# WHO GAVE YOU THE CAUCHY-WEIERSTRASS TALE? THE DUAL HISTORY OF RIGOROUS CALCULUS

Abstract. In 1821, Cauchy wrote that a variable quantity tending to zero (generally understood as a null sequence) "becomes" infinitesimal; and in 1823, he wrote that it "becomes" *an* infinitesimal. How do Cauchy null sequences become Cauchy infinitesimals? In 1829, Cauchy developed a detailed theory of infinitesimals of arbitrary order (not necessarily integer). How does his theory connect with the work of later authors? We examine several views of Cauchy's foundational contribution in analysis.

## Contents



## 1. Introduction

When the author first came across the title of the recent book *Who Gave You the Epsilon? And Other Tales of Mathematical History* [1], he momentarily entertained a faint glimmer of hope. The book draws

2000 *Mathematics Subject Classification.* Primary 26E35; Secondary 97A20, 97C30.

*Key words and phrases.* Archimedean property, Bernoulli, Cauchy, continuum, hyperreals, infinitesimal, Stolz, sum theorem, ultraproduct.

its title from an older essay, entitled *Who Gave You the Epsilon? Cauchy and the origins of rigorous calculus* [34]. The faint hope was that the book would approach the thesis expressed in the title of the older essay, in a critical spirit, namely, as a mathematical *tale* in need of a re-examination. Anderson, Katz, and Wilson not having undertaken the latter task, such an attempt is made here.

Cauchy's foundational stance has been the subject of an ongoing controversy. Why Cauchy, of all the 18th and 19th century mathematicians? What's interesting about Cauchy is that he develops some surprisingly modern mathematics using infinitesimals. In this he surpasses earlier authors. Several items deserve to be mentioned:

(1) *Cauchy's "sum theorem"*. This result asserts the convergence of a series of continuous functions under a suitable condition of convergence, and has been the subject of a historical controversy, ever since A. Robinson proposed a novel reading of the sum theorem that would make it correct. The controversy centers on the question whether the condition was meant by Cauchy to hold at the points of an Archimedean continuum, or at the points of a Bernoullian continuum (i.e., an infinitesimal-enriched continuum).[1] Lakatos presented a paper in 1966 (published posthumously in 1978 by Cleave [46]), where he argues that the 1821 result is correct as stated (he thereby reversed

[1]In more detail, let $x$ be in the domain of $f$, and consider the following condition, which we will call *microcontinuity*:

> "if $x'$ is in the domain of $f$ and $x'$ is infinitely close to $x$, then $f(x')$ is infinitely close to $f(x)$".

Then ordinary continuity of $f$ is equivalent to $f$ being microcontinuous on the Archimedean continuum (A-continuum for short), i.e., at every point of its domain in the A-continuum. Meanwhile, uniform continuity of $f$ is equivalent to $f$ being microcontinuous on the Bernoulian continuum (B-continuum for short), i.e., at every point of its domain in the B-continuum (the relation of the two continua is discussed in more detail in Appendix B). Thus, the function $\sin(1/x)$ for positive $x$ fails to be uniformly continuous because microcontinuity fails at a positive infinitesimal $x$. The function $x^2$ fails to be uniformly continuous because of the failure of microcontinuity at an infinite member of the B-continuum.

A similar distinction exists between pointwise convergence and uniform convergence, see e.g. Goldblatt [33, Theorem 7.12.2, p. 87]. Which condition did Cauchy have in mind in 1821? Abel interpreted it as convergence on the A-continuum, and presented "exceptions" (what we would call today counterexamples) in 1826. After the publication of additional such exceptions by Seidel and Stokes in the 1840s, Cauchy clarified/modified his position in 1853. In his text [13], he specified a stronger condition of convergence on the B-continuum, including at $x = 1/n$ (explicitly mentioned by Cauchy). The stronger condition bars Abel's counterexample. See more in footnote 2.

his position presented in *Proofs and Refutations* [45]). Laugwitz concurs, see e.g. his 1989 text [48]. Traditional historians tend both (a) to reject Cauchy's infinitesimals, claiming they are merely shorthand for limits, and (b) to claim that his 1821 sum theorem was false.[2] The 1821 formulation may in the end be too ambiguous to know what Cauchy's intention was at the time, if Cauchy himself knew.

(2) *Cauchy's proof of the binomial formula (series) for arbitrary exponents.* Laugwitz [47, p. 266] argues this to be the first correct proof of the formula (except for Bolzano's proof in 1816, see [50, p. 657]).

