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Uncertainty Rules in Talmudic Reasoning

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The Babylonian Talmud, compiled from the 2nd to 7th centuries C.E., is the primary source for all subsequent Jewish laws. It is not written in apodeictic style, but rather as a discursive record of (real or imagined) legal (and other) arguments crossing a wide range of technical topics. Thus, it is not a simple matter to infer general methodological principles underlying the Talmudic approach to legal reasoning. Nevertheless, in this article, we propose a general principle that we believe helps to explain the variety of methods used by the Rabbis of the Talmud for resolving uncertainty in matters of Jewish Law (henceforth: Halakhah). Such uncertainty might arise either if the facts of a case are clear but the relevant law is debatable or if the facts themselves are unclear.

1. A formal model

Roughly speaking, the principle we argue for is that, in general, halakhic rules for dealing with uncertainty are not probabilistic, but rather are action rules telling us what to do.

Thus, suppose that in situation S we have that

1. If a_1 we do x_1
2. If a_2 we do x_2
3. $\neg x_1 \wedge \neg x_2$ implies y

If there is a 50% doubt about a_i we formally decide $\neg x_i$. Assume that there is such a 50% doubt. Having formally decided $\neg x_1 \wedge \neg x_2$, we now get y. This conclusion holds even if we know that $a_1 \vee a_2$ must logically hold.

Let us now consider one such model.

Our starting point is a language for describing states and actions. Our language has constants for states, $s_1, \ldots, s_k, \ldots$ constants for actions $a_1, a_2, \ldots$ and notation for sets of actions, e.g. $\mathbb{A} = \{a_1, a_2\}$.

We can take predicates like $P(s,x)$ meaning $P(x)$ holds at state s, and predicates like $\mathbf{move}(\mathbb{A}, s, s')$ reading: the set of actions $\mathbb{A}$ moves us from state s to state s'. Our axioms have the form:

$$\bigwedge_i (P_i(s,x) \wedge \mathbf{move}(\mathbb{A}, s, s')) \rightarrow \bigwedge_j P'_j(s', x).$$

If state s satisfies P_i for x and we move to s' by doing $\mathbb{A}$, then s' satisfies P'_j for x.

We also have a language with meta-predicates $\Psi(P(s, x))$ reading: the property P is classified as an instance of Ψ, at the state s for the individual x.

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A history **s** is a sequence of states $\mathbf{s} = (s_1, s_2, \ldots, s_k)$ such that **move**$(\{a_i^j\}, s_1, s_{i+1})$ holds for some $j = 1, 2, \ldots, m(i)$. This means the actions $\{a_i^j\}, j = 1,2, \ldots, m(i)$ were taken at s_i and we shifted from state s_i to state s_{i+1}. We also have a language with O_T and F_T. $O_T\Psi$ means Ψ is obligatory and $F_T\Psi$ means Ψ is forbidden. (For the nature of rules and character of O and F, see Abraham *et al.* 2010.[1])

A halakhic decision takes the following form. Suppose we moved along the states $s_1, \ldots, s_r$. Suppose at state s_i we have $P_{i,j}(s_i, x_i)$ holding, $j = 1, \ldots, m(i)$. Suppose actions $\{a_i^k\} = \mathbb{A}_i$, $k = 1, \ldots, n_i$, are responsible for moving us from state s_i to s_{i+1}. Then the Halakhah might stipulate that $\Psi(P(s_r, x))$ holds. We write this as follows:

$$\left(\bigwedge_i \mathbf{move}(\mathbb{A}_i, s_i, s_{i+1})\right) \wedge \bigwedge_{i,j}[P_{i,j}(s_i, x_i) \wedge \Psi_{i,j}(P_{i,j}(s_i, x_i))]) \Rightarrow !\Psi(P(s_r, x)). \qquad (1)$$

The arrow '⇒!' symbolises halakhic stipulation and we allow for some of the $\Psi_{i,j}$ not to appear in Equation (1).

Thus, for example, the Bible forbids doing any work on the Sabbath (*šabbat*). Call this $F_T\Psi_1$ where $\Psi_1(P)$ means that P is a 'work' predicate. A fellow bought a complicated do-it-yourself cupboard and wants to slot all pieces together on the Sabbath. Is this considered work? Here $P(x)$ is to slot x together' and we are asking whether $\Psi_1(P(x))$ holds. Once a ruling is given, then the ruling holds from then on, and $P(x)$ is forbidden.

Let **HR** (s) be the set of halakhic rulings of Equation (1) available at state s. When we move from state s to state s', we carry the halakhic rulings with us and may add some new rules. Thus, **HR**(s) is a subset of **HR**(s'). When in state s' a question arises as to the status of some predicate $\Psi(P(s, x))$, we check whether some ruling of Equation (1) can be instantiated to give an answer. If not, we require a ruling and the new ruling of Equation (1) is added to **HR**(s'). This is how the sets **HR** grow and evolve.

