Voting Power: An Information Theory Approach

Abstract

In recent years, there has been increasing awareness of the importance of formal measures of voting power and of the relevance of such measures to real life political issues. Nevertheless, existing measures have been criticized, especially because of their dependence on the unrealistic assumption that different coalitions have equal probabilities.

In this paper we show that the classical problem of measuring voting power can be naturally embedded in information theory. This perspective on voting power allows us to extend measures of voting power to cases in which there are dependencies among voters. In doing so, we distinguish between two different notions of a given voter's power – 'control' and 'informativeness' – corresponding, respectively, to the average uncertainty regarding the outcome of a vote that remains when all others have voted and the average uncertainty that is eliminated when only the given voter has voted. This distinction settles a number of well-known paradoxes and that enables the study of voting power on the basis of actual political behavior at all levels.

Voting Power: An Information Theory Approach

1. Introduction

In recent years the study of voting power faced two conflicting developments. On the one hand, an increasing number of political scientists emphasized its value, borrowed its ideas, and repeatedly demonstrated its relevance to real life political issues. On the other hand, some criticized “this branch of probability theory”, claiming that “it can safely be ignored by political scientists” (Albert 2003).

The expansion of the European Union, for example, raised many questions regarding the appropriate weight to be assigned to the different members in its different institutions. This problem was investigated by a number of leading experts on voting power (e.g., Felsenthal and Machover 1997, Garrett and Tsebelis 1999 a and b, Holler and Widgrén 1999, Lane and Berg 1999, Laruelle and Widgren 1998, Nurmi 1997, Nurmi and Meskanen 1999, Steuneberg, Scmidtchen and Koboldt 1999). Voting power of different nations in a number of other international organizations has also been examined (e.g., Lane 2005, Leech 2002, Rablen 2005). The relevance of the study of voting power is also evident in many other spheres of political science, such as parliamentary affairs (e.g., Aleskerov et al. 2004), electoral competition (e.g., Feix et al. 2002), the question of apportionment and electoral districting (e.g., Leech 2002), and electoral systems such as approval voting (e.g., Brams and Sanver 2003) and two-tier voting systems (e.g., Maaser and Napel, 2005).

The sharpest criticism on the study of voting power was made by Albert in two articles (Albert 2003, 2004). The main basis for Albert's criticism is that measures of voting power ignore political realities as they usually assume equal probabilities of different possible coalitions. Since this is not the case in reality, Albert claimed that the 'empirical', 'positive' and 'theoretical' values of the voting power approach should be questioned. This criticism did not remain unanswered (e.g., Felsenthal and Machover 2003), but even those who held opposite views did not reject all of his arguments. Thus, the most forgiving responder, List (2003), accepts the claim that given its probabilistic assumptions, the voting power approach "is not a free-standing (positive or normative) theory". Most recently, Laruelle and Valenciano (2008) present a book-length critique of traditional voting power theory.

In this paper we show that the classical problem of measuring voting power can be naturally embedded in information theory. This perspective on voting power allows us to extend measures of voting power to cases in which there are dependencies among voters. In doing so, we also distinguish between two different notions of voting power in a manner that settles a number of paradoxes that have been recently analyzed in the literature.

Voting power for the case we consider here, in which there might be dependencies among voters, is sometimes called *a posteriori* voting power. We emphasize, however, that the question of how the existence of such dependencies is ascertained, empirically or otherwise, is not a concern of ours. Moreover, we acknowledge that when notions of *a priori* voting power are invoked for normative purposes, the possible existence of such dependencies might be an entirely irrelevant consideration and different types of analysis might be called for.

In Section 2, we briefly describe the problem of measuring voting power and define the Banzhaf voting measure and its obvious generalization to cases involving party inter-dependence. In Section 3, we show how the obvious generalization leads to counter-intuitive results and introduce some possibilities for remedying the definition. In Sections 4 and 5, we offer two different information-theoretic generalizations of voting power. In Section 6, we consider a variety of representative examples and compute each measure of voting power for each of them. In Section 7, we make a few observations about merged and split parties. Finally, in Section 8, we offer some conclusions.

2. Voting Power

To explain the problem of voting power, let us first consider a simple example.

