Measuring Disproportionality, Volatility and Malapportionment:

Axiomatization and Solutions

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Abstract

The measurement of disproportionality, volatility and malapportionment often employ similar indices. Yet the debate on the issue of adequate measurement has remained open. We offer a formal and rigorous list of properties that roughly subsume those of Taagepera and Grofman (2003). One of these properties, Dalton's principle of transfers, is formalized in a manner that resolves the ambiguity associated with it in previous studies. We show that the *cosine measure* satisfies all the properties. We also show how the Gallagher index can be modified to satisfy all the properties. The cosine measure and the modified Gallagher measure are co-monotone.

Key words: proportionality, disproportionality, volatility, malapportionment, Gallagher measure, PR systems

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Introduction

Among the different features of electoral systems and electoral results, the one most investigated is probably the degree of (dis)proportionality. Among the different features of political change, a frequently investigated variable is the volatility of the party system. A critical practical problem, which applies to almost all democratic countries, is the possibility of malapportionment of electoral districts. The measurement of disproportionality, volatility and malapportionment often employ similar indices (Monroe, 1994), yet the debate on the issue of adequate measurement has remained open (Lijphart, 1999). The measurement question underlying these problems is clearly relevant to a variety of economic and social issues such as the allocation of utilities to different players proportionally to their initial 'weights', the change of weights over time and so forth.

Formally, let $\mathbf{x} = \langle x_1, x_2, \ldots, x_i\ldots, x_n\rangle$ represent the distribution of votes among n political parties and let $\mathbf{y} = \langle y_1, y_2, \ldots, y_i\ldots, y_n\rangle$ represent the distribution of seats in the elected body among the respective parties. It is convenient to think of **x** and **y** as points in the space $\mathbf{R}^n$. The quantity we wish to measure is a function of these two points.

It should be noted that this representation of the problem is not unique. For example, in their seminal work on apportionment, Balinski and Young (2001) consider a closely related measure that is a function of the single point $\langle x_1/y_1,\ldots,x_n/y_n\rangle$. Furthermore – again unlike Balinski and Young – this measure relates only to two specific points (that is, a particular empirical result) but not to an apportionment system as a whole.

When volatility is measured, **x** represents the distribution of votes (or seats) at any given point of time while **y** represents the distribution of votes (or seats) at another point of time. In the case of malapportionment, **x** represents the distribution of (eligible) voters among electoral districts and **y** represents the distribution of seats allocated to these constituencies. Perfect partisan proportionality (or proportional apportionment, or lack of volatility) exists when there exists some a such that for all $1 \leq i \leq n$, $y_i = ax_i$.

Some Standard Indices

Perhaps the three most popular indices of disproportionality are those of Duncan and Duncan (1955), Rae (1967), and Gallagher (1991). Each of these measures assumes that $\Sigma x_i = \Sigma y_i = 1$.

Duncan and Duncan's measure is:

$$D(\mathbf{x},\mathbf{y}) = \Sigma |x_i - y_i| / 2$$

It was later implemented in the field of electoral proportionality by Loosemore and Hanby (1971) and as a measure of volatility by Pedersen (1979).

The closely related measure suggested by Rae is:

$$R(\mathbf{x},\mathbf{y}) = \Sigma |x_i - y_i| / n$$

Rae's measure was very popular for a while, but is rarely used today because its sensitivity to the number of participating political parties seems counter-intuitive. We shall formalize the underlying intuition below.

The Gallagher index is usually defined as follows:

$$G(\mathbf{x},\mathbf{y}) = (1/2 * \Sigma(x_i - y_i)^2)^{1/2}$$

Note that this is simply a multiple of Euclidean distance.

It can be easily shown that the Gallagher index is not co-monotone with either of the other indices. That is, there exist pairs (**x**,**y**) and (**x'**,**y'**) such that D(**x**,**y**)≤D(**x'**,**y'**), while G(**x'**,**y'**)≤G(**x**,**y**). Likewise, there exist pairs such that R(**x**,**y**)≤R(**x'**,**y'**), while G(**x'**,**y'**)≤G(**x**,**y**). R and D are in fact co-monotone.

Furthermore, as we shall see, these measures do not satisfy a number of essential properties that should be required from a disproportionality measure.

Desirable Properties

Taagepera and Grofman (2003), following Monroe (1994), enumerate a list of 12 properties that a disproportionality measure should satisfy. They conclude that none of the 19 measures examined by them satisfies all these properties. Of the measures they consider, the one that comes closest is the Gallagher index.

We offer a more formal and rigorous list of properties that roughly subsume those of Taagepera and Grofman. We will show that the *cosine measure*, cos(**x**,**y**), not considered by Taagepera and Grofman, satisfies all the properties. We will also show how the Gallagher index can be modified to satisfy all the properties. The cosine measure and the modified Gallagher measure are co-monotone.

