ denotes the cardinality of X. If X is a set and κ is a cardinal, then $[X]^\kappa$ denotes the collection of all subsets of X of cardinality κ.

1. Compactness and reflection principles

Questions about compactness (and, dually, reflection) are among the most fundamental that can be asked about classes of mathematical structures and typically take the following general form:

> Suppose a structure is such that all of its "small" substructures satisfy a certain condition. Must the entire structure satisfy the same condition?

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Let us look at some interesting specific instances of this question at a given cardinal κ. For notational ease, if $\mathfrak{A}$ is a structure and P is a property, we say that $\mathfrak{A}$ is *almost* P if $\mathfrak{B}$ has property P for every substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $|\mathfrak{B}| < |\mathfrak{A}|$.

Question 1.1. (1) *Is every almost free abelian group of size κ free?*
(2) *Is every almost metrizable first countable topological space of size κ metrizable?*
(3) *Is every almost countably chromatic graph of size κ countably chromatic?*

The study of compactness principles has played a major role in modern mathematics, in fields such as logic, set theory, topology, graph theory, and algebra. A number of seminal theorems of the twentieth century center around instances of compactness in a finitary setting. These include:

- König's infinity lemma, which asserts that every infinite tree with finite levels has an infinite branch;
- the compactness theorem for first-order logic;
- Tychonoff's theorem, which states that any product of compact topological spaces is compact;
- the de Bruijn-Erdős compactness theorem: if k is a natural number and $\mathcal{G}$ is a graph, all of whose finite subgraphs have chromatic number at most k, then $\mathcal{G}$ has chromatic number at most k.

Taken together, these theorems can be seen to assert that the cardinal $\aleph_0$ exhibits a high degree of compactness.

When the cardinal context of these theorems is shifted up one level, though, they typically become false:

- there is always a tree of height $\aleph_1$ with countable levels and no branch of size $\aleph_1$;
- the logic $L_{\omega_1\omega}$, which is obtained from first order logic by allowing countably infinite disjunctions and conjunctions, fails to be compact;
- there are two Lindelöf topological spaces whose product fails to be Lindelöf;
- under rather mild hypotheses, there is a graph $\mathcal{G}$ of size $\aleph_2$ such that all subgraphs of size $\aleph_1$ have a countable chromatic coloring number but $\mathcal{G}$ has uncountable chromatic number.

These results, in turn, can be seen to assert that the cardinal $\aleph_1$ exhibits a high degree of *incompactness.*

For cardinals greater than $\aleph_1$, questions of compactness become more complicated, and they are inextricably linked with large cardinals and independence over ZFC. For example, the assertion that every tree of height $\aleph_2$ with levels of size $\aleph_1$ has a branch of size $\aleph_2$ is equiconsistent over ZFC with the existence of a weakly compact cardinal. For another, if κ is an uncountable cardinal, then the compactness of the logic $L_{\kappa\kappa}$, which extends first-order logic by allowing conjunctions, disjunctions, and chains of quantifiers of any length less than κ, is equivalent to κ being a strongly compact cardinal.

1.1. **Square principles.** Square principles can be seen as canonical examples of set-theoretic incompactness, and they have played a role in a large portion of my research. The most fundamental square principles were introduced by Jensen and Todorcevic; let us recall the latter here.

Definition 1.2. Suppose that β is a limit ordinal and $X \subseteq \beta$.

(1) $\mathrm{acc}(X) = \{\alpha < \sup(X) \mid \sup(X \cap \alpha) = \alpha\}$;
(2) X is *closed* if $\mathrm{acc}(X) \subseteq X$;
(3) X is *unbounded in* β if $\sup(X) = \beta$;
(4) X is *club* in β if it is closed and unbounded in β.

Definition 1.3 (Todorcevic)**.** Suppose that κ is an uncountable cardinal. $\square(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