Example 1: Imagine a parliament of 101 delegates in which three political parties are represented: A with 48 seats, B with 47 seats and C with 6 seats. Each of the parties votes *en bloc* either ‘yes’ or ‘no’ with no abstentions possible and with a majority of 51 votes necessary to either pass or block any resolution. In this case, any pair of parties, regardless of size, can get a bill passed or blocked. Thus, it is obvious that if the parties can be presumed to vote independently, the political power of all three parties is equal.

Plainly, it is not the case, then, that the voting power of a party is proportional to the number of seats held by that party. Thus some more sophisticated measure of voting power is necessary that yields the desired result that all three parties hold equal power. The problem of identifying such a measure dates back at least to the 1787 Constitutional

Convention (see Riker 1986) and a number of possible solutions have been suggested over the years (as summarized by Felsenthal and Machover (1998, 2005)).

One way to measure voting power was suggested by Shapley and Shubik (1954). Using the Shapley value for cooperative n-person games (Shapley 1953), they measured the relative share of a given 'prize' to be allocated to each voter. As noted by Coleman (1971), however, this approach is somewhat problematic. Compare, for instance, the case of a committee of 5 in which an ordinary majority is necessary to pass a bill to the case of a committee of 5 in which a unanimous vote is required. According to the Shapley-Shubik measure, the power of each player is identical in each situation (1/5). This correctly captures the symmetry among the voters but fails to capture the fact that in the first case a given voter's chances of getting a bill he supports passed are much greater than they are in the second case.

As a result of this apparent flaw (which some do not regard as an actual flaw, e.g., Berg 1999), a measure more commonly used these days is one originally suggested by Penrose (1952) and subsequently by Banzhaf (1966, 1968). The "Banzhaf measure" is simply the proportion of cases in which that a player (e.g., a political party) will cast a deciding vote in a committee (e.g., a parliament) with a given size, given shares of the different players, and a given required majority, when total independence between players is assumed, that is, when the probability of all possible coalitions is equal. (Note that we use the term "coalition" to refer to the set of voters supporting a given bill; the term is not intended to imply any prior coordination among the voters.)

More formally, suppose that a set, V, of n voters is asked to vote for (+1) or against (-1) some proposed bill. (Following most work in this area, we assume that abstaining is not permitted.) Given the votes of the voters, we need to aggregate them to determine which issues are approved and which are rejected. Let B be the set of voters who vote for the bill. Let f be an aggregation function that determines if B is a winning coalition (f(B)=+1) or a losing coalition (f(B)=-1). We require that f be *monotonic* (that is, $B' \supseteq B$ implies $f(B') \geq f(B)$).

For a given voter v, there are 2^{n-1} subsets of V that don't include v. We denote by v' the set of voters other than v. Let $D_v = \{B \subseteq v' \mid f(B)=-1 \text{ and } f(B \cup v)=+1\}$. That is, D_v consists of the coalitions for which the vote of v is decisive. The Banzhaf measure of the power of v is $Bz(v) = |D_v|/2^{n-1}$.

Assuming that all coalitions are equally likely, the Banzhaf measure is simply the probability that v will cast a deciding vote. When this assumption does not hold, however, the Banzhaf measure, as defined, does not necessarily correspond to any meaningful definition of power. The precise conditions under which the Banzhaf measure does correspond to some notion of power and the extent of the measure's bias when these conditions fail to hold has been the subject of intense study (Straffin 1977, 1978, 1988, Gelman et al. 2002, Kaniovski 2008).

There is one quite obvious generalization of the Banzhaf measure to cases in which different possible coalitions have different probabilities. For any disjoint sets of voters, B and N, let $p(B^+,N^-)$ be the probability that the members of B vote +1 and the members of N vote –1 (and we don't care about the other voters). These probabilities can be easily computed from the probabilities of coalitions, $p(B^+,[V-B]^-)$.

Note that while it is often the case in actual applications that the probability of any given coalition can be estimated empirically, we simply assume that these probabilities are given and make no assumptions regarding how they were determined. Moreover, we do not assume that these probabilities are a by-product of possible negotiations among voters or of constraints resulting from distributions of voter preferences in some Euclidean space. Thus, our approach differs fundamentally from that of preference-based methods for generalizing the Banzhaf measure (Braham and Holler 2005, Napel and Widgren 2005).

