# Generalized Raiffa solutions

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**Abstract**

We define a family of solutions for $n$-person bargaining problems which generalizes the discrete Raiffa solution and approaches the continuous Raiffa solution. Each member of this family is a *stepwise solution*, which is a pair of functions: a *step-function* that determines a new disagreement point for a given bargaining problem, and a *solution function* that assigns the solution to the problem. We axiomatically characterize stepwise solutions of the family of generalized Raiffa solutions, using standard axioms of bargaining theory.

# 1 Introduction

## 1.1 Stepwise bargaining solutions

Bargaining is in many cases a process in which the parties achieve interim settlements step by step, where each settlement is a starting point for further negotiations. It is possible to describe such a graduated process, in the framework of Nash's bargaining theory (Nash (1950)), as a change of the disagreement, or status quo point, while keeping the set of possible utilities—the bargaining set—fixed. Raiffa (1953) suggested two variants of a solution of this kind, for two-player bargaining problems.

In the first solution, the step—the interim agreement —is discrete. Given a disagreement point $d$, the most preferred outcome for a player is the one that gives her the maximal utility while keeping the other player at her disagreement utility. The interim agreement is the average of these two, most preferred points. By repeating these steps, using each interim agreement as a new disagreement point, the process converges to a Pareto optimal point of the bargaining set.

The second solution suggested by Raiffa is continuous. At each point in time, the process moves continuously in the *direction* of the average of the two most preferred points, rather than to this point itself.

Here, we propose a family of discrete solutions that bridges Raiffa's discrete solution and the continuous one. A generalized Raiffa solution is indexed by $p \in (0, 1]$. The interim agreement for the $p$-solution is the convex combination between the old disagreement point, and the interim agreement point of the Raiffa solution, where $p$ is the weight of the latter.

For $p = 1$, we have, of course, the discrete Raiffa solution. Furthermore, as we will show, when $p$ approaches 0, the solutions approach the continuous

Raiffa solution. All the solutions we discuss are defined for $n$ players.

### 1.2 Axiomatization

We provide an axiomatization of the family of generalized Raiffa solutions. The novelty of our axiomatization lies in the explicit expression of the graduated aspect of the bargaining. We propose a *stepwise solution* for bargaining problems which consists of *two* solution functions: a solution function that specifies an interim agreement and a solution function that specifies the terminal agreement arrived at. The relation between the two functions is formulated as an axiom that says that the terminal agreement in a bargaining problem is obtained by changing the disagreement point to the interim agreement.

The axioms used are the most basic and standard axioms of bargaining theory. Only one axiom is required from the solution function that specify the terminal agreement: that it is individually rational. The main axiomatic thrust involves the solution function that determines the interim agreement, that drives the bargaining process. It is required to be strongly individually rational, symmetric, scale covariant, independent of the non-individually rational points, and monotonic in the bargaining set.

These five axioms are almost the same as the axioms defining the Kalai and Smorodinsky (1975) solution (KS), which also requires Pareto optimality, something that is obviously not required for an interim disagreement point. Symmetry, scale covariance, individual rationality and independence of non-individually rational points are the same in both axiomatizations (though, the last of these is listed by KS as an "assumption" rather than an axiom). Finally, the monotonicity axiom used here is the same as the simple axiom used in Kalai (1977), and does not require the caveats used in the monotonicity axiom of the KS solution.

### 1.3 Implementation

Myerson (1997) describes an implementation of each of the interim agreements points reached in the discrete Raiffa solution for two-player bargaining problems. For the $k$-th interim agreement in the sequence, a game with perfect information is constructed, with $k$ rounds, that has a unique subgame-perfect equilibrium, and the expected payoff vector of which is this interim agreement. Trockel (2009) describes a game the payoff vector of which is the discrete Raiffa solution, namely, the limit of the sequence of interim agreements.

We extend Myerson's games to implement the interim agreements of the generalized Raiffa solutions for $n$-player problems. In these games, the parameter $p$ of the solution is the probability that the bargaining game does not end and continues to the next round. A similar extension is possible for the game described in Trockel (2009).

A disadvantage of these implementations is that like in many other non-cooperative bargaining an agreement is reached immediately. It is desirable to find implementations for the generalized Raiffa solutions in which the players actually reach the interim agreements, as in the games in John and Raith (1999).