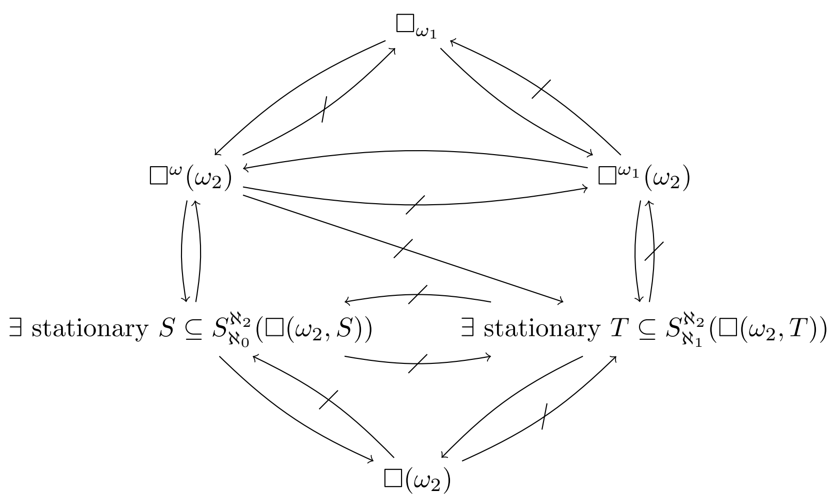
(1) for every limit ordinal $\alpha < \kappa$, C_α is club in α;
(2) (coherence) for all limit ordinals $\alpha < \beta < \kappa$, if $\alpha \in \mathrm{acc}(C_\beta)$, then $C_\beta \cap \alpha = C_\alpha$;
(3) (non-triviality) there is no club D in κ such that, for all $\alpha \in \mathrm{acc}(D)$, we have $D \cap \alpha = C_\alpha$.

$\square(\kappa)$ is manifestly an incompactness principle: a witness to $\square(\kappa)$ is a coherent sequence of clubs that, by non-triviality, cannot be extended by placing another club on top. In addition, if $\kappa = \mu^+$, then $\square(\kappa)$ is a significant weakening of Jensen's original principle $\square_\mu$, which strengthens Clause (3) of Definition 1.3 by requiring that $\mathrm{otp}(C_\alpha) \leq \mu$ for all $\alpha < \mu^+$.

$\square(\kappa)$ has a number of consequences in many fields of mathematics. One of its uses is as a tool to show that the consistency of certain mathematical statements implies the consistency of large cardinals. Jensen

[15] proved that, if κ is a regular uncountable cardinal and $\square(\kappa)$ *fails*, then κ is a weakly compact cardinal in Gödel's constructible universe, L. Therefore, if one can show that $\square(\kappa)$ implies a statement, then the consistency of the negation of that statement implies the consistency of the existence of a weakly compact cardinal.

Square principles have played an important role in much of my work. In [17], I investigate the implications and non-implications between various square principles intermediate between $\square_\mu$ and $\square(\mu^+)$, in particular obtaining the following complete picture at $\mu = \aleph_1$.



1.2. **Compactness for chromatic and coloring numbers of uncountable graphs.** In work with Rinot [22], we investigate compactness for the chromatic and coloring numbers of uncountable graphs. Our main result indicates that rather mild assumptions imply a maximal amount of incompactness for the chromatic number.

Theorem 1.4 (LH-Rinot, [22])**.** *Suppose that the Generalized Continuum Hypothesis holds and λ is an uncountable cardinal such that $\square(\lambda^+)$ holds. Then there is a graph $\mathcal{G}$ of size λ^+ such that:*

(1) *every subgraph of $\mathcal{G}$ of size less than λ^+ has countable chromatic number;*
(2) *the chromatic number of $\mathcal{G}$ is λ^+.*

Since $\square(\lambda^+)$ is compatible with certain compactness principles, we obtain surprising corollaries stating that the maximal amount of incompactness for the chromatic number is compatible with large amounts of compactness in other areas. For example, it is compatible with compactness for the coloring number.

On the other hand, we show that the coloring number cannot admit arbitrarily large compactness gaps.

Theorem 1.5 (LH-Rinot, [22])**.** *Suppose that μ is a cardinal, $\mathcal{G}$ is a graph, and every subgraph of $\mathcal{G}$ of strictly smaller cardinality than $\mathcal{G}$ has coloring number at most μ. Then $\mathcal{G}$ has coloring number at most μ^{++}.*

It is known that one-cardinal incompactness gaps in coloring number can be achieved. The question of whether two-cardinal gaps are possible remains open. Its solution is likely to involve deep combinatorial questions, and I plan to continue to investigate it. The most prominent special case of this question is the following:

Question 1.6. *Is it consistent that there is a graph $\mathcal{G}$ of size $\aleph_{\omega+1}$ such that every smaller subgraph of $\mathcal{G}$ has countable coloring number but $\mathcal{G}$ has coloring number $\aleph_2$?*

1.3. **Trees with ascent paths and productivity of chain conditions.** The notion of chain conditions of partially ordered sets has played a major role in modern set theory, both in providing an impetus for set theoretic investigation and in being a key tool itself in the development of the technique of forcing.