Recall that D_v consists of the set of coalitions in v' which leave v with a decisive vote. Then the probability that v casts a decisive vote is simply $\sum_{B \in D} p(B^+,[v'-B]^-)$. In the ordinary case considered by Banzhaf, each such coalition B has probability of precisely $1/2^{n-1}$ so that the sum is simply $|D_v|/2^{n-1}$. We think of this formula (see Heard and Swartz 1999; Gelman et al 2002, Laruelle and Valenciano 2005) as the obvious generalization of Bz to cases where different coalitions might be assigned different probabilities and we refer to it as Bz*(v).

Note that, like the original definition of Bz, Bz* captures the probability that the *other* voters will vote in such a way that v is decisive. Bz* is entirely indifferent to the probability that v will vote one way or the other. We will argue in the next section that this is a flaw of Bz* and we will suggest new measures of voting power that remedy this flaw. We will show that the classical theory of voting power can be naturally generalized by embedding it in information theory (see Shannon 1948). This follows naturally from interpreting the voting power of a voter, v, as the amount of information we obtain about the outcome of a vote (given the *a priori* probability of any coalition) by ascertaining the vote of v. We offer several information-theoretic generalizations of the classic Banzhaf measure that apply to situations where the probabilities of different coalitions are not necessary equal. We show that in such situations the notion of power bifurcates into two measures that we call *control* and *informativeness*, respectively. These correspond to the uncertainty regarding the outcome that remains when we know the votes of all voters other than v (control) and the uncertainty that is removed when we know only the vote of v (informativeness). We show the precise relationship between these measures and the original Banzhaf measure.

3. Extending the Banzhaf Measure

Let us begin by considering a simple example of a situation in which different coalitions occur with different probabilities.

Example 2: Consider A, B and C of Example 1, but now suppose that A supports every bill and B votes against every bill. Suppose further that C votes half of the time with A and half of the time with B. Under such conditions, it is apparent that C is the only player to have power: its decision always dictates the outcome.

Example 2a: More generally, suppose that A and B might support or oppose a bill but that they never agree. (This scenario is not imaginary at all. Suppose, for instance, that A is a right-wing party, B is a left-wing party, and C is a centrist party. This example has been considered in the context of voting power by Kilgour (1974). The power of "pivotal" parties to attract voters (e.g., Downs 1957) and to participate in coalitions (e.g., De Swaan 1973) has been noted many times. These are different kinds of power than the ones we consider here.) Note, however that in a case of dependency between C and the other players its power decreases; thus, for example if C almost always votes with A, the power of C is diminished. We will discuss this in detail below.

While the measure Bz* considered above is certainly a natural extension of Bz, consideration of these two examples suggests that it is somewhat problematic. In Example 2, the vote of A is decisive whenever B and C do not vote the same way. Since C votes independently of B, this means that Bz*(A) = ½. This is entirely counter-intuitive, however, since, as we have seen, all the power lies with C and none with A, which is entirely deterministic.

This counter-intuitive aspect of Bz* is further borne out in two further examples.

Example 3. Consider the case where there are five equal voters and a majority-wins system, but in which voter A is guaranteed to vote in favor of every single bill no matter what it is. All other voters vote independently and with equal probability of voting in favor or against any bill.

Example 4. Consider again five equal voters and a majority-wins system, but where the probability of any coalition of exactly 3 (or exactly 2) out of 5 is nil. That is, there are no possible coalitions for which any single voter is decisive. (Let's call this the "no close calls" case.)

In Example 3, the probability of the completely predictable voter A having a decisive vote is identical to that of any other voter, namely, 3/8. As a result, Bz* assigns the same value to A as to the other voters. Nevertheless, since A is entirely deterministic, it is counterintuitive to say that A has as much power as the other voters.

In Example 4, Bz* assigns every voter the value 0, since there are no cases in which that voter's vote is decisive. Nevertheless, it seems plain that that the players must have *some* power (Machover 2007).

Before tinkering with the definition of Bz, though, it is worth considering what answer we would prefer for Example 4. In fact, it is not hard to see that *any* answer would be somewhat counterintuitive. To see why, consider the following two scenarios.

- Scenario 1: Voter v is an extremely persuasive politician and therefore always succeeds in persuading at least three of the other four voters of his view.
- Scenario 2: Voter v and whoever is sitting closest to him are both very impressionable and once they know the majority view among the other three voters, they always vote accordingly.