### 1.4 Related work

An axiomatic expression of the *process* of bargaining was given first in the axiomatization of the proportional solution in Kalai (1977). The axiom of step-by-step negotiation in Kalai's paper states that certain types of decompositions of the bargaining set result in the same solution to the problem. Later work emphasizes axioms that involve the change of a disagreement point while keeping the bargaining set fixed (Thomson (1987), Peters and van Damme (1991), Livne (1989), Anbarci and Sun (2009), Trockel (2009)). In particular, properties of the set of disagreement points that do not change the solution were studied in some of these works. In all these papers no assumptions are made regarding a specific process like the one in Raiffa (1953).

Salonen (1988) axiomatize the discrete Raiffa solution for $n$-person bargaining problems. Livne (1989) and Peters and van Damme (1991) axiomatize the continuous Raiffa solution for two-person bargaining problems. More recently, Anbarci and Sun (2009) propose an axiomatization of the discrete Raiffa solution for the two-person case. These works axiomatize the solution, but not the process itself.

The continuous Raiffa solution is described rigourously and axiomatized by Livne (1989) and Peters and van Damme (1991) for two players only. The path that leads to the solution is described in the two dimensional space by specifying the utility of one player in terms of the utility of the other. We, in contrast, describe a path in the bargaining set parameterized by *time*. This captures more clearly the process aspect of the bargaining and is naturally extended to any number of players. Although our axioms cannot capture the continuous Raiffa solution, since it does not have a "next disagreement point", we show that the continuous solution is the limit of the discrete generalized Raiffa solutions that we characterize. A time parameterized path of partial agreements is described in O'Neill *et al* (2004), but bargaining is described there by a continuum of Pareto frontiers rather than one bargaining problem.

## 2 Preliminaries

Let $N$ be a finite set of players of size $n$. Utility vectors for $N$ are points in $R^N$. For $i \in N$ and $x = (x_i)_{i \in N}$ in $R^N$ we write $x_{-i}$ for the projection of $x$ on $R^{N \setminus \{i\}}$. We write $x = (x_i, x_{-i})$. For $x, y \in R^N$, $x \geq y$ means that for each $i \in N$, $x_i \geq y_i$, and $x > y$ means that $x_i > y_i$ for each $i$. When we use elements of $R^N$ as linear functionals on $R^N$ we denote them in bold face. Thus, for fixed $\mathbf{a} \in R^N$ and $\alpha \in R$, $\mathbf{a}x = \alpha$, where $\mathbf{a}x = \sum_{i \in N} \mathbf{a}_i x_i$, describes a hyperplane in $R^N$. A subset $X$ of $R^N$ is *comprehensive* if for each $x \in X$, $\{y \mid y \leq x\} \subseteq X$. The interior of a comprehensive set is not empty. The set $X$ is *positively bounded*, if there exist a linear functional $\mathbf{a} \in R^N$ and a constant

$\alpha \in R$, such that $\mathbf{a} > 0$ and $\mathbf{a}\,x \leq \alpha$ for each $x \in X$. A point $x \in X$ is *Pareto optimal* in $X$, if $y \geq x$ for $y \in X$ implies $y = x$. The set $X$ is *symmetric* if it is invariant under permutations of the players. A vector in $R^N$ is *symmetric* if all its coordinates are the same.

A *bargaining set* for $N$ is a nonempty subset $S$ of $R^N$ which is closed, convex, comprehensive, and positively bounded. In addition we require that all the boundary points of $S$ are Pareto optimal in $S$.

A *bargaining problem* (or a *problem*, for short), for $N$ is a pair $(S, d)$, where $S$ is a bargaining set and $d \in S$. The point $d$ is called the *disagreement point*. The *individually rational* points for a bargaining problem $(S, d)$ are the points in $S_d = \{x \mid x \in S,\ x \geq d\}$. The problem $(S, d)$ is *symmetric* if both $S$ and $d$ are symmetric.

The set of all bargaining problems is denoted by $\mathcal{B}$.

# 3 Stepwise-bargaining axioms

A *stepwise solution* is a pair of solution functions, $(\delta, \sigma)$, $\delta \colon \mathcal{B} \to R^N$, $\sigma \colon \mathcal{B} \to R^N$, such that for each problem $(S, d)$, $\delta(S, d)$ and $\sigma(S, d)$ are in $S$. The solution function $\delta$ assigns to each problem an interim agreement while the solution function $\sigma$ assigns to each problem a terminal agreement. The following axioms should be read with the quantifier "for each $(S, d)$ in $\mathcal{B}$".