(3) *Cauchy's use of the Dirac delta function.* Over a century before Dirac, Cauchy used such functions to solve problems in Fourier analysis and in the evaluation of singular integrals. This is analyzed by Laugwitz [49].

(4) Perhaps most controversial of all, *Cauchy's definition of continuity in terms of infinitesimals.* Many historians have interpreted Cauchy's definition as a proto-Weierstrassian definition of continuity in terms of limits. Thus, Smithies [67, p. 53, footnote 20] cites the *page* in Cauchy's book where Cauchy gave the infinitesimal definition, but goes on to claim that the concept of *limit* was Cauchy's "essential basis" for his concept of continuity [67, p. 58]. Smithies looked in Cauchy, saw the infinitesimal definition, and went on to write in his paper that he saw a limit definition. Such automated translation has been prevalent at least since Boyer [7, p. 277].

## 2. How does a null sequence become a Cauchy infinitesimal?

The nature of Cauchy's infinitesimals has been the subject of an ongoing debate for a number of decades. Traditional historians tend to dismiss Cauchy's *infiniment petits* as merely a linguistic device masking Cauchy's use of the limit concept, an early anticipation of the more rigorous methods developed in the second half of the 19th century,

[2]Cauchy himself published a clarification/modification in 1853, which amounts to requiring convergence on the B-continuum (see footnote 1). Traditional historians acknowledge his clarification/modification, and interpret it as the addition of the condition of uniform convergence, even though Cauchy states it in terms of a *single* variable, whereas the traditional definition of uniform continuity or convergence in the context of an A-continuum necessarily requires *a pair* of variables. Cauchy specifically evokes $x = 1/n$ where Abel's "exception/counterexample" fails to converge. The matter is discussed in detail by Bråting [9].

namely epsilontic analysis. Hourya Benis Sinaceur [65] presented a critical analysis of the traditional approach to Cauchy in 1973. Other historians have taken Cauchy's infinitesimals at face value, see e.g., Lakatos [46], Laugwitz [48]. The studies in the past decade include Sad, Teixeira, and Baldino [60], as well as Bråting [9].

To what extent did Cauchy intend the process that he described as a null sequence "becoming" an infinitesimal, to involve some kind of a collapsing?

Cauchy did not have access to the modern set theoretic mentality (currently dominant in the area of mathematics), where equivalence relation and quotient space constructions are taken for granted. One can still ponder the following question: to what extent may Cauchy have anticipated such collapsing phenomena?

Part of the difficulty in answering such a question is Cauchy's foundational stance. Cauchy was less interested in foundational issues than, say, Bolzano. To Cauchy, getting your sedan out of the garage was the only justification for shoveling away the snow that blocks the garage door. Once the door open, Cauchy is in top gear within seconds, solving problems and producing results. To him, infinitesimals were the asphalt under the snow, not the snow itself. Bolzano wanted to shovel away *all* the snow from the road. Half a century later, the triumvirate[3] shovel ripped out the asphalt together with the snow, intent on consigning the infinitesimal to the dustbin of history.

For these reasons, it is not easy to gauge Cauchy's foundational stance precisely. For instance, is there evidence that he felt that two null sequences that coincide except for a finite number of terms, would "generate" the same infinitesimal?[4] Support for this idea comes from several sources, including a very unlikely one, namely Felscher's essay [28] in *The American Mathematical Monthly* from 2000, where he attacks both Laugwitz's interpretation and the idea that infinitesimals play a foundational role in Cauchy.

In his zeal to do away with Cauchy's infinitesimals, Felscher seeks to describe them in terms of the modern terminology of *germs*. Here two sequences are in the same germ if they agree at infinity, i.e., for all sufficiently large values of the index $n$ of the sequence $\langle u_n \rangle$, that is,

[3]Boyer [7] refers to Cantor, Dedekind, and Weierstrass as "the great triumvirate".

[4]Note that the term "generate" was used by K. Bråting [9] to describe the passage from sequence to infinitesimal in Cauchy.

are equal almost everywhere.[5] Felscher's maneuver successfully eliminates the term "infinitesimal" from the picture, but has the effect of undermining Felscher's own thesis, by lending support to the presence of such "collapsing" in Cauchy.

Indeed, the idea of reading germs of sequences into Cauchy is precisely Laugwitz's thesis in [47, p. 272]. Germs of sequences are also the basis of Laugwitz's $\Omega$-calculus [61], a non-Archimedean field constructed using a Fréchet filter.[6]

In an editorial footnote to Lakatos's essay, J. Cleave outlines a non-Archimedean system developed by Chwistek [14], involving a quotient by a Fréchet filter (similarly to the Schmieden–Laugwitz construction), and concludes that the relation of such infinitesimals to Cauchy's is "obvious".[7] To us it appears that the said relation requires additional argument.