How much power shall we assign to v in each of these cases? Perhaps v should be assigned much power in the first case and little power in the second case? But note that in our problem description above, we are given only the probability of each coalition and the result in each case; we deliberately ignore the question of the dynamics that create such dependencies. The "no close calls" case can be instantiated by either one of these scenarios. Thus, there could not possibly be a single "right" answer to the question of how much power v has.

The above examples and discussion lead us to reconsider what is really being measured when we quantify voting power. When we ask, in the case where all coalitions are equiprobable, for what proportion of cases is v decisive, we are in fact asking this: how much *information* about the outcome can be found in the vote of v? And this question can be formalized in two different ways:

1. Once we know how everyone but v has voted, how much uncertainty remains regarding the outcome?

Let's call this kind of power "control". This is essentially what Bz measures in the restricted case. See also Luce and Rogow's (1956) "locations of power" (Brams, 1975, 202-213).

2. How much uncertainty regarding the outcome is removed once we know how (only) v votes?

Let's call this kind of power "informativeness". Note that informativeness is not the same as influence: it is a measure of what we can learn about the outcome from knowing how v alone votes. Ironically, despite the monotonicity of the voting rule, when voters are inter-dependent, knowing that v voted in favor of a bill might be helpful in determining that the bill will fail, and vice versa (see Example 8 below).

Thus, for example, in the "no close calls" case (Example 4), each individual voter has 0 control; once we know how all the others have voted, there is no doubt left as to the outcome. Yet, each voter in that case has considerable informativeness; once we know how any individual voter votes, the probability that the outcome will be in accord with that vote is extremely high.

In the original case addressed by Banzhaf, in which every coalition is equally probable, control roughly equals informativeness (the exact relationship is discussed below). For this reason, those who have considered only the canonical case have conflated what, in other circumstances, are actually two distinct measures of power. Ignoring either one of them leads inevitably to unsatisfying results.

What remains now is to formalize the two notions of power just described. Conveniently, information theory provides us with precisely the tools we need to formalize these notions.

4. An information-theoretic generalization of the Banzhaf measure: *control*

There is a lovely correspondence between the Banzhaf measure and some basic ideas in information theory. For a discrete random variable, X, consider the standard entropy function

$$H(X) = -\sum_{x \in X} p(x) \log p(x)$$

This measures the amount of "uncertainty" or "information" inherent in X. Thus, for example, suppose X can take two values, +1 and –1. If the probability of each of these is ½, then H(X) = 1, the highest possible value. (All logs in this paper are base 2.) If the probability of +1 is either 0 or 1 (so that the probability of –1 is 1 or 0), then H(X) = 0, the lowest possible value, reflecting the fact that there is actually no uncertainty regarding the value of X. (Summands corresponding to zero probabilities are taken as zero.) For intermediate values, H(X) ranges between 0 and 1; the closer the probabilities to ½, the greater the uncertainty, that is, the greater is H(X).

The conditional entropy of X relative to Y is given as

$$H(X|Y) = -\sum_{y \in Y} p(y) \sum_{x \in X} p(x \mid y) \log p(x \mid y)$$

H(X | Y) is a formalization of the average amount of uncertainty regarding X that is left once we are given the value Y. If, once we are given Y, we can completely determine the value of X, then H(X | Y) = 0. If Y tells us nothing about X that we didn't know already, then H(X | Y) = H(X).

Thus, the closely related notion of "mutual information" is given by the formula H(X) – H(X | Y). This is a measure of how much uncertainty regarding X is reduced by knowing the value of Y. In other words, it is the amount of information that can be obtained about X by observing Y.

Returning to our voters, let $p(F=1 \mid B^+,N^-)$ represent the probability that the outcome is +1, given that members of B vote +1 and the members of N vote –1. The unconditional probability that the outcome of a random vote will be +1 is simply $p(F=1) = \sum_{f(B)=+1} p(B^+,[V\text{-}B]^-)$.

Note that we can think of F and (with a bit of convenient imprecision) v' as discrete random variables, where F takes the values +1 and –1 with probability p(F=1) and 1-p(F=1), respectively, and v' takes 2^{n-1} possible values corresponding to each of the n-1 voters in v' voting +1 or –1.

Now we can easily formalize the notions of control and informativeness introduced above in terms of information.