Definition 1.7. Suppose $\mathbb{P}$ is a partial order and κ is a cardinal.

(1) If $p, q \in \mathbb{P}$, then p and q are *compatible* if there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.

(2) An *antichain* in $\mathbb{P}$ is a subset $A \subseteq \mathbb{P}$ such that the elements of A are pairwise incompatible.
(3) $\mathbb{P}$ satisfies the *κ-chain condition* (κ-c.c.) if, for every antichain A in $\mathbb{P}$, we have $|A| < \kappa$.
(4) $\mathbb{P}$ is κ-Knaster if, whenever $A \subseteq \mathbb{P}$ and $|A| = \kappa$, there is a subset $B \subseteq A$ such that $|B| = \kappa$ and B consists of pairwise compatible elements.

Clearly, if a partial order $\mathbb{P}$ is κ-Knaster, then it satisfies the κ-c.c. One reason the κ-Knaster condition is useful is that it is *productive*: if $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster partial orders, then the product $\mathbb{P} \times \mathbb{Q}$ is also κ-Knaster. This is not necessarily true for the κ-chain condition, and the study of the productivity of chain conditions has led to much deep work in set theory.

Though the κ-Knaster condition is *finitely* productive, it is not necessarily *infinitely* productive, i.e., an infinite product of κ-Knaster partial orders may not be κ-Knaster. If κ is a weakly compact cardinal, then it is in fact the case that any product of fewer than κ κ-Knaster partial orders is again κ-Knaster. In joint work with Lücke, we give a partial converse to this by showing that, if the κ-Knaster condition is even *countably* productive, then κ is weakly compact in L.

Theorem 1.8 (LH-Lücke, [20])**.** *Suppose that κ is an uncountable regular cardinal and $\square(\kappa)$ holds. Then there is a κ-Knaster partial order $\mathbb{P}$ such that $\mathbb{P}^{\aleph_0}$ does not satisfy the κ-chain condition.*

The question of the consistency of infinite productivity of the κ-Knaster property for small values of κ remains open and is something I plan to continue to investigate. The most prominent instance of this question is the following:

Question 1.9. *Is it consistent that, whenever $\mathbb{P}$ is an $\aleph_2$-Knaster partial order, $\mathbb{P}^{\aleph_0}$ is also $\aleph_2$-Knaster?*

Cox [5] introduced the notion of a *κ-stationarily layered* poset, which is a further, somewhat technical strengthening of the κ-Knaster condition. If κ is weakly compact, then in fact every κ-Knaster partial order (and even every κ-c.c. partial order) is stationarily layered. We also provide a partial converse to this, answering a question of Cox and Lücke.

Theorem 1.10 (LH-Lücke, [20])**.** *Suppose that κ is an uncountable regular cardinal and $\square(\kappa)$ holds. Then there is a κ-Knaster partial order that is not κ-stationarily layered.*

This work with Lücke also concerns the existence of trees with ascent paths, which are generalizations of cofinal branches. In one result, we show that the assertion that all κ-trees have narrow ascent paths is strictly weaker than the tree property at κ.

Theorem 1.11 (LH-Lücke, [20])**.** *The following statement is equiconsistent with the existence of a weakly compact cardinal: There are $\aleph_2$-Aronszajn trees, but every $\aleph_2$-tree has an $\aleph_0$-ascent path.*

We also provide a complete picture of the relationships between the existence of special trees and the existence of Aronszajn trees with narrow ascent paths at successors of regular cardinals.

1.4. **Simultaneous stationary reflection and square sequences.** Just as square principles provide canonical instances of set-theoretic incompactness, stationary reflection principles provide canonical instances of compactness. There is therefore an inherent tension between square principles and stationary reflection, and the investigation of this tension has proved to be quite fruitful (cf. [6]). Roughly speaking, a *stationary* subset of a cardinal κ is one that is "non-negligible," and a stationary reflection principle is a statement asserting that all stationary sets have proper initial segments that are themselves stationary.

Definition 1.12. Suppose κ is a regular, uncountable cardinal and $S \subseteq \kappa$.
(1) S is *stationary* in κ if it has non-empty intersection with every closed, unbounded subset of κ.
(2) If S is stationary, then S *reflects* if there is $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α.
(3) *Stationary reflection* holds at κ (denoted $\mathrm{Refl}(\kappa)$) if every stationary subset of κ reflects.