For formal reasons, our presentation differs slightly from earlier ones. First, we place no restrictions on $x_1,..,x_n$, $y_1,..,y_n$ other than that all are greater than or equal to 0. In particular, we do not insist, as is often done, that $\Sigma x_i = 1$ or that $\Sigma y_i = 1$. Also, we consider a proportionality index in which the least proportional pair obtains the lowest value. The indices enumerated above are <u>dis</u>proportionality indices in which the least proportional pair is assigned the highest value. This is merely a matter of convention. If we wish to switch conventions, we need only replace a disproportionality index F with the corresponding proportionality index IF = 1-F and vice versa. (Thus, in the language of disproportionality employed by Taagepera and Grofman and others above, the measure we suggest here is 1-cos(**x**,**y**) rather than cos(**x**,**y**).)

A proportionality index, F, should satisfy the following properties:

1. *Continuity*: F(**x**,**y**) is a continuous function of each variable $x_1,..,x_n$, $y_1,..,y_n$.
2. *Irrelevance of virtual parties*:

   $F(\langle x_1,x_2,\ldots,x_n,0\rangle,\langle y_1,y_2,\ldots,y_n,0\rangle) = F(\langle x_1,x_2,\ldots,x_n\rangle,\langle y_1,y_2,\ldots,y_n\rangle)$
3. *Symmetry*: F(**x**,**y**) = F(**y**,**x**)
4. *Insensitivity to order*: For any permutation p, F(p(**x**),p(**y**)) = F(**x**,**y**)
5. *Scale invariance*: For any positive real number a, F(a**x**,**y**) = F(**x**,**y**)
6. *Specified limits*:
   a. *Orthogonality*: If for all $1 \leq i \leq n$, $x_i=0$ if and only if $y_i>0$, then F(**x**,**y**) = 0
   b. *Identity*: F(**x**,**x**) = 1
   c. *Range*: For all other **x** and **y**, 0 < F(**x**,**y**) < 1
7. *Dalton's principle of transfers*: Let **w** be any point (other than **y**) on the straight line connecting **x** and **y**. Then, F(**x**,**y**) < F(**x**,**w**).
8. *Optimality of equality*: Let $x=\langle x_1,x_2,\ldots,x_n\rangle$ and $y=\langle y_1,y_2, \ldots,y_n\rangle$ be such that $x_1=x_2$ and $y_1=y_2$ and let $y'==\langle y'_1,y'_2,y_3\ldots,y_n\rangle$ be such that $y'_1+y'_2 = y_1+y_2$ and $y'_1 \neq y'_2$. Then F(**x**,**y'**) < F(**x**,**y**). This simply means that if two parties have equal votes and share a fixed number of seats, F is strictly maximized when they share those seats equally.

Taagepera and Grofman noted that informal presentations of Dalton's principle of transfers (Dalton 1920), discussed by Monroe (1994), left room for considerable ambiguity. We believe that, taken together, the last two properties presented here resolve that ambiguity in an elegant manner.

The Cosine Measure

The proportionality (i.e., inverted) versions of the disproportionality measures, ID, IR and IG, mentioned above, can all be adjusted to satisfy *scale invariance* by dividing each x_i by Σx_i and each y_i by Σy_i. However, IR and IG both fail to satisfy *orthogonality*. In addition, IR fails to satisfy *irrelevance of virtual parties* and IR and ID fail to satisfy *optimality of equality*.

Consider now the cosine measure, defined as follows:

$$\cos(\mathbf{x},\mathbf{y}) = (\Sigma\ (x_i * y_i)) / (\Sigma x_i^2 * \Sigma y_i^2)^{1/2}$$

It can easily be shown that the function $\cos(\mathbf{x},\mathbf{y})$ satisfies all the above properties (as well as those of Taagepera and Grofman). Indeed, in the information retrieval literature, where a very similar proportionality problem (the similarity between two documents) is faced, the cosine function has been the standard measure of proportionality for over three decades (Salton 1971).

Note that for the two dimensional case, the cosine function defined here is equivalent to the usual trigonometric cosine function. In fact, this trigonometric interpretation of cosine facilitates intuition for why cosine makes sense as a measure of proportionality, as can be seen in Figure 1. Consider points lying on the vectors **x** and **y**, respectively. By scale invariance, we don't care where on the vectors the points lie. Choose the point, A, on **x** that coincides with the unit circle. Draw a line from A that is perpendicular to the vector **y** and call the point where this line meets y, B. Then $\cos(\mathbf{x},\mathbf{y})$ is the length of the line segment OB. Clearly, when x and y are proportional they lie on the same vector, so the length of OB is 1. As can be seen in Figure 1, for **y'** that forms a larger angle with **x** than **y**, the segment OB' is shorter than OB.

Now, let us deal with the modification of the inverse Gallagher index, IG(**x**,**y**) = $1-(1/2*\Sigma(x_i-y_i)^2)^{1/2}$. As noted above, the inverse Gallagher index fails to satisfy both scale invariance and orthogonality. A simple amendment to the usual definition solves both problems.