We argue that Cauchy's published work contains evidence that he intuitively sensed a "collapsing" involved in the passage from a null sequence to an infinitesimal. The second chapter of the *Cours d'Analyse* [10] of 1821 contains a series of theorems (eight of them) whose main purpose appears to be to emphasize the importance of the *asymptotic* behavior of the sequence (i.e., as the index tends to infinity). Furthermore, he refers to his infinitesimals as "quantities", a term he uses in the context of an ordered number system, as opposed to the complex numbers which are always "expressions" but never "quantities". In fact, the recent English translation [8] of the *Cours d'Analyse* erroneously translates one of Cauchy's complex "expressions" as a "quantity", and the reviewer for *Zentralblatt* dutifully notes this error.

A second aspect of collapsing is the mental transformation of the process of "tending to zero" into a concept/noun (as a null sequence is

[5]Equality "almost everywhere" is a 20th century concept; Felscher seems to suggest that Cauchy thought of the relation between his variable quantities and his infinitesimals in a way that would be later described as equality almost everywhere.

[6]In the Schmieden–Laugwitz construction, a Fréchet filter is used where an ultrafilter would be used in a hyperreal construction; see Appendix B.

[7]Cleave writes in his footnote 32*: "A construction of non-standard analysis is given in Chwistek [14] (1948) which is derived from a paper published in 1926. It is basically the reduced power $\mathbb{R}^{\mathbb{N}}/F$ where $F$ is the Fréchet filter on the natural numbers (the collection of cofinite sets of natural numbers) (see Frayne, Morel, and Scott [31]) [...] This particular construction is not an elementary extension of $\mathbb{R}$ but there are sufficiently powerful transfer properties to enable some non-standard analysis to be performed. It may be observed that the elements of $\mathbb{R}^{\mathbb{N}}/F$ are equivalence classes of sequences of reals, two sequences $s_1, s_2 \ldots$ and $t_1, t_2, \ldots$ being counted equal if for some $n$, $s_m = t_m$ for all $m \geq n$. The relation of these classes to Cauchy's variables is obvious" [46, p. 160].

transformed into an infinitesimal), thought of as an encapsulation of a highly compressed process. In the successive theorems in his Chapter 2, Cauchy seeks to collapse the initial perception of his infinitesimal $\alpha$ as a temporally-deployed *process*, by deliberately leaving out the implied index (i.e. label of the terms of the sequence), and by explicitly specifying and emphasizing a rival index: the exponent in a power $\alpha^n$ of the infinitesimal, thought of as an infinitesimal of a higher and higher order.[8]

## 3. An analysis of *Cours d'Analyse* and its infinitesimals

We will refer to the pages in the *Cours d'Analyse* [10] itself, rather than the collected works.

Cauchy's Chapter 2, section 1 starts on page 26. Here Cauchy writes that a variable quantity becomes *infinitely small* if, etc. Here "infinitely small" is an adjective, and is not used as a noun-adjective pair.

On page 27, Cauchy employs the noun-adjective combination, by referring to "infinitely small quantities" (*quantités infiniment petites*, still in the feminine). He denotes such a quantity $\alpha$. Note that the index in the implied sequence is suppressed (namely, $\alpha$ appears without a lower index).[9]

On page 28, he introduces a competing numerical index, namely the exponent, by forming the infinitesimals

$$\alpha, \alpha^2, \alpha^3, ...$$

By the time we reach the bottom of the page (fourth line from the bottom), he is already employing "infinitely small" as a *noun* in its own right: *infiniment petits* (in the masculine plural).

On page 29, Theorem 1 asserts that a highest-order infinitesimal will be smaller than all the others (infinitesimals are consistently referred to in the masculine). The theorem has not yet chosen a letter label for the competing index (i.e. order of infinitesimal).

Still on page 29, Theorem 2 for the first time introduces a label for the order of the infinitesimal, namely the letter $n$, as in

$$\alpha^n.$$

On page 30, Theorem 3 introduces several different letter indices:

$$n, n', n'' \ldots$$

[8] In the education literature, such a compressing phenomenon is studied under the name of procept (process+concept) [36], encapsulation [22], and reification [63].