Definition. CON(v) = H(F | v').

That is, the amount of *control* belonging to a voter v, CON(v), is the average amount of uncertainty remaining after the votes of all the other voters are known.

For all the cases for which Bz(v) was originally defined, CON(v) = Bz(v). In fact, CON(v) = Bz*(v) for a slightly broader class than that. Let $p(v=1 \mid B^+,N^-)$ represent the probability that v votes +1, given that members of B vote +1 and the members of N vote –1. We say that v is a *free* voter if for any coalition, B, of voters other than v, we have $p(v=1 \mid B^+,[v'\text{-}B]^-) = ½$. Obviously, if all coalitions are equally probable, all voters are free voters.

Theorem 1. For any free voter v, CON(v) = Bz*(v).

Proof. According to the formula above, we need to compute the conditional probability of each possible outcome given the votes of all voters but v. For simplicity of notation, for any value $0 \le x \le 1$, let h(x) = -xlog x - (1-x) log (1-x). Then, by definition, CON(v) =

$$\sum_{B \subseteq v'} p(B^+,[v'-B]^-) * h(p(F=1 \mid B^+,[v-B]^-)).$$

If B is such that v is not decisive (i.e., if B is not in D_v, as defined at the end of Section 2 above), then the probability that the outcome is +1 is either 1 or 0, so that $h(p(F=1 \mid B^+,[v'-B]^-)) = 0$. Thus, we can rewrite the above as

$$CON(v) = \sum_{B \in D} p(B^+,[v'-B]^-) * h(p(F=1 \mid B^+,[v-B]^-))$$

Finally, if v is a free voter, then $h(p(F=1 \mid B^+,[v-B]^-)) = h(p(v=1 \mid B^+,[v-B]^-)) = 1$. In such cases we have $CON(v) = \sum_{B \in D} p(B^+,[v'-B]^-) = Bz^*(v)$. QED

It is worth contemplating carefully the cases in which CON(v) differs from Bz*(v). Consider the problematic Example 3, in which A is deterministic. Bz* gave A as much power as the other voters. However, in that example we have CON(A)=0, since "discovering" A's vote does not reduce the uncertainty of F at all; we already knew how A would vote. In short, CON depends on the probability of v to act one way or the other, while Bz* depends only on the others but not on the propensity of v. In Section 6, we will see that in a variety of cases, Bz* assigns plainly misleading degrees of power to voters.

Actually, in some sense Bz* is a crude variation on CON. To see the precise connection between these measures, we define a slightly different distribution over different possible coalitions than the original one. For every $B \subseteq v'$, define

$p'([B \cup v]^+,[V- (B \cup v)]^-) = p'(B^+,[V-B]^-) = p(B^+,[v'-B]^-)/2$. That is, the probabilities of all coalitions remain the same as given, except that we regard v as a free voter.

Now, we define $CON'(v) = \sum_{B \in D} p'(B^+,[v'-B]^-) * h(p'(F=1 \mid B^+,[v-B]^-)$

Then we have:

<u>Theorem 2.</u> For all v, CON'(v) = Bz*(v).

<u>Proof.</u> This is immediate from the fact that for all B in D_v,

$p'(B^+,[v'-B]^-) = p(B^+,[v'-B]^-)$ and

$h(p'(F=1 \mid B^+,[v'-B]^-) = h(p'(v=1 \mid B^+,[v'-B]^-) = 1$.

We can thus think of Bz*(v) as a variant of CON(v) in which we treat v as a free voter, regardless of whether this is in fact the case. This makes some intuitive sense. Power is, after all, a measure of potential, so it is meaningful to ask what the potential control of voter v is, given only the constraints on the other voters.

To summarize, we have introduced a new measure, CON(v), which we call *control*, that is a natural information-theoretic generalization of the original Banzhaf measure. Unlike Bz*, CON takes into account the predictability of v, regarding the control of v as diminished to the extent that v is predictable. In Section 6, we will consider the CON measure for each of the voters that participate in the examples we have introduced in the course of the paper.

5. A second information-theoretic measure: *informativeness*

As we already noted, there are two measures of power, control and informativeness. We have already formalized control in information-theoretic terms. In this section we do the same for informativeness.