By weakening the definition of $\square(\kappa)$ to allow multiple club subsets of each ordinal $\alpha < \kappa$, one obtains a hierarchy of square principles, $\square(\kappa, \lambda)$ for $0 < \lambda < \kappa$, decreasing in strength as λ increases. Similarly, by strengthening the definition of stationary reflection to require that multiple stationary subsets of κ reflect simultaneously, one obtains a hierarchy of stationary reflection principles, $\mathrm{Refl}(\kappa, \lambda)$ for $0 < \lambda < \kappa$, increasing in strength as λ increases.

In joint work with Hayut [14], we investigate the interactions between these two hierarchies, showing that they are tightly correlated: roughly speaking, we show that, for infinite, regular cardinals $\lambda < \kappa$, $\square(\kappa, \lambda)$ is compatible with $\mathrm{Refl}(\kappa, \mu)$ for all $\mu < \lambda$ but is *incompatible* with $\mathrm{Refl}(\kappa, \lambda)$. This is in sharp contrast to the situation for the analogous hierarchy of principles $\square_{\kappa,\lambda}$, which were investigated in [6].

1.5. **Constructions with small approximations.** For an infinite cardinal κ, it is often helpful when constructing incompact objects of size κ^+ to be able to piece together these objects from small approximations, i.e., approximations of size $< \kappa$. Such constructions were done, making use of combinatorial objects known as *morasses*, by Shelah and Stanley [28, 29] and by Velleman [30]. Shortly thereafter, similar constructions were carried out using variants of the guessing principle known as *diamond* by Shelah et al., culminating in [27].

One area in which these constructions are relevant concerns generalizations of the Souslin Hypothesis (SH). SH asserts that every complete, dense linear order without endpoints and with the countable chain condition is isomorphic to the real numbers. SH has inspired a tremendous variety of research in set theory and was eventually found to be independent of ZFC. SH can naturally be seen as $\mathrm{SH}_{\aleph_1}$ and has natural generalizations SH_κ for regular cardinals $\kappa > \aleph_1$.

In joint work with Rinot [21], we develop a new technique for carrying out such constructions, using a combination of a square principle and a diamond principle. We encapsulate this technique in a new principle (of a sort known as a *forcing axiom*), which we call $\mathrm{SDFA}(\mathcal{P}_\kappa)$. We show that, for uncountable successor cardinals κ, the principle $\mathrm{SDFA}(\mathcal{P}_\kappa)$ is equivalent to an instance of the Generalized Continuum Hypothesis together with a particular square principle. In our main application, we prove that, for an infinite cardinal λ, $\mathrm{SDFA}(\mathcal{P}_{\lambda^+})$ implies that $\mathrm{SH}_{\lambda^{++}}$ fails. This leads to the following corollary regarding the large cardinal strength of the failure of the generalized Souslin Hypothesis, improving upon a thirty-year-old result of Shelah and Stanley [28].

Theorem 1.13 (LH-Rinot [21])**.** *Suppose that λ is an uncountable cardinal, $2^\lambda = \lambda^+$, and $\mathrm{SH}_{\lambda^{++}}$ holds. Then λ^{++} is a Mahlo cardinal in L.*

We suspect that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ will have a number of other applications in a variety of fields of mathematics, and I plan to continue to investigate this. For example, we feel it is likely to entail the existence of exotic varieties of superatomic Boolean algebras or topological spaces, such as positive solutions to the Arhangel'skii Problem. As is the case with the generalized Souslin Hypothesis, this could provide better lower bounds for the large cardinal strength of the non-existence of these objects. We are also interested in removing the cardinal arithmetic assumptions from the statement of Theorem 1.13. The eventual goal would be an answer to the following question.

Question 1.14. *Is the existence of an infinite cardinal λ for which $\mathrm{SH}_{\lambda^{++}}$ holds equiconsistent with the existence of a weakly compact cardinal?*

1.6. **Set theory and cohomology.** One of the appealing aspects of the study of compactness and incompactness in set theory is that it has a number of applications to other fields of mathematics. In an instance of this, in joint work with Bergfalk, we have been considering connections with questions in homological algebra, particularly pertaining to the Čech cohomology groups of regular, uncountable cardinals. Bergfalk, in [3], shows that, for a natural number n and a regular, uncountable cardinal κ, the assertion that the n^{th} Čech cohomology group of κ, $\check{H}^n(\kappa)$, is non-trivial is equivalent to a purely combinatorial set-theoretic statement and can be naturally interpreted as an assertion of set-theoretic incompactness. In joint, ongoing work, we show, among other things, that Gödel's universe L exhibits the maximum possible amount of incompactness in this direction.