[9] Note that Cauchy uses lower indices to indicate terms in a sequence in his proof of the intermediate value theorem [10, Note III, p. 462].

| | independent variable increment ($\Delta x$) | dependent variable increment ($\Delta y$) |
|---|---|---|
| Cauchy's first definition | infinitesimal | variable tending to zero |
| Cauchy's second definition | infinitesimal | infinitesimal |

Table 1. Cauchy's first two definitions of continuity in 1821 are of the form *"if $\Delta x$ is ..., then $\Delta y$ is ..."*. Note the prevalence of the term "infinitesimal".

for the orders of his infinitesimals. The theorem concerns the order of the sum, again forcing the student to focus on the competing orders (at the expense of the suppressed index of the "variable quantity" itself).

Still on page 30, Theorem 4 introduces the terminology of *polynomials* in $\alpha$, and describes their orders $n, n', n'' \ldots$ for the first time as a "sequence". We now have two "sequences": (the variable quantity) $\alpha$ itself, whose index is implicit (was never labeled), and the sequence of orders, which are both emphasized and elaborately labeled using "primes" $'$ and double primes $''$.

In all these theorems, it is the asymptotic behavior of null sequences that is constantly emphasized, which suggests that Cauchy might have found it perfectly natural to identify/collapse sequences that agree almost everywhere. Terminology *finit par être*, *finit par devenir* (suggestive of such collapse) is employed repeatedly.

On pages 31 and 32, three additional theorems and one corollary are stated, for a total of eight results on the asymptotic behavior of infinitesimals.

By the time Cauchy reaches Section 2 of Chapter 2 on page 34 (concerning continuity of functions), he has already encapsulated the *process* implied in the notion of "variable quantity", into the concept/masculine noun *infiniment petit*. When he evokes an infinitely small $x$-increment $\alpha$, only a stubborn Weierstrassian will refuse to interpret his $\alpha$ as a concept/noun. The *second* definition (out of the three definitions of continuity given here) is the one Cauchy italicizes, implying it is the main one. Here both the $x$-increment and the $y$-increment are described as infinitely small increments (see Figure 1).

In this context, the verb "become" is being used in two different senses:

(a) the terms in the sequence *become* smaller than any number;
(b) the encapsulating sense of a process being compressed into (and thus *becoming*) a concept/noun,

as analyzed by Sad, Teixeira, and Baldino [60], who employed the terminology of a *transformation of essence*. It is interesting to note that Bolzano fought against the sense (a), by suppressing the parametrisation and viewing the null sequence as a set (perhaps he was influenced by Zeno paradoxes), but not against the sense (b).

## 4. Theories of infinitesimals from Cauchy to Stolz

In 1823, Cauchy's emphasis on the *noun* aspect of his infinitesimals is even more pronounced than in 1821:

> Lorsque les valeurs numériques[10] successives d'une même variable décroissent indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme *un infiniment petit* ou une quantité infiniment petite. Une variablé de cette espèce a zéro pour limite [11, p. 4].

The use of the noun, *un infiniment petit*, makes it difficult to interpret the "becoming" in the sense (a) above; rather, the definition requires sense (b) to be grammatically coherent. Here the variable [quantity] *becomes* a masculine noun: *un infiniment petit* (recall that in 1821 the variable [quantity] *became* a feminine adjective: *infiniment petite*). Cauchy is very precise here: it is the *limit* of the variable that's zero. The variable itself *becomes* an *infiniment petit*. Cauchy wrote neither that a variable *is* an infinitesimal, nor that the limit of the infinitesimal is zero, but rather that the limit of the *variable* is zero, cf. [60, p. 301-302].

Cauchy's *Cours d'Analyse* presented only a theory of infinitesimals of polynomial rate of growth as compared to a given $\alpha$. The shortcoming of such a theory is its limited flexibility. Since Cauchy only considers infinitesimals behaving as polynomials of a fixed infinitesimal (called the "base" infinitesimal in 1823), his framework imposes obvious limitations on what can be done with such infinitesimals. Thus, one typically can't extract the square root of a "polynomial" infinitesimal.

What is remarkable is that Cauchy did develop a theory to overcome this shortcoming. Cauchy's astounding theory of infinitesimals of arbitrary order (not necessarily integer) is analyzed by Laugwitz [47, p. 271].

[10] The meaning of the expression *valeur numérique* is subject to debate; see next section and footnote 15.

In 1823, and particularly in 1829, Cauchy develops a more flexible theory, where an infinitesimal is represented by an arbitrary *function* (rather than merely a polynomial) of a base infinitesimal, denoted "$i$". This is done in Cauchy's textbook [12, Chapter 6]. The title of the chapter is significant. Indeed, the title refers to the *functions* as "representing" the infinitesimals; more precisely, "fonctions qui représentent des quantités infiniment petites". Here is what Cauchy has to say in 1829:

> Designons par $a$ un nombre constant, rationnel ou irrationnel; par $i$ une quantite infiniment petite, et par $r$ un nombre variable. Dans le systeme de quantités infiniment petites dont $i$ sera la base, une fonction de $i$ representée par $f(i)$ sera un infiniment petit de l'ordre $a$, si la limite du rapport $f(i)/i^r$ est nulle pour toutes les valeurs de $r$ plus petite que $a$, et infinie pour toutes les valeurs de $r$ plus grandes que $a$ [12, p. 281].