2. Majority rules in the Talmud

In what follows we treat various uncertainty examples in the Talmud and show that the considerations involved are not probabilistic but operational rulings for the practicing individual to take action.

One of the Talmud's guiding principles for dealing with uncertainty is 'follow the majority' (*Ḥulin* 11a). As we shall see, this rule is applied in a variety of ways. Perhaps the canonical form of the rule concerns the oft-cited (e.g. *Ketubot* 15a) case in which an unlabelled piece of meat is found on the street in a town with p

[1] Note that here we use the meta-predicates Ψ as objects of obligations and prohibitions $O_T\Psi$ and $F_T\Psi$. We can use predicates $P(s,x)$ as well as meta-predicate defining themselves., i.e.

$$\Psi_{P(s,x)}(Q(s', y))$$

holds iff

$$Q(s', y) = P(s, x)$$

holds. Thus, we can also have O_TP and F_TP

kosher (*kašer*) butcher shops and q non-kosher butcher shops. All other considerations (such as proximity and size of the shops) being equal, the meat is deemed kosher if and only if $p > q$. We note as an aside that will be of some importance below that if $p = q$, the meat is deemed 'in doubt' by this decision method and a secondary decision method must be invoked to resolve the matter.

This sounds rather straightforward. However, scattered around the Talmud we find a number of extensions of this rule as well as a number of limitations. Let us now consider the complete picture.

The Talmud (*Ḥulin* 11a) states that there are two distinct principles of 'follow the majority' that cannot be inferred one from the other. The first principle is typified by the example of the meat we just considered in which the majority is said to be 'present'. The second principle involves what is called an 'absent' majority and is typified by the following example. The milk of a cow suffering from some life-threatening illness is not kosher. Such an illness might be completely undetectable unless we slaughter the cow and perform an autopsy. Nevertheless, we can drink milk from a random cow, despite the inevitable uncertainty regarding its health, because most cows are healthy.

The Talmud does not define the difference between present and absent majorities but it is worth attempting to define that difference since there are important differences in the respective 'follow the majority' principles. For example, the 2nd century scholar, Rabbi Me'ir, asserts that an absent majority does not constitute sufficient grounds to overturn an existing status quo but implies that a present majority does (*Yevamot* 67b). Conversely, in capital cases, where we require something approximating certainty to convict, a present majority never constitutes grounds for conviction but an absent majority might. For example, in a case that rests on establishing the identity of a defendant's mother, the fact that 99 out of 100 candidate mothers would satisfy the conditions for conviction (a present majority) is inadequate grounds to convict, but the fact that (in the absence of countervailing evidence) most apparent family relationships represent biological relationships (an absent majority) is sufficient grounds to convict.

The examples cited in the Talmud of each type of majority as well as the above rules suggests that the difference is that a present majority entails a closed set of objects the proportion of which have some relevant status is known. An absent majority entails some empirical claim regarding the proportion of a population that has some relevant status, where the claim is based on some sample. Since this sample might not be currently present, such a majority is regarded as absent. (The distinction between present and absent majorities can be fruitfully compared with the distinction between the classical interpretation of probability, motivated by gambling applications, and the frequentist interpretation of probability, motivated by insurance applications, see *von Mises 1928*.)

Consequently, the rule that we follow a present majority is regarded as formal and procedural. It is treated identically whether the majority is 0.51 or 0.99. Furthermore, the conclusion to which it leads is never regarded as a certainty sufficient for convicting in a capital case. By contrast, an absent majority is tied to an underlying empirical claim and hence the rule that we follow an absent majority is linked to the strength that the Rabbis wished to assign to that claim. Rabbi Me'ir always regards empirical claims as sufficient only to yield a default rule, which in turn he regards as no stronger than a different default rule that presumes that the last known status quo continues. On the other hand, those who do not accept

Rabbi Me'ir's view hold that a sufficiently strong empirical claim be treated as a certainty for legal purposes.

Let us now consider a simple paradox that arises in the use of the rule that we follow an absent majority.

1. Suppose that known kosher milk and known non-kosher milk (call this state s), have been inadvertently mixed. Call this action of mixing action a, resulting in a new state s'. In symbols, we have

   $\neg$ Kosher (s, unit of milk with label number i) $i = 1,2,3, \ldots, n$, and Kosher (s, unit of milk with label number j), $j = n+1, n+1, \ldots, m$ and action **Mix** applied to the units takes us from state s to the new mixed state s' where the numbering labels on the units is lost.