The first axiom describes the role of the interim agreement as a status quo point for the rest of the bargaining process.

**Axiom 1 (Step)** $\sigma(S, d) = \sigma(S, \delta(S, d))$.

There is only one requirement of the solution function: that it selects individually rational points.

**Axiom 2 ($\sigma$-Individual rationality)** $\sigma(S, d) \in S_d$.

We now consider several standard axioms, though they involve $\delta$ rather than $\sigma$.

Strong individual rationality requires that the new disagreement point is at least as desirable for all players as the old one, and if there is room for improvement on $d$, then the new disagreement point should improve upon $d$.

**Axiom 3 ($\delta$-Strong individual rationality)** *$\delta(S, d) \in S_d$, and if $d$ is not Pareto optimal in $S$, then $\delta(S, d) \neq d$.*

**Axiom 4 ($\delta$-Symmetry)** *If all players are symmetric in $(S, d)$ then they are also symmetric in $\delta(S, d)$.*

**Axiom 5 ($\delta$-Scale covariance)** *If $a, b \in R^N$, $a > 0$ and $f(x) = (a_i x_i + b_i)_{i \in N}$, then $\delta(f(S), f(d)) = f(\delta(S, d))$.*

**Axiom 6 ($\delta$-Monotonicity)** *If $S \subseteq T$ then $\delta(S, d) \leq \delta(T, d)$*

**Axiom 7** **($\delta$-Irrelevance of non-individually rational points)** *If $S_d = T_d$ then $\delta(S, d) = \delta(T, d)$*

The family of generalized Raiffa stepwise solutions is introduced in the next section. But we already state our main theorem here.

**Theorem 1** *A stepwise bargaining solution $(\delta, \sigma)$ satisfies axioms 1-7 if and only if it is a generalized Raiffa solution.*

# 4 Generalized Raiffa solutions

## 4.1 Defining the solutions

The set of generalized Raiffa solutions is a family of stepwise bargaining solutions $(\delta^p, \sigma^p)$ for $p \in (0, 1]$. The solution function $\sigma^1$ for two players, was introduced in Raiffa (1953).

For a bargaining set $S$ and $x \in S$ the set $\{y_i \mid (y_i, x_{-i}) \in S\}$ is not empty since $x \in S$. The positive boundedness of $S$ guarantees that the set is bounded from above, and by the closedness of $S$ the set is closed. Thus, for a fixed bargaining set $S$ and $i \in N$, the real valued function $m_i(S, \cdot)$ on $S$, described next, is well defined:

$$m_i(S, x) = \max\{y_i \mid (y_i, x_{-i}) \in S\}.$$

The *Utopia point* for a bargaining problem $(S, d)$ is $m(S, d) = (m_i(S, d))_{i \in N}$. Obviously, $m(S, x) \geq x$ for all $x$ in $S$.

**Definition 1** *For $p \in (0, 1]$ the function $\delta^p : \mathcal{B} \to R^N$ is defined by*

$$\delta^p(S, d) = d + (p/n)(m(S, d) - d).$$

To see that $\delta^p(S, d) \in S$, consider for each $i$ the point $e^i = (m_i(S, d), d_{-i})$ in $S$. It is easy to see that $\delta^1(S, d) = (1/n) \sum_{i \in N} e^i$, and as $S$ is convex, $\delta^1(S, d) \in S$. For $p \in (0, 1]$, $\delta^p(S, d) = (1-p)d + p\,\delta^1(S, d)$, and thus $\delta^p(S, d) \in S$.

To define $\sigma^p(S, d)$, we construct inductively a sequence $d^k = d^k(S, d)$ of points in $S$: $d^0 = d$, and for each $k \geq 0$, $d^{k+1} = \delta^p(S, d^k)$. By the definition of $\delta^p$, $d^{k+1} \geq d^k$, for all $k \geq 0$. Since $S$ is positively bounded, the sequence is bounded and therefore there exists a point $d^\infty = d^\infty(S, d)$ such that $d^k \to d^\infty$.