Laugwitz [47, p. 271] explains this to mean that the order $a$ of the infinitesimal $f(i)$ is the uniquely determined real number (possibly $+\infty$, as with the function $e^{-1/t^2}$) such that $f(i)/i^r$ is infinitesimal for $r < a$ and infinitely large for $r > a$.

Laugwitz [47, p. 272] notes that Cauchy provides an example of functions defined on positive reals that represent infinitesimals of orders $\infty$ and 0, namely

$$e^{-1/i} \quad \text{and} \quad \frac{1}{\log i}$$

(see Cauchy [12, p. 326-327]).

Note that according to P. Ehrlich [23], the development of non-Archimedean systems based on orders of growth was pursued in earnest at the end of the 19th century by such authors as Stolz and du Bois-Reymond. These systems appear to have an anticedent in Cauchy's theory of infinitesimals as developed in his texts dating from 1823 and 1829.

## 5. *Bestiarium infinitesimale*

An interpretation of Cauchy's foundational stance endeavoring to take Cauchy's infinitesimals at their face value has not been without its detractors. A decade ago, W. Felscher [28] set out to investigate Cauchy's continuity, in an 18-page text, marred by an odd focus on d'Alembert.[11] To be sure, it is both legitimate and necessary to examine Cauchy's predecessors, including d'Alembert, if one wishes to understand Cauchy himself. Indeed, a debate of long standing (over a century long, in fact) had opposed two rival methodologies in the study of the foundations of the new science of Newton and Leibniz:

(A) a methodology eschewing infinitesimals; and
(B) a methodology favoring them.[12]

It is legitimate to ask which of the two methodologies underpins Cauchy's oeuvre.

However, Felscher's conceptual framework is flawed in a fundamental way. The outcome of his investigation is predetermined from the outset by the following two factors:

(1) Felscher's exclusive focus on d'Alembert,[13] one of the radical adherents of the A-methodology, and
(2) Felscher's *postulating* a methodological continuity between the work of d'Alembert and Cauchy.[14]

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[11]Felix Klein [43, p. 103] discusses the error in d'Alembert's proof of the fundamental theorem of algebra, first noticed by Gauss.

[12]Having outlined the developments in real analysis associated with Weierstrass and his followers, F. Klein pointed out that "The scientific mathematics of today is built upon the series of developments which we have been outlining. But *an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries*" [43, p. 214] [emphasis added–authors]. Such a different conception, according to Klein, "harks back to old metaphysical speculations concerning the *structure of the continuum* according to which this was made up of [...] *infinitely small* parts" [43, p. 214] [emphasis added—authors].

[13]Felscher mentions Euler and the Bernoullis in his section entitled *D'Alembert's program*, but says not a word about them in his article.

[14]Felscher describes d'Alembert as "one of the mathematicians representing the heroic age of calculus" [28, p. 845]. Felscher buttresses his claim by a lengthy, and perhaps even visionary, quotation concerning the definition of the limit concept, from the article *Limite* from the *Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts et des Métiers* (volume 9 from 1765):

> On dit qu'une grandeur est la limite d'une autre grandeur, quand la seconde peut approcher de la premiere plus près que d'une grandeur donnée, si petite qu'on la puisse supposer, sans pourtant que la grandeur qui approche, puisse jamais surpasser la grandeur dont elle approche; ensorte que la différence d'une pareille quantité à sa limite est absolument inassignable (Encyclopédie, volume 9, page 542).

However, what Felscher overlooked is the fact that the article *Limite* was written by two authors, and the above passage defining the concept of "limit" (as well as the two propositions on limits) does not stem from d'Alembert but from the encyclopedist Jean-Baptiste de La Chapelle, who was recruited by d'Alembert to write 270 articles for the Encyclopédie. Indeed, the section of the article contaning these items is signed (E) (at bottom of first column), known to be de La Chapelle's

Displaying a masterly command of scholarly Latin, Felscher offers the reader a glimpse of the *bestiarium infinitesimale* in section 6 of his essay, starting on page 856. The punchline comes in the middle of page 857, where Felscher points out that Cauchy refers specifically to *numerical* values of his variables, the latter being described by Cauchy as *becoming* infinitesimals.