Theorem 1.15 (Bergfalk-LH)**.** *Suppose V=L, let $n \geq 1$ be a natural number, and let $\kappa \geq \aleph_n$ be a regular cardinal that is not weakly compact. Then $\check{H}^n(\kappa) \neq 0$.*

I find these connections with other areas of mathematics to be among the most interesting aspects of this area of study and plan to continue to pursue their investigation. I also feel that these connections make for appealing topics of study for advanced undergraduates, as they can deepen their understanding of the familiar fields of, for example, algebra or topology, while providing a natural entry point to fairly sophisticated set theoretic ideas.

2. Singular cardinal combinatorics

An infinite cardinal κ is *singular* if a set of size κ can be written as a union of fewer than κ sets, each of size less than κ. Many interesting and deep questions in set theory revolve around combinatorics and cardinal arithmetic at singular cardinals and their successors. For a variety of reasons, results at singular cardinals and their successors are typically more difficult to obtain than the analogous results at regular cardinals and their successors and often require stronger large cardinal assumptions.

Some of the most fundamental tools of research into singular cardinal combinatorics are the so-called *Prikry-type* forcing notions. A typical Prikry-type forcing notion allows one to pass from a model of set theory with a suitably large cardinal, κ, to an outer model in which κ is a singular cardinal. We say that, in this situation, κ has become *singularized* in the outer model.

In joint work with Ben Neria and Unger [2], we introduce a new Prikry-type forcing notion, known as *diagonal supercompact Radin forcing*, and use it to prove the following global consistency result about the failure of the Singular Cardinals Hypothesis and the non-existence of special Aronszajn trees.

Theorem 2.1 (Ben Neria-LH-Unger, [2]). *If there are a supercompact cardinal κ and a weakly inaccessible cardinal $\theta > \kappa$, then there is a forcing extension in which κ is inaccessible and there is a club $E \subseteq \kappa$ of singular cardinals ν at which* SCH *and* AP *both fail.*

This is a step in the direction of a possible proof of the consistency of every regular cardinal $\kappa > \aleph_1$ satisfying the tree property, which would provide a striking example of global compactness and would answer a long-standing and central question of Magidor.

In other recent work [19], I have extended and generalized previous results [11, 7, 24] indicating that Prikry-type forcing notions are in some sense the *only* way to singularize cardinals. More precisely, I prove a quite general theorem stating that, if κ is a regular cardinal that becomes singular in some outer model of set theory, then, as long as a certain amount of cardinal structure is maintained, then there must be an object in the outer model which resembles a generic object for a Prikry-type forcing notion. Results such as this play a key role in providing limits on what kind of consistency results we can hope to prove.

3. Infinitary Ramsey theory

Other of my work deals with infinitary Ramsey theory, which addresses generalizations of the infinite Ramsey theorem:

> For all natural numbers m and n and all functions $c : [\mathbb{N}]^m \to \{0, \ldots, n\}$, there is an infinite subset H of $\mathbb{N}$ such that c is constant on $[H]^m$.

Beginning with the work of Erdős and Rado [9], the study of generalizations of Ramsey's theorem, both to larger sets and to more structured sets, has become a major avenue of inquiry in set theory. In [23], Thilo Weinert and I investigate Ramsey-like statements for certain classes of linear orders. Recall that a linear order is *scattered* if it does not contain an order-isomorphic copy of the rational numbers. The class of scattered linear orders can be seen as the simplest natural class of linear orders extending the class of well-orders and has thus played a central role in the study of general linear orders.

In [8], Džamonja and Thompson introduce generalizations of the class of scattered orders, namely the classes of *κ-scattered* and *weakly κ-scattered* linear orders, where κ is an infinite cardinal. Briefly, a linear order is κ-scattered if it does not contain a κ-dense suborder and is weakly κ-scattered if it does not contain a κ-saturated suborder. The classes of $\aleph_0$-scattered and weakly $\aleph_0$-scattered linear orders coincide and are both simply the classical class of scattered linear orders, but, for uncountable κ, there are weakly κ-scattered linear orders that are not κ-scattered. One of the primary results of [23] indicates that, for uncountable cardinals κ such that $\kappa^{<\kappa} = \kappa$, the classes of κ-scattered and weakly κ-scattered linear orders have quite different Ramsey-theoretic properties. In particular, the class of weakly κ-scattered linear orders exhibits a type of closure under Ramsey-type statements that can consistently fail for the class of κ-scattered linear orders. On the way to this result, we also obtain a generalization of the Milner-Rado Paradox for the class of κ-scattered linear orders.