**Definition 2** *For $p \in (0, 1]$ the function $\sigma^p : \mathcal{B} \to R^N$ is defined by $\sigma^p(S, d) = d^\infty(S, d)$.*

Since $S$ is closed $\sigma^p(S, d) \in S$. Thus, the pair $(\delta^p, \sigma^p)$ is a stepwise bargaining solution. We call such a pair a *generalized Raiffa solution.*

The reason for restricting $p$ to the interval $(0, 1]$ is obvious. For $p = 0$, $\delta^p(S, d) = d$ and thus no new disagreement point arises. For $p > 1$, $\delta^p(S, d)$ may fail to be in $S$ as is the case for $(S, 0)$ where $S = \{x \mid \sum_i x_i \leq 1\}$.

## 4.2 Properties of the solutions

**Theorem 2** *For each $p \in (0,1]$ and $(S,d) \in \mathcal{B}$, $\sigma^p(S,d)$ is Pareto optimal in $S$.*

To prove this theorem we first make a simple observation.

**Observation 1** *For a bargaining set $S$ and $x \in S$, $m(S,x) = x$ iff $x$ is Pareto optimal in $S$.*

**Proof.** If $x$ is Pareto optimal in $S$ then for each $i$ there is no point $(y_i, x_{-i})$ in $S$ with $y_i > x_i$. Hence, $m(S,x) = x$. Conversely, if there exists $y \in S$ such that $y \neq x$ and $y \geq x$, then for some $i$, $y_i > x_i$. By comprehensiveness, $(y_i, x_{-i}) \in S$ and therefore $m_i(S,x) > x_i$. ∎

We also need the next proposition which will serve us in several theorems.

**Proposition 1** *For a fixed bargaining set $S$, $m(S,\cdot)$ is continuous on $S$.*

**Proof of Theorem 2.** For a given $(S,d)$, $\delta_p(S,d)$ is by definition the limit point $d^\infty$ of the sequence $d^k$ in $S$ defined by $d^{k+1} = \delta^p(S,d^k)$. Thus, $d^{k+1} = d^k + (p/n)(m(S,d^k) - d^k)$. Taking the limit and using the continuity of $m(S,\cdot)$ we have $d^\infty = d^\infty + (p/n)(m(S,d^\infty) - d^\infty)$. Since $p > 0$ it follows that $m(S,d^\infty) = d^\infty$, and by Proposition 1, $d^\infty$ is Pareto optimal in $S$. ∎

Using the continuity of $m(S,\cdot)$ we can show that generalized Raiffa solutions are continuous in the disagreement point and the parameter $p$.

**Theorem 3** *For each bargaining problem $S$, $\delta^p(S,d)$ and $\sigma^p(S,d)$ are continuous in $(p,d) \in (0,1] \times S$.*

Except for the first two, all the axioms involve only the step function $\delta$. This makes sense, as each stage of the bargaining results solely in reaching a partial agreement over a new disagreement point. Nevertheless, the function $\sigma$ inherits all the properties of $\delta$ but monotonicity. The proof of the following theorem is straightforward and is omitted.

**Theorem 4** *For each $p \in (0,1]$, the solution function $\sigma^p$ satisfies axioms 3,4,5, and 7,* mutatis mutandis.

Proposition 1 and Theorem 3 hinge crucially on the requirement that all boundary points are Pareto optimal, as demonstrated by Example 1 below. However, some continuity properties hold even without this requirement. Indeed, the convexity and closedness of $S$ alone imply that $m_i(S,\cdot)$ is a proper concave function on $S$ with a closed hypograph. It follows from Theorem 10.2 in Rockafellar (1970), that $m_i(S,\cdot)$ is continuous on the interior of $S$ and on any simplicial subset of $S$. The continuity of $\delta_p(S,\cdot)$ on these sets follows immediately. This continuity suffices to prove Theorem 1.

**Example 1** We construct a set $S$ for $n = 3$ that satisfies all the requirements of a bargaining set, except that it has boundary points which are not Pareto optimal. For this set, $m(S, \cdot)$ is not continuous on $S$. The construction is based on an example in Rockafellar (1970) (p. 83).