   We need a rule which will say whether Kosher (s', unit of milk without a number) is true or not in state s'.
2. The mixture is kosher if the proportion of kosher to non-kosher units of milk in the mixture is greater than 60:1. This is the rule applied in his case which decides whether the milk is kosher. So the rule is

   $\wedge_i \neg$ Kosher (s, unit of milk with label number i) $i = 1,2,3, \ldots, n$, and $\wedge_j$ Kosher (s, unit of milk with label number j), $j = n+1, n+1, \ldots, m$, and $n/m < 1/61$ and action **Mix** applied to the units takes us from state s to the new mixed state s' where the liquids are mixed $\Rightarrow$ Kosher (s', mixed liquid).

3. Now suppose that it is known in general that 5% of all cows are non-kosher due to various endemic illnesses, though these cannot be identified through external examination. Call this state t.
4. Now we take the combined milk of a huge herd of cows, as is common in the dairy industry. Call this action b, resulting in state t'.

We need a rule to tell us whether this milk kosher or not. The probability that less than 1/61 of this milk is non-kosher is vanishingly small, so that one might think that it is non-kosher by the rule used in 2 above. Nevertheless, the vast majority of commentators do not rule this way. The principle is that by the rule that we follow an absent majority, it has already been decided (as we saw above) that each individual cow is kosher. Once that decision has been made, the matter is settled. The mixture is regarded as consisting of 100% kosher milk. Formally, the rule is as follows:

> **If** in state s [less than 50% of cows are unhealthy] **and** [milk from an unhealthy cow is non-kosher] **and** [we take action **Mix** of all milk from all the cows at state s and thus move by **Mix** action to state s' in which the milk from all the cows is mixed], **then** at s' the milk is kosher.

We add the rule above to $\mathbf{HR}(s')$. This rule is of the correct form Equation (1). Let us write it more carefully:

> **If** in state s [less than 50% of cows are unhealthy] (call this $P(s, \text{cows})$) **and** [milk from an unhealthy cow is non-kosher] (i.e. $\neg\Psi(P'(s, \text{milk}))$ where

'Kosher' = 'Ψ') **and** [we take action **Mix** of all milk from all the cows at state s (call this **move** (**Mix**, s, s')) and thus move by **Mix** action to state s' in which the milk from all the cows is mixed], **then** $\Psi(Q(s'$ milk)), where $Q(s'$ milk) is the mixture.

If we write the ruling only in symbols we get

$$[P(s, \text{cows}) \wedge \neg\Psi(P'(s, \text{milk})) \wedge \mathbf{move}(\mathbf{Mix}, s, s')] \Rightarrow !\Psi(Q(s' \text{ milk})).$$

3. Extensions and limitations of the present majority rule

In what follows we discuss extensions and limitations on the formal decision rule that we follow a present majority. We will see that it too is quite different than what we customarily think of as probabilistic reasoning.

Suppose we have three pieces of identical meat of which one unidentified piece is non-kosher (call this state x). Using the 'follow the (present) majority' rule, the Talmud states (*Giṭṭin* 54b) that each of the pieces is regarded as kosher. More remarkably, the 15th-century commentator Rabbi 'Ašer (in gloss 37 to *Ḥulin*, chapter 7) interprets this to mean that we are permitted to eat all three pieces simultaneously. Indeed, he rules that if the three pieces are liquefied, the liquid mixture can be drunk even though it is known with certainty that 1/3 of the mixture is non-kosher, far in excess of 1/61. The principle is quite clear. Once some rule has been invoked (in this case, to treat each piece as kosher), the matter is settled and can be applied even after subsequent state transitions occur such that the prior decision leads to absurd conclusions (in this case, that all the pieces can be eaten).

It is worth noting that if there are two pieces of identical meat of which one unidentified piece is non-kosher, the majority rule is obviously inapplicable. In such case, we have an 'unresolved set' and some secondary decision method must be invoked to resolve the matter. However, the secondary method invoked in the case of an unresolved set is different than the secondary method invoked in the case we saw above in which an isolated piece of meat is found in a town with an equal number of kosher and non-kosher butcher shops. The Talmud (*Kritut* 17b) does not treat an item from an unresolved set in the same way as it treats an item that is in doubt.

Another example of the present majority rule, indeed its purported source according to the Talmud (*Ḥulin* 11a), is the rule that when there is disagreement among a panel of judges, the ruling is according to the majority. The critical point to note is that in this case there is no uncertainty at all regarding facts and hence interpreting the present majority rule as a probabilistic method for resolving uncertainty regarding facts is, ipso facto, too narrow. Rather, the rule must be interpreted as concerning the treatment of mixed sets and can be stated as follows:

> Given a set of objects the majority of which have the property P and the rest of which have the property not-P, we may, under certain circumstances, regard the set itself and/or any object in the set as having property P.