Recall that we informally defined informativeness as the amount of uncertainty regarding the outcome that is removed once we know how (only) v votes. F and v are discrete random variables. Thus, we can formalize "informativeness" in information-theoretic terms as follows:

Definition. INF(v) = H(F) – H(F | v)

That is, INF(v) is the mutual information between F and v. The first term, H(F), is simply the uncertainty about the outcome before we know anything about the voters' preferences. If the aggregation function f is symmetric (in the sense that if all votes are reversed, the outcome is reversed) and all coalitions are equally probable (that is, if the outcomes +1 and –1 are equiprobable), then H(F) = h(p(F=1)) = 1. The second term, H(F | v), is the weighted average of the uncertainty about the outcome given, respectively, that v votes +1 and that v votes –1. *Thus, the difference between the two terms precisely measures the informativeness of v regarding the outcome.*

We saw above that CON(v) is a generalization of Bz(v). That is, in ordinary cases, CON(v) = Bz(v). Is it also the case that INF(v) is a generalization of Bz(v)? Not exactly. In fact, for all free v, Bz(v) is equal to the mutual information between v and the joint of F and v'.

Theorem 3. For any free voter v, $Bz^*(v) = H(F,v') – H(F,v' \mid v) \geq INF(v)$.

Proof. To prove the equality, note that from Theorem 1, we have that for all free v, $Bz^*(v) = H(F \mid v')$. But for any free v,

$H(F \mid v') =$

$H(F,v') – H(v') =$

$H(F,v') – H(v' \mid v) =$

$H(F,v') – [H(v',v) - H(v)] =$

$H(F,v') – [H(F,v',v) - H(v)] =$

$H(F,v') – H(F,v' \mid v)$.

To prove the inequality, note that

$H(F,v') – H(F,v' \mid v) =$

$INF(v) + [H(v' \mid F) – H(v' \mid F,v)]$

where the term in brackets is always non-negative. QED

From this we conclude that $Bz^*(v)$ actually consists of two parts: the mutual information between v and F and an extra part that is not of genuine interest. INF(v) is a measure of the interesting part only. Thus, in the canonical case where all coalitions have equal probability, $INF(v) \leq Bz(v)$; that is, informativeness is bounded by control. This is not, however, true in the general case.

One way to better appreciate the connection between Bz(v) and INF(v) is to consider the function a_v, which represents the *satisfaction* of v, namely, the proportion of coalitions for which the vote of v is identical with the outcome. As originally observed by Penrose (1952) and discussed by Laruelle and Valenciano (2005), for the case where the probability of all coalitions are equal, Bz(v) = $2a_v$-1. In this case, if we assume further that our aggregation function is symmetric, it follows that the two outcomes, +1 and –1, are equiprobable. Thus, we have that INF(v) = 1-h(a_v), where h is the entropy function defined in the proof of Theorem 1. Both 1-h(a_v) and $2a_v$-1 are strictly increasing functions that map ½ to 0 (reflecting that being on the winning side half the time amounts to being entirely uninformative) and map 1 to 1 (reflecting that always being on the winning side amounts to being maximally informative), but for all $½ < a_v < 1$, we have that $1\text{-}h(a_v) < 2a_v\text{-}1$.

6. Applications to some representative examples

We can get a better intuitive understanding of the measures we have introduced here by considering a number of examples of aggregation functions and probability distributions over coalitions. In Table 1, we show for each voter in each of the cases we have seen (and a few more we consider below), Bz*(v), CON(v) and INF(v). Let's discuss each of these examples.

Example 1. All three voters are free voters. For each of them, Bz*(v) = CON(v) = ½ and INF(v) = 1-h(3/4) ≈ 0.19. Note that, as is always the case for free voters, Bz*(v) = CON(v) ≥ INF(v).

Examples 2 and 2a. In this case, A and B always cancel each other out and C is decisive. Once we know the vote of C, we know the outcome and hence neither A nor B has any control. Additionally, the individual votes of A and B, respectively, give us no information about the outcome, so both have 0 informativeness. The probability that C will disagree with B is ½, so that the vote of A *appears* to be decisive ½ of the time and hence Bz*(A) = ½. As noted above, Bz*(A) is a measure of the control of A, if we regard A as free and the other voters as constrained. In fact, however, in all cases where A appears to be decisive, his vote is fully determined (in Example 2 it is always 1 and in Example 2a it must be the opposite of B). Therefore, CON(A) = INF(A) = 0. The values of all measures are the same for B as for A. C is in fact a dictator and Bz*(C) = CON(C) = INF(C) = 1.