Consider first the set $S_0 = \{(x_1, x_2, x_3) \mid x_2 < 0, x_3 \leq x_1^2/x_2\}$. It is straightforward to check that the set is convex. The intersection of $S_0$ with the plane $x_2 = a$ is the hypograph of the parabola $x_1^2/a$ that attains its maximum at 0. As $a$ approaches 0 the parabolas get narrower, converging to the line $\{(0, 0, b) \mid b \leq 0\}$. The closure of $S_0$ is the union of $S_0$ with this line. Let the bargaining set $S$ be the comprehensive hull of the closure of $S_0$.

Any point of the form $(x_1, x_2, x_1^2/x_2)$ with $x_2 < 0$ is Pareto optimal in $S$. In particular if we choose $x_1 = 1/n$, and $x_2 = 1/(bn^2)$ for some $b < 0$, the point $y^n = (1/n, 1/(bn^2), b)$ is Pareto optimal in $S$. Thus, $m(S, y^n) = y^n$, and $\lim m(S, y^n) = (0, 0, b)$, while $m(S, \lim y^n) = m(S, (0, 0, b)) = (0, 0, 0)$.

# 5 The continuous Raiffa solution

Generalized Raiffa solutions are obtained by moving from a given disagreement point $d$ to a new one a fixed fraction $p/n$ of the way from $d$ to the Utopia point $m(S, d)$. When this fraction becomes smaller and smaller the discrete moves converge to a smooth move. Thus, we can think of a path $\pi(t)$ in $S$, which describes a continuous change of disagreement points over time $t$, and where the direction of movement at time $t$ is from $\pi(t)$ towards $m(\pi(t))$. Such a path does indeed exist and it is unique.

**Theorem 5** *For each problem $(S, d)$ there exists a unique function $\pi\colon [0, \infty) \to S$, that solves the differential equation*

$$\pi'(t) = m(S, \pi(t)) - \pi(t),$$

*with the initial condition $\pi(0) = d$. The limit $\lim_{t\to\infty} \pi(t)$, denoted $\sigma^0(S, d)$, exists and it is Pareto optimal in $S$. The function $\sigma^0$ is called* the continuous Raiffa solution.

**Theorem 6** *The continuous Raiffa solution is the limit of the the generalized Raiffa solutions. That is, for each problem $(S, d)$, $\lim \sigma^p(S, d)_{p\to 0} = \sigma^0(S, d)$.*

# 6 Non-cooperative implementation

Myerson (1997) describes a simple non-cooperative implementation of the interim agreement points of the discrete Raiffa solution for two-player bargaining problems. We extend this implementation for the generalized Raiffa solutions for $n$-player bargaining problems.

Let $(S, d)$ be an $n$-player bargaining problem, and for $p \in (0, 1]$, let $(d^k)_{k\geq 0}$ be the sequence of interim agreement points defined by the generalized Raiffa solution function $\delta^p(S,d)$, namely, $d^0 = d$ and $d^{k+1} = \delta^p(S, d^k)$. For each $k \geq 0$ we define a game with perfect information $G^k$ of $k$ rounds as follows.

In the game $G^0$ players have no choices and the payoff vector is $d^0$. In the first round of the game $G^k$, for $k \geq 1$, nature chooses with probability $1-p$ to play $G^0$. With probability $p$, a random order of the players is selected. The first player in this order proposes a payoff vector in $S$. After that, each of the rest of the players announces, according to the selected order, if she accepts the proposed payoff or rejects it. After the first rejection the players play $G_{k-1}$. If all accepted, the game terminates with the proposed payoff vector.

**Theorem 7** *For each $k \geq 0$, there exists a unique subgame-perfect equilibrium in $G^k$. In this equilibrium if $i$ is selected as a proposer in the first round she proposes $(m_i(S,d), d_{-i})$ and all other players accept it. The expected payoff vector in $G^k$ is $d^k$.*

**Proof.** To present a subgame-perfect equilibria in the games $G^k$ it is enough to define the strategies in the first round of $G^k$ for each $k \geq 1$. In this round each player $i$ accepts any proposal $x$ when $x_i \geq d_i^k$ and rejects it otherwise, and proposes $(m_i(S,d), d_{-i})$. Thus, all proposals are accepted in the first round and the expected payoff vector is $d^k$.