The adjective *numerical* is linked etymologically to the noun *number*. Cauchy's numbers (unlike his *quantities*) are certainly appreciable (i.e., neither infinitesimal nor infinite, nor even negative), as can be seen by reading the first page *Préliminaires* of his 1821 *Cours D'Analyse*. Felscher concludes that the variables assume only appreciable values, but not non-Archimedean ones.[15]

Felscher's etymological insight appears to have eluded some of the earlier commentators. It offers a refutation of the Luzin hypothesis,[16] to the effect that Cauchy variables may pass through non-Archimedean values on their way to zero. Has Felscher shown that Cauchy's *bestiarium infinitesimale* is in fact uninhabited?

Hardly so. While chasing out the infinitesimal mouse of Luzin's hypothesis[17] for the Cauchy variable, Felscher missed the hippopotamus of the possibility of the variable *becoming* an infinitesimal (see the introduction).

On page 846, Felscher quotes an agitated passage from d'Alembert's 1754 article. D'Alembert attacks the *obscurity*, and even the *falsehood* of a definition of infinitesimals attributed to unnamed geometers, and

"signature" in the *Encyclopedie*. Felscher had already committed a similar error in his 1979 work [25]. We are grateful to D. Spalt for this historical clarification. Note that Robinson [59, p. 267] similarly misattributes this passage to d'Alembert.

[15]Note that Schubring [62, p. 446], in footnote 14, explains Cauchy's term *numerical value* as what we would call today the *absolute value*. Fisher [29, p. 262] interprets Cauchy's definition accordingly, so as to allow room for infinitesimal values of Cauchy's variables. See also footnote 17 on Cleave.

[16]Luzin himself, in fact, similarly rejected non-Archimedean time, as discussed by Medvedev [57].

[17]Luzin was probably not the first and surely not the last to formulate a hypothesis to the effect that Cauchy's variable quantities pass through infinitesimal values on their way to zero. Lakatos [46, p. 153] speculates that Cauchy's variables "ran through Weierstrassian real numbers *and* infinitesimals", while J. Cleave (who edited Lakatos's essay for publication in the *Mathematical Intelligencer*) in footnote 18* in [46, p. 159] disagrees, limiting Cauchy variables to sequences of Weierstrassian reals (Cleave quotes the relevant passage on *numerical values* but does not analyze it here). Cleave alludes to the etymological point in [15, p. 268], where he disagrees with Fisher on this point.

sums up his thesis by accusing such geometers of *charlatanerie*, a term ably translated as *quackery* by Felscher, who sums up as follows:

> Reading these words today we may get the impression that they were written at the time of Weierstrass or Cantor,[18] or even by a contemporary mathematician.[19]

It is sobering to realize that, forty years after A. Robinson, a logician named Walter Felscher still conceived of the history of analysis in terms of a triumphant march out of the dark ages of infinitesimals, and toward the yawning heights of Weierstrassian epsilontics.

d'Alembert's verbal excesses merely put in relief the fact that no such rhetoric is to be found anywhere in either Cauchy or Bolzano. Felscher presents a convincing case that d'Alembert was opposed to infinitesimals. Felscher's title *Bolzano, Cauchy, epsilon, delta*[20] could therefore have pertinently been replaced by *D'Alembert, Weierstrass, epsilon, delta*, as the case for Bolzano's opposition to infinitesimals can similarly be challenged.[21] Indeed, as Lakatos [46, p. 154] points out,

> [Bolzano] was possibly the only one to see the problems related to the difference between the two continuums: the rich Leibnizian continuuum and, as he called it, its

[18]Cantor was indeed a worthy heir to d'Alembert's anti-infinitesimal vitriol. Cantor dubbed infinitesimals the *cholera bacillus* of mathematics, see J. Dauben [17, p. 353] and [18, p. 124]. This was perhaps the most vitriolic opposition to the B-continuum (see Appendix B) before Errett Bishop's *debasement of meaning*, a term he applied to classical mathematics as a whole in 1973 [3], and to infinitesimal calculus à la Robinson, in 1975 [2].

[19]It is worth pondering which contemporary mathematician (known for anti-infinitesimal vitriol) Felscher may have had in mind here, given his interest in intuitionistic logics [26, 27].

[20]Apparently, a kind of a mantra: *Bolzano, Cauchy, epsilon, delta; Bolzano, Cauchy, epsilon, delta; ...* which, repeated sufficiently many times, would lead one to accept Felscher's reduction of Cauchy's continuum to an A-continuum (see Appendix B).