Example 3. Since A is predictable, CON(A) = INF(A) = 0. However, Bz*(A) = 3/8. For all other voters, Bz*(v) = CON(v) = 3/8. INF(v) = h(11/16) – ½[1+h(1/8)] ≈ 0.13. For illustrative purposes, we show how INF(v) is calculated. First we compute H(F), the uncertainty of the outcome. Since A will certainly vote in favor, the probability of a bill passing is the probability that at least two of the remaining voters will vote in favor, namely, 11/16. Thus H(F) = h(11/16). For any voter v, other than A, p(v=1) = p(v=-1) = ½. Furthermore, p(F=0 | v=1) = 1/8 (since then the bill can only be blocked if all three other voters vote against) and p(F=1 | v=-1) = ½ (since the vote comes down two a simple majority vote among the other three voters. Thus, H(F | v) = ½*h(1/8) + ½*h(1/2) = ½[1+h(1/8)].

Example 4. All voters are identical. Since no single voter is ever decisive, for every voter v, Bz*(v) = CON(v) = 0. Recall, though, that there is something counterintuitive about this fact. After all, there were two possible outcomes so *somebody* must have decided the matter. The simple resolution of this paradox lies in the fact that while none of the voters has control, each of them has informativeness: for every voter v, knowing how v votes gives us a great deal of information about the outcome. To be precise, using the same method of calculation as above, we obtain INF(v) = 1-h(15/16) ≈ 0.66. (Note that this illustrates that our observation above that INF(v) is bounded by CON(v) does not necessarily hold when there are dependencies among voters.) Other generalizations of the Banzhaf measure (Beisbart 2007; Bovens and Beisbart 2007) have been considered that provide different solutions to this problematic example.

Let's now consider an even more extreme version of Example 4.

Example 5. Consider again five equal voters and a majority-wins system, but where the probability of any coalition of 3 out 5 or 4 out of 5 is nil. That is, every vote is unanimous, with equal chance of going either way. As in the less extreme example, Bz*(v) = CON(v) = 0. But, since knowing the vote of any single voter tells us the final result, we have for every v, INF(v) = 1.

7. Merged and split parties

The possibility of dependencies between parties raises some questions about merged and split parties (Leech and Leech 2006).

We assume, as a matter of definition, that all the components of a single party vote the same way. However, we have not considered the converse: if two parties are guaranteed to always vote identically, they are effectively a single party. What is the relationship between the control and informativeness of each of the constituent parties in such an arrangement and the control and informativeness of the entire unified party? Ideally, we would want these to be as similar as possible since the difference between two parties that always vote the same way and a single party consisting of both of them is a purely semantic one.

Furthermore, if two previously independent voters decide to become completely aligned – that is, they merge into a single party that votes as a bloc – does this guarantee that the control and informativeness of the merged party is at least as great as was that of each of the constituent parties prior to the merger? Certain measures have been criticized in the past because they did not guarantee a positive answer to this question (e.g., Brams 1975).

The answer to these questions is as follows. Each of the constituent parties in a merger has exactly the same degree of informativeness as the merged party. However, each of the constituent parties has 0 control, since once we know the votes of the other voters (including the other constituents in the merged party) there remains no uncertainty at all with regard to the outcome. Furthermore, a merged party always has at least as much control and informativeness as each of the constituent parties had prior to the merger.

To illustrate these points, let's consider three more examples.

Example 6: Imagine that, as in example 2 above, we have a parliament of 101 delegates in which A has 48 seats and B has 47 seats and A and B never agree. However, now party C in Example 2 splits into two entirely independent components: C1 with 4 seats and C2 with 2 seats. In effect, C1 decides every vote and C2 is a dummy. Therefore, CON(C1) = INF(C1) = 1 and CON(C2) = INF(C2) = 0.