We prove the uniqueness of the subgame-perfect equilibrium by induction on $k$. The case $k=0$ is trivial. Suppose we proved it for $k-1$ and consider the game $G^k$. By the induction assumption, the expected payoff of round $k-1$ in the unique subgame-perfect equilibrium in $G^{k-1}$ is $d^{k-1}$, as claimed above. In a subgame-perfect equilibrium the strategies of all players other than $i$ should be to accept any proposal $x$ of $i$ such that $x_{-i} > d_{-i}^{k-1}$. In such an equilibrium, player $i$ cannot propose $x$ such that $x_{-i} \geq d_{-i}^{k-1}$ and $x_{-i} \neq d_{-i}^{k-1}$, because $i$ can do better by proposing $x'$ such that $x'_i > x_i$ and $x'_{-i} > d_{-i}^{k-1}$, and $x'$ will be accepted. Thus in any subgame-perfect equilibrium $i$ should propose $(m_i(S,d), d_{-i})$. It is impossible in equilibrium that the proposal is rejected, because in this case the expected payoff vector is $d^{k-1}$, but $i$ can do better by proposing $x > d^{k-1}$, which will be accepted. ∎

# 7 Proofs

**Proof of Proposition 1.** As argued before, the properties of $S$ guarantee that $m_i(S,x)$ is a finite real valued function on $S$. Moreover, by the convexity of $S$, $m_i(S,x)$ is a concave function on $S$. Therefore it is continuous at each point in the interior of $S$ (Theorem 10.1 in Rockafellar (1970)). Assume that $x$ is Pareto optimal in $S$, then for each $i$, $m_i(S,x) = x_i$. Let $x^k \to x$. Since $m_i(S,x^k) \geq x_i^k$ it follows that $\liminf m_i(S,x^k) \geq x_i$. Assume that $\limsup m_i(S,x^k) = y_i > x_i$. Then, $(y_i, x_{-i})$ is a partial limit of $(m_i(S,x^k), x_{-i}^k)$, and as $S$ is closed it is in $S$. This contradicts the assumption that $x$ is Pareto optimal in $S$. We conclude that $\lim m_i(S,x^k) = x_i = m_i(S,x)$. Since each point is either in the interior of $S$ or Pareto optimal in $S$, the proof is complete. ∎

**Lemma 1** *If $x^0$ is a boundary point of the bargaining set $S$ and $\mathbf{b}x \le \mathbf{b}x^0$ for all $x \in S$, then $\mathbf{b} > 0$.*

**Proof.** Let $x_i < x_i^0$. Then, by comprehensiveness, $x = (x_i, x_{-i}^0) \in S$. Since $x_i$ can be negative with a large absolute value, $\mathbf{b}_i \ge 0$. Suppose $\mathbf{b}_i = 0$, then $\mathbf{b}x = \mathbf{b}x^0$. Therefore, $x$ is on the boundary of $S$. But $x$ is not Pareto optimal, contradicting the requirement that all boundary points of $S$ are Pareto optiaml. ∎

**Proof of Theorem 1.** The function $\delta^p$ satisfies the monotonicity axiom (6) since by construction, $m(\cdot, d)$ is monotonic. It is easy to see that $(\delta^p, \sigma^p)$ satisfies the rest of the axioms.

To show the converse, we first consider bargaining sets defined by hyperplanes. Let $S^0 = \{x \mid \sum_i x_i \le 1\}$, and consider the bargaining problem $(S^0, 0)$. By the symmetry axiom (4), there exists $p \in R$ such that $\delta_i(S^0, 0) = p/n$ for each $i \in N$. By the strong individual rationality axiom (3), $p > 0$. Since $\delta(S^0, 0)$ is feasible, $p \le 1$. Thus, $\delta(S^0, 0) = \delta^p(S^0, 0)$.

Let $\mathbf{b} > 0$ be a functional in $R^N$ and $\beta \in R$. Then $S = \{x \mid \mathbf{b}x \le \beta\}$ is a bargaining set. Consider the problem $(S, d)$ with $d$ that satisfies $\mathbf{b}d < \beta$. Let $a_i = (\beta - \mathbf{b}d)/\mathbf{b}_i$. Then $a_i > 0$ and therefore $a_i x_i + d_i$ is a scale transformation. It is easy to see that it transforms $(S^0, 0)$ into $(S, d)$. Moreover, the point $\delta^p(S^0, 0)$ is transformed to $d + (p/n)(m(S, d) - d)$. Thus, by the scale covariance axiom (5), $\delta(S, d) = \delta^p(S, d)$.