An important extension of the present majority rule applies to cases in which the set in question does not consist of objects but rather of abstract possibilities.

For example, in a civil case involving a man who accuses his just-wed wife of infidelity (based solely on the uncontested fact that at the time of their marriage she was not a virgin), the Talmud (*Ketubot* 9a) argues on her behalf that (a) it is not known if she was raped or had intercourse of her own volition and (b) in either case, it is not known if the event occurred subsequent to her contracting marriage with her husband. Since the husband would prevail in the case only for one of the four possibilities in the Cartesian product (she willingly had intercourse subsequent to contracting marriage), he loses the cases on grounds of the formal present majority rule.

Formally the problem is that we know that at the time s' of the marriage x was not a virgin; call this $Q(s',x)$. Call her previous state s and assume that in that state she was a virgin, but it is not clear what action moved her from state s to state s'. It could have been rape (action a_1), it could have been consent (action a_2) and either case could have been before or after contracting marriage, $P(s, x)$ or $\neg P(s, x)$. It is clear the formal pattern is the following:

- $(P_i(s, x) \wedge \mathbf{move}(a_i, s, s')) \rightarrow Q(s', x)$, for $i = 1, \ldots, k + m$.

  We also know that

- $(P_i(s, x) \wedge \mathbf{move}(a_i, s, s')) \Rightarrow\ !\Psi(Q(s', x))$, if $i \leq k$

  and

- $(P_i(s, x) \wedge \mathbf{move}(a_i, s, s')) \Rightarrow\ !\neg\Psi(Q(s', x))$, if $k < i \leq k + m$.

We observe $Q(s',x)$ but we do not know which action a_i was taken. What is the ruling Ψ or not Ψ?

The commentators note that no claim has been made that the probabilities of her having been raped or of her having had intercourse subsequent to the contract, respectively, are precisely $\frac{1}{2}$. Rather the claim is that nothing is known about these probabilities at all (and indeed if the probabilities were known, different decision method would be invoked). Thus, the method is vulnerable to manipulation in a manner somewhat reminiscent of Bertrand's paradox. For example, we could artificially collapse the majority argument by restating the crucial issue as 'infidelity or not infidelity' and, conversely, we could artificially strengthen the majority argument (to seven out of eight, rather than three out of four) by distinguishing between violent rape and statutory rape. The argument is thus seen to rest rather formally on assumptions about what categories are natural kinds (e.g., rape) and which are not (e.g., statutory rape).

Finally, we turn to a crucial limitation on the application of the present majority rule. Suppose we have a set of 10 pieces of meat, 9 of which are kosher and 1 of which is non-kosher and identifiable as such (say, by its position in the pile). Now we randomly choose a piece without paying attention to which one, and, having done so, it is no longer possible to determine whether it was one of the kosher pieces or the non-kosher piece. In contrast with the canonical case of a piece of meat found in the street in which we assign a status according to the majority of the sample from which it is drawn, in this case the Talmud rules that the proportion of kosher and non-kosher pieces is irrelevant. Rather, the set is treated as an unresolved set, precisely as in the case above in which two pieces of meat, one kosher and one non-kosher, are mixed.

The principle is this. An isolated item such as one found on the street must be assigned some status applicable to an individual item, e.g., kosher or non-kosher, and hence the majority rule is invoked to resolve the matter. An unresolved set might, however, be assigned a third status appropriate to a set, namely, neither kosher nor non-kosher but, rather, mixed. An item that is taken from an unresolved mixed set simply inherits the mixed status of the set from which it is taken; it is treated like a chip off the old block. Plainly, from a probabilistic point of view, it is hard to distinguish the case of the found piece from the case of the selected piece (and although it is tempting to suggest psychological explanations, these do not hold water when the full range of examples is carefully examined).

4. Conclusion

We have seen that the Talmudic way of dealing with uncertainty is pragmatic and not probabilistic

1. Given a situation s arising from action x, resulting in situation s' where some uncertainty occurs, the Talmud makes a decision that sticks and allows life to continue.
2. Given a situation x which could have arisen from one of actions $a_1, \ldots, a_n$, we may have uncertainty as to the nature of the situation depending on which action a_i gave rise to it. Again, we use a rule to make a decision.

The above rules are not probabilistic because in a sequence of actions, we draw conclusions that persist even in cases where judging the final state in isolation might have lead to very different conclusions.

References

Abraham, M., Gabbay, D. and Schild, U. 2010. 'Obligations and prohibitions in Talmudic deontic logic', in G. Governatori and G. Sartor, eds., *Deon 2010 Proceedings*, Springer, LNCS.

von Mises, R. 1928 [translated by Dover, 1957]. *Probability, Statistics and Truth*, Springer.