Example 7. Suppose now that in spite of the split, C1 and C2 always vote together – half of the time with A and half of the time with B. That is, we have returned to Example 2 except that C1 and C2 are *nominally* distinct, but in fact vote as a bloc. We already know, from Example 2, that for the unified party C, we have CON(C) = INF(C) =

1. What about each of the constituents, C1 and C2? As in Example 6, once we know how C1 votes, we know the outcome, so INF(C1) = 1, just as it was prior to the merger and just as holds for the unified party C of which it is a constituent. However, unlike in Example 6, once we know how everyone other than C1 has voted, we also know the outcome (since once we know the vote of C2, we also know the vote of C1). Thus, CON(C1) = 0; by merging, and hence creating dependency with C2, C1 has sacrificed control. Note further, that, although C2 is a dummy in the usual sense, C2 has as much informativeness as C1 in the sense that once we know how C2 votes, we know the outcome. Thus, INF(C2) = 1; the dependency with the dictator C1 has increased the informativeness of C2. Of course, CON(C2) remains 0.

Note that in all the above, we deliberately ignore the question of how it happens that C1 and C2 always vote the same way: is it C1 that follows the lead of C2 or vice versa? (Indeed, in many practical cases we wouldn't know the answer to this question.)

Finally, we consider a cautionary example.

Example 8. Let the parties be as in Example 6 except that the dummy C2 always votes exactly *opposite* of C1. Then, for each party, each measure yields the same result as in Example 7. In particular, note that CON(C2) = 0, as is intuitive, but that INF(C2) = 1, even though C2 loses every vote! This simply reflects the fact that knowing the vote of C2 alone is sufficient to be able to determine the outcome. This extreme example should serve as a warning not to misinterpret the kind of power that INF represents.

- Table 1 about here -

8. Conclusions

In the words of Max Weber, "Power is the probability that one actor within a social relationship will be in a position to carry out his own will despite resistance, regardless of the basis on which this probability rests." (Weber, 1978). But Bertrand Russell (1938, 4) notes that: "…the fundamental concept in social science is Power, in the same sense in which Energy is the fundamental concept in physics. Like energy, power has many forms, such as wealth, armaments, civil authority, influence on opinion. No one of these can be regarded as subordinate to any other, and there is no one form from which the others are derivative. The attempt to treat one form of power, say wealth, in isolation, can only be partially successful, just as the study of one form of energy will be defective at certain points, unless other forms are taken into account." At the same time, one cannot escape the realization that, as Bachrach and Baratz (1962) emphasized, there is more than one face to 'power'.

In this article we examined only one form of power – voting power. There is no doubt, however, that this 'form' of power does have outstanding importance in modern politics – especially in democracy, as well as in international organizations. Nevertheless, the study of voting power has tended to concentrate on one meaning of the concept – the probability of one to be 'decisive' – especially as measured by Bz. What Russell said about different forms of 'power' can be said about the different forms of 'voting power': i.e. the attempt to treat one form of voting power, say Banzhaf's voting power, in isolation, can only be partially successful.

Once we remove the assumption of party independence, we reveal two different forms of voting power – 'control' and 'informativeness'. These forms of power can be very different from each other under quite common circumstances, a fact that has been relatively neglected. Once we tease apart these two measures of voting power a number of "paradoxes" are easily understood.

Moreover, the generalization of measures of voting power to cases of party inter-dependency permits the application of voting power to realistic situations, and consequently heads off the criticism that the study of voting power "can safely be ignored by political scientists". Voting power can now be computed even for those common cases in which actual behavior of parties makes apparent that different coalitions occur with different frequencies.

Acknowledgment. The authors wish to express their deep gratitude to Dan Felsenthal, an outstanding scholar and a dear friend, for many helpful discussions concerning the ideas in this paper.

Table 1: Bz*, 'Control' and 'Informativeness' measures of all players in Examples 1-8

Example	Player	Bz*	Control	Informativeness
1	All	1/2	1/2	1-h(3/4) ≈ 0.19
2/2a	A, B	1/2	0	0
	C	1	1	1
3	A	3/8	0	0
	others	3/8	3/8	h(11/16)-½[1+h(1/8)] ≈ 0.13
4	all	0	0	1-h(15/16) ≈ 0.66
5	all	0	0	1
6	A, B	1/2	0	0
	C1	1	1	1
	C2	0	0	0
7	A, B	1/2	0	0
	C1	1	0	1
	C2	0	0	1
	C(=C1+C2)	1	1	1
8	A, B	1/2	0	0
	C1	1	0	1
	C2	0	0	1

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