Next consider any problem $(S, d)$ where $d$ is in the interior of $S$. This implies that $m_i = m_i(S, d) > d_i$ for each $i \in N$. Let $\mathbf{b}$ be the linear functional $\mathbf{b} = (1/(m_i - d_i))_{i \in N}$ and let $\beta = 1 + \mathbf{b}d$. Consider the problem $\hat{S} = \{x \mid \mathbf{b}x \le \beta\}$ for which $d$ is an interior point, and $m_i(\hat{S}, d) = m_i$. Let $S^- = \hat{S} \cap S$. It is easy to check that $S^-$ is a bargaining set. Moreover, $S_d^- = \hat{S}_d$ and therefore, by axiom (7), $\delta(S^-, d) = \delta(\hat{S}, d)$. We have shown that $\delta(\hat{S}, d) = \delta^p(\hat{S}, d)$, and thus, by the definition of $\delta^p$, $\delta^p(S^-, d) = d + (p/n)(m(S, d) - d)$. We conclude by the monotonicity axiom (6), that

$$\delta(S, d) \ge d + (p/n)(m(S, d) - d).$$

Fix $i$ in $N$. We construct a problem $(S^+, d)$ such that $S \subseteq S^+$ and $\delta_i(S^+, d) = \delta_i^p(S^+, d)$. Since $e^i (= e^i(S, d))$ is a boundary point of $S$ there exists $\mathbf{b}$ such that $\mathbf{b}x \le \mathbf{b}e^i$ for each $x \in S$. By Lemma 1, $\mathbf{b} > 0$. Consider the bargaining set $S^+ = \{x \mid \mathbf{b}x \le \mathbf{b}e^i\}$. As $d$ is an interior point, $\mathbf{b}d < \mathbf{b}e^i$, and therefore, as we have shown, $\delta_i(S^+, d) = \delta_i^p(S^+, d) = d_i + (p/n)(m_i(S_i^+, d) - d_i)$. Note that $e^i$ is also a boundary point of $S^+$, and therefore, $m_i(S^+, d) = m_i(S, d)$. Thus, $\delta_i(S^+, d) = d_i + (p/n)(m_i(S_i, d) - d_i)$. But, $S \subseteq S^+$, and therefore by the monotonicity axiom (6),

$$\delta_i(S, p) \le d_i + (p/n)(m_i(S, d) - d_i).$$

Thus, $\delta_i(S, d) = d_i + (p/n)(m_i - d_i) = \delta_i^p(S, d)$. This is true for each $i$, and therefore $\delta(S, d) = \delta^p(S, d)$.

If $d$ is on the boundary of $S$, then by the definition of a bargaining problem $d$ is Pareto optiaml in $S$. By the individual rationality axiom (2), $\delta(S,d) = d$ and by the definition of $\delta^p$, $\delta^p(S,d) = d$.

We need to show that $\sigma = \sigma^p$. For $(S,d)$ define a sequence $d^k$ by $d^0 = d$ and $d^{k+1} = \delta(S,d^k)$. By applying the step axiom (1) repeatedly, $\sigma(S,d) = \sigma(S,d^k)$ for each $k$. By the individual rationality axiom (2), $\sigma(S,d^k) \geq d^k$. Hence, $\sigma(S,d) \geq d^k$ for each $k$. We have shown that $\delta(S,d^k) = \delta^p(S,d^k)$. Therefore, $d^{k+1} = \delta^p(S,d^k)$, and $d^k$ converges to $d^\infty = \sigma^p(S,d)$. We conclude that $\sigma(S,d) \geq \sigma^p(S,d)$. But $\sigma(S,d) \in S$ by definition, and $\sigma^p(S,d)$ is Pareto optimal by Proposition 2, hence, $\sigma(S,d) = \sigma^p(S,d)$. ∎

**Proof of Theorem 3.** In view of the continuity of $m(S,\cdot)$ the function $\delta^p(S,d) = d + (p/n)(m(S,d) - d)$ is continuous in $(p,d)$. Since $S$ is fixed we may omit it from some functions. For a given $d \in S$ we denote by $d^k = d^k(p,d)$ the the elements of the sequence defined by $d^0(p,d) = d$ and $d^{k+1}(p,d) = d^k + (p/n)(m(d^k) - d^k)$. Obviously, $d^0$ being constant is continuous in $(p,d)$, and by induction for each $k$, $d^k(p,d)$ is also continuous in $(p,d)$.