Define **x'** = **x**/ $(\Sigma x_i^2)^{1/2}$ and **y'** = **y**/ $(\Sigma y_i^2)^{1/2}$. Now let the modified inverse Gallagher index be IG'(**x**,**y**) = IG(**x'**,**y'**). It can easily be shown that IG' satisfies all of the above properties (as well as those of Taagepera and Grofman). In fact, cos(**x**,**y**) = 1 - (1 - IG'(**x**,**y**))^2, so that IG' and cosine are co-monotone. Indeed, for any continuous increasing function f that leaves 0 and 1 invariant, f(cos(**x**,**y**)) will be a measure co-monotone with cos(**x**,**y**) that satisfies all the above properties.

Examples

In the following table we consider seven illustrative examples. For each, we consider the standard disproportionality measures D, R and G, as well as 1-cos and the adjusted Gallagher measure G' = $(1\text{-cos})^{1/2}$.

We note several points. Example 2 shows that R does not satisfy *irrelevance of virtual parties*. Example 3 shows that both R and G do not satisfy *orthogonality*. Examples 4 and 5 show that both D and R do not satisfy *optimality of equality*. The co-monotone G' and 1-cos satisfy all the properties.

The properties that we consider here all concern only the *ordering* of pairs according to their degree of disproportionality. Other than requiring that the values lie between 0 and 1, the properties do not concern the specific values assigned to a given pair. In fact, as we noted, any increasing function of cosine that left 0 and 1 invariant would satisfy all the properties. Nevertheless, as is evident in Examples 6 and 7, some such functions assign values more intuitively than others. In Example 6, half the seats are allocated proportionally and half perfectly disproportionally. Thus, a measure of 0.5, as assigned by 1-cos, is an intuitively more appealing result than the value 0.71 assigned by G'. Similarly, in Example 7, 0.25, as assigned by 1-cos, is a more intuitive result than 0.5. Although two examples hardly constitute proof, it would appear that among the measures that satisfy all the properties, 1-cos is scaled more naturally than G'. With regard to the other examples in Table 1, it is not apparent that for any of them, apart from Example 3, there is some obviously "right" answer, though this point could be debated. Thus, for Examples 1, 2, 4 and 5, 1-cos yields values that some might regard as counter-intuitively small. Hence, it seems that some may prefer G' for practical measurements.

In conclusion, we have found a class of disproportionality measures that, unlike all previously proposed measures, satisfy a reasonable set of properties. Of these the simplest and apparently best scaled is the 1-cos measure, although, in view of some of the examples mentioned above, some may prefer G' for practical measurements.

Table 1. **Disproportionality measures for several examples***

	‘Votes’	‘Seats’	D	R	G	G’	1-cos
Example 1	<50,50>	<60,40>	.10	.10	.10	.14	.02
Example 2	<50,50,0>	<60,40,0>	.10	.07	.10	.14	.02
Example 3	<50,50,0>	<0,0,100>	1.00	.67	.87	1.00	1.00
Example 4	<25,25,25,25>	<30,30,20,20>	.10	.05	.07	.14	.02
Example 5	<25,25,25,25>	<35,25,20,20>	.10	.05	.09	.17	.03
Example 6	<50,50,0>	<50,0,50>	.50	.33	.50	.71	.50
Example 7	<25,25,25,25,0>	<25,25,25,0,25>	.25	.10	.25	.50	.25

* Results are rounded.

Figure 1. OA is of unit length. The line AB is perpendicular to **y**. cos(**x**,**y**) is the length of AB.

Bibliography

Balinski ML, Young HP (2001) Fair representation: meeting the ideal of one man, one vote. 2nd ed. Brookings Institution Press, Washington DC

Dalton H, (1920) Measurement of the inequality of income. Economic Journal 30: 348-361

Duncan OD, Duncan B (1955) A methodical analysis of segregation indexes. American Sociological Review 20: 210-217

Gallagher M, (1991) Proportionality, Disproportionality and Electoral Systems. Electoral Studies 10: 33-51

Lijphart A, (1994) Electoral Systems and Party Systems. Oxford, New York and Oxford

Loosemore J, Hanby V, (1971) The theoretical limits of maximum distortion: some analytical expressions for electoral systems. British Journal of Political Science 1: 467-477

Monroe BL, (1994) Disproportionality and malapportionments: measuring electoral inequity Electoral Studies. 13: 132-149

Pedersen MN, (1979) The dynamics of European party systems: changing patterns of electoral volatility European Journal of Political Research, 7/1: 1-26

Rae DW, (1967) The political consequences of electoral laws, Yale, New Haven.

Salton G, (1971) The SMART retrieval system—experiments in automatic document *processing*, Prentice-Hall, Englewood Cliffs, NJ.

Taagepera R Grofman B, (2003) Mapping the indices of seats–votes disproportionality and inter-election volatility, Party Politics, 9(6): 659-677