[21]In discussing Bolzano's attitude toward infinitesimals, we have to distinguish between the early Bolzano and the late Bolzano. The early Bolzano defines the "Infinitely small" as "variable quantities" in the following terms: A quantity is infinitely small if it gets less than any given quantity (here Bolzano does not speak of "values" [but naturally he thought of them]). The late Bolzano definies infinitely small (and infinitely large) *numbers*; one of them is $1/(1+1+1+\ldots$ (infinitely many terms). We are grateful to D. Spalt for this historical clarification.

> 'measurable' subset–the set of Weierstrassian real numbers. Bolzano makes it very clear that the field of 'measurable numbers'[22] constitutes only an Archimedean subset of a continuum enriched by non-measurable - infinitely small or infinitely large - quantities.

D. Kurepa [44, p. 664] provides some details on Bolzano's use of infinitesimals.

Felscher's intriguing parenthetical remark indicates that he was more sensitive to Cauchy's language than numerous Cauchy historians:

> it is left open whether a *quantité variable*, with an assignment converging to zero, actually *is* or only *becomes* a *quantité infiniment petite* [28, p. 850].

On page 851, Felscher presents an analysis of Cauchy's use of an infinitesimal quantity, denoted $i$, in differentiating an exponential function. Here Felscher's additional parenthetical remark, to the effect that

> notational confusion arises from denoting both the variable $i$ and its values by the same letter,

is an unjustified criticism of Cauchy, and underscores Felscher's blindness toward the dynamic aspect of Cauchy's infinitesimal $i$, when individual values are irrelevant in the context of the dynamism of the encapsulation taking place whenever Cauchy evokes an infinitesimal. Otherwise Felscher's analysis is unexceptionable, save for a *non-sequitur* of a conclusion:

> "No 'infinitesimal' non-Archimedean numbers are ever used by Cauchy for his *quantités infiniment petites*."

In reality, Cauchy's discussion of the derivative of the exponential function admits a number of possible interpretations.

On page 852, Felscher analyzes Cauchy's infinitesimals in modern terms:

> Using today's terminology, one would describe Cauchy's forms to be filled by assignments as functions, but in order to distinguish them from the actual functions subsequently considered by Cauchy, one might call them *functional germs*. [Emphasis in the original–authors]

Felscher mentions Cauchy's use of functional germs again on page 855. In his zeal to rename Cauchy's infinitesimals by using a modern notion, so as bashfully to avoid the distasteful *infi* term, Felscher comes close

[22] *Measurable number* is Bolzano's term for appreciable number (no relation to Lebesgue-measurability). Here Bolzano foreshadows Björling's dichotomy (see [9]), which can be analyzed in terms of A- and B-continua (see Appendix B).

to endorsing Laugwitz's controversial "Cauchy numbers",[23] similarly defined in terms of germs.[24]

On page 853, Felscher *omits* a crucial first phrase used by Cauchy in formulating his first definition of continuity. Namely, he omits Cauchy's phrase *stating that* $\alpha$ *is an infinitesimal*, corresponding to the upper-left entry in Table 1.[25]

On pages 854-855, Felscher makes the following statement:

> Both Bolzano and Cauchy gave definitions of continuity which express today's [...]continuity. Both made their definitions precise and used them in today's sense; both

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[23]In [50, p. 659], Laugwitz wrote: "Every real function $f(u)$ defined on an interval $0 < u < p$ represents a Cauchy number. Two such functions $f(u)$ and $g(u)$ represent the same Cauchy number if and only if there is an interval $0 < u < q$ in which $f(u) = g(u)$." Note that Laugwitz essentially defines his "Cauchy numbers" by exploiting the concept of the germ of a function; Laugwitz explicitly mentions *function germs* in [47, p. 272]. Felscher [28, p. 858] leaves very little doubt as to how he felt with regard to Laugwitz infinitesimals:

> In this connection one must also mention certain articles and books by D. Laugwitz, in which [...] he develops his own 'mathematics of the infinitesimal' and uses it to interpret skillfully various aspects of the mathematics of the period from Euler to Cauchy. And so we have glanced at the *bestiarium infinitesimale*.