Ramsey's theorem was initially used as a tool to obtain a result in formal logic, and, since then, results and techniques from Ramsey theory have continued to be valuable throughout the study of mathematical logic. In [16], Kolesnikov and I address a model-theoretic conjecture of Grossberg concerning the amalgamation property in abstract elementary classes. We obtain results extending and improving upon those of Baldwin, Kolesnikov, and Shelah [1], in particular proving that Grossberg's conjecture is optimal. The bulk of our proof consists of a careful analysis of certain functions from the set of finite subsets of an uncountable set X into a set of logical relation symbols and is purely combinatorial in nature.

The area of infinitary combinatorics provides a fertile ground for possible undergraduate research. There are a number of reasons for this. First, the statements and, in many cases, the proofs of non-trivial theorems require a minimal amount of background knowledge, so students can get to interesting results relatively quickly. Second, infinitary combinatorics can provide students with a bridge between the worlds of finite combinatorics on one hand and set theory and logic on the other. Basic concepts of combinatorics and graph theory are familiar to or easily grasped by undergraduate mathematics students, and their generalizations to the infinitary realm lead to problems which look similar to those in finite combinatorics but may have radically different solutions or admit radically different proof techniques. This provides an attractive way for students to become introduced to basic ideas and tools in set theory and logic.

4. Current and future work

I plan to continue my work in set-theoretic compactness. Questions 1.6 and 1.9, left open from my work with Rinot and Lücke, respectively, remain very much of interest and provide interesting test questions regarding the extent of compactness and incompactness that can exist at small cardinals, and I plan on continuing to investigate them.

Question 1.14, regarding the consistency strength of Souslin's Hypothesis at $\aleph_2$, remains a major open question in the field. It has led to many fruitful developments in set theory already, and I plan on continuing to work with Rinot towards its solution.

Many interesting compactness statements regarding the non-stationary ideal, including stationary reflection and the failure of square, can be generalized to statements about other interesting ideals, such as the weakly compact ideal or the indescribability ideals. Such generalizations have not been widely studied, and I plan on investigating them with Brent Cody and other potential collaborators, including Bagaria, Gitman, Magidor, and Sakai.

More generally, my work with Rinot [22] and work of Fuchino et al. [10] point toward the possibility that there may be interesting dividing lines between various classes of reflection principles that help explain the ease or difficulty of achieving these reflection principles at small cardinals. An investigation in this direction could shed much light on reflection principles, large cardinals, and forcing axioms, and we plan on pursuing these ideas.

There are a number of major open questions regarding the combinatorics of singular cardinals or their successors that motivate my research and which I plan on continuing to investigate. Chief among these are the following three questions regarding the relationship between $\aleph_{\omega+1}$ and $\aleph_2$.

Question 4.1. *Are any of the following consistent?*

(1) *The Chang's Conjecture variation,* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$.
(2) *The existence of an infinite* $A \subseteq \omega$ *and a scale* $\vec{f}$ *in* $\prod_{n \in A} \aleph_n$ *of length* $\aleph_{\omega+1}$ *such that* $\vec{f}$ *has stationarily many bad points of cofinality* ω_2.
(3) *Transitive models of set theory* $V \subseteq W$ *such that* $(\aleph_{\omega+1})^V = (\aleph_2)^W$.

Question 4.1 motivated a large part of my work in [18] regarding interactions between square principles and covering properties. In particular, I give there an alternate prove of the fact that, under a suitable stationary reflection hypothesis at $\aleph_2$, all clauses of 4.1 must fail (this also follows from earlier work of Sharon and Viale in [26]. It seems likely that further investigations into these covering properties will shed more light on the question.

Another pair of major motivating questions are the following two, regarding Jónsson cardinals.

Question 4.2. *Is it consistent that* $\aleph_\omega$ *is a Jónsson cardinal?*

Question 4.3. *Is it consistent that there is a Jónsson cardinal that is the successor of a singular cardinal?*

Question 4.3 partially motivated my work on bounded stationary reflection in [17] and [18], and further progress will likely require further understanding of strong reflection principles at singular cardinals and their successors.

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