Consider neighborhoods in $S$ of the form $S_z^+ = \{y \mid y \in S, z < y\}$. For each $p$, and $d \in S_z^+$, $\delta^p(S,d) \in S_z^+$, and therefore also $\sigma^p(S,d) \in S_z^+$. Note that for each $y \in S_z^+$, $y \leq m(z)$ by comprehensiveness and as all boundary points are Pareto optimal, strict inequality must hold. Therefore, $S_z^+ = S \cap \{y \mid z < y < m(z)\}$. When $x$ is a boundary point, the open boxes $\{y \mid z < y < m(z)\}$ that contain $x$ form a basis for the neighborhoods of $x$, since when $z$ approaches $x$ from below, $m(z)$ approaches $m(x) = x$ by the continuity of $m$.

Fix $\hat{p} \in (0,1]$ and $\hat{d} \in S$. We show that $\sigma^p(S,d)$ is continuous at $(\hat{p},\hat{d})$. Let $\mathcal{N}$ be a neighborhood of $\sigma^{\hat{p}}(S,\hat{d})$. Then, $\mathcal{N}$ contains a neighborhood of $\sigma^{\hat{p}}(S,\hat{d})$ of the form $S_z^+$. There is a $k$ for which $d^k(\hat{p},\hat{d}) \in S_z^+$. Choose a neighborhood $\mathcal{N}'$ of $d^k(\hat{p},\hat{d})$, contained in $S_z^+$. By the continuity of $d^k$, there exists a neighborhood $\mathcal{O}$ of $(\hat{p},\hat{d})$ such that for each $(p,d) \in \mathcal{O}$, $d^k(p,d) \in \mathcal{N}' \subseteq S_z^+$. Thus, $\sigma^p(S,d^k) \in S_z^+ \subseteq \mathcal{N}$. The proof is complete, as $\sigma^p(S,d) = \sigma^p(S,d^k)$. ∎

**Proof of Theorem 5.** If $d$ is a boundary point, then $\pi(t) = d$ for each $t$ and the claim holds trivially. Let $d$ be a point in the interior of $S$. The continuity of $m$ on the interior of $S$ guarantees by Peano's theorem the existence of a solution $\pi$ on some interval $[0,\omega)$ (Hartman (1982), Theorem 2.1). For each $t$ in this interval, the trajectory from 0 to $t$ is included in a closed bounded subset of the interior of $S$. But in each such set, each function $m_i$ satisfies the Lipschitz condition, by Theorem 10.4 in Rockafellar (1970). Thus, $m(S,x) - x$ satisfies the Lipschitz condition on this set, and by the Picard-Lindelöf Theorem, the solution is unique (Hartman (1982)), Theorem 1.1). Finally, let $[0,\omega)$ be the maximal interval for the existence of the solution $\pi$ (where $\omega$ is possibly $\infty$). Then, $\pi(t)$ converges to the boundary of $S$ when $t \to \omega$, by Theorem 3.1 in Hartman (1982). ∎

**Proof of Theorem 6.** Let $\mathcal{N}$ be a neighborhood of $\sigma^0(S,d)$. Then, $\mathcal{N}$ contains a neighborhood of $\sigma^0(S,d)$ of the form $S_z^+$. There is a $\hat{t}$ for which $\pi(\hat{t}) \in S_z^+$.

Choose a neighborhood $\mathcal{N}'$ of $\pi(\hat{t})$, contained in $S_z^+$. Note, that the sequence $d^k(S,d)$ defined by $d^0 = d$ and $d^{k+1} = \delta^p(S,d^k) = d^k + (p/n)(m(S,d^k) - d^k)$ is obtained by applying the Euler method to approximate $\pi(t)$, where $p/n$ is the time increment. By the continuity of $m(S,x) - x$ the Euler's method converges to a solution of the differential equation in the interval $[0,\hat{t}]$ (See Coddington and Levinson (1955), Theorem 1.2), and since the solution is unique, it converges to $\pi$. In particular, for a small enough $p$, there exists $k$ such that $d^k$ is in $\mathcal{N}' \subseteq S_z^+$. Therefore, for such $p$, $\delta^p(S,d^k) \in S_z^+$. But $\delta^p(S,d) = \delta^p(S,d^k)$, which completes the proof. ∎

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