[24]An instructive case study in a *bestiarium*-consignment attitude toward infinitesimals is the review of Felscher's text for Math Reviews, by one Nicole Brillouët-Belluot. The review contains not an inkling of the fact that the text in question is a broadside attack on scholars attempting to analyze Cauchy's infinitesimals seriously. She mentions the "epsilon-delta technique", and notes that Cauchy made his "definitions [of continuity] precise and used them in today's sense", but fails to mention that the definitions in question are *infinitesimal* ones, a fact not denied by Felscher (at least in the case of one of the definitions). Her review mentions d'Alembert, who does not appear in Felscher's summary, indicating that she had read the body of Felscher's text itself (rather than merely Felscher's summary). She notes that Felscher reports on how "limit was explained and used by d'Alembert and Cauchy". She reports neither on Felscher's extensive, and vitriolic, quotes from d'Alembert (including the dramatic phrase "the metaphysics and the infinitely small quantities, whether larger or smaller than one another, are totally useless in the differential calculus"), nor d'Alembert's colorful epithets like *charlatanerie*/quackery. The reviewer completely identified with Felscher's conclusion to such an extent that she chose to spare the Math Reviews reader the burden of infinitesimal quackery, judging that Felscher carried that burden once and for all, for the rest of us. Similar remarks apply to the review of Felscher's text by Reinhard Siegmund-Schultze for Zentralblatt Math.

[25]This crucial detail leads Felscher to a further error of a conceptual nature, discussed below.

> employed them by comparing numbers and their distances with the help of inequalities in order to prove important theorems in analysis. However, Cauchy defined and used the notion of limit, whereas Bolzano did not.

Felscher's assertion concerning continuity and inequalities is misleading. While it is true that Cauchy gave a definition of *limit* using such inequalities (see, e.g., Grabiner [34]), he never gave such a definition of *continuity*.

On page 855, line 9, Felscher alleges that Cauchy's first definition of continuity is similar to Bolzano's, with the implication that the terminology of "infinitesimal" is not employed by Cauchy. Now the form in which Cauchy's definition was quoted by *Felscher* two pages earlier (see our comment above concerning Felscher's page 853) did not employ infinitesimals. But the form in which it appears in *Cauchy* did employ infinitesimals.[26]

The most remarkable aspect of Felscher's, unfortunately seriously flawed, essay is how close he comes to sensing the cognitive view of compression/encapsulation outlined in our introduction:

> We speak of a function or a variable approaching some value *indéfiniment* (indefinitely); we imagine a limiting process. [...] Thus far, $\epsilon$ and $\delta$ (and in case of sequences also $n$ and $N$) appear as handles affixed to the stages of those infinite processes. It seems that if appropriately handled in our mental exercises, they enable us to use finitely many arguments to prove statements that, in the end, speak about all the stages of the infinite process [28, p. 858].

## 6. Was epsilontics inevitable?

Meanwhile, the foremost Leibniz historian H. Bos acknowledged that Robinson's hyperreals provide a

> preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities [6, p. 13].

F. Medvedev further points out that nonstandard analysis

[26]See Table 1 (for a summary of Cauchy's definitions) and footnote 25. Schubring [62, p. 465] writes that J. Lützen's is the best analysis of continuity in Cauchy. Meanwhile, Lützen [53, p. 166] states: "Cauchy [...] gives two definitions, first one without infinitesimals, and then one using infinitesimals." The second claim is correct, but not the first.

> makes it possible to answer a delicate question bound up with earlier approaches to the history of classical analysis. If infinitely small and infinitely large magnitudes are [to be] regarded as inconsistent notions, how could they [have] serve[d] as a basis for the construction of so [magnificent] an edifice of one of the most important mathematical disciplines? [56, 58]

A powerful question, indeed. How do historians answer Medvedev's question?

Not all scholars are satisfied with the *amazing-intuition-and-deep-insight* answer offered by J. Grabiner who writes:

> [M]athematicians like Euler and Laplace had a deep insight into the basic properties of the concepts of the calculus, and were able to choose fruitful methods and *evade pitfalls* [34, p. 188] [emphasis added–authors]

How can deep insight manage to "evade pitfalls" if the foundations are regarded as inconsistent? Grabiner [34, p. 189] further claims that,

> [s]ince an adequate response to Berkeley's objections would have involved recognizing that an equation involving limits is a shorthand expression for a sequence of inequalities—a subtle and difficult idea—no eighteenth century analyst gave a fully adequate answer to Berkeley.

This is an astonishing claim, which amounts to reading back into history, developments that came much later. Such a claim amounts to postulating the inevitability of a triumphant march, from Berkeley onward, toward the radiant future of Weierstrassian epsilontics. The claim of such inevitability in our opinion is an assumption that requires further argument.

Berkeley was, after all, attacking the coherence of *infinitesimals*. He was not attacking the coherence of some kind of incipient form of Weierstrassian epsilontics and its inequalities. Isn't there a simpler answer to Berkeley's query, in terms of a passage from a point of B-continuum (see Appendix B), to the infinitely close point of the A-continuum, namely passing from a variable quantity to its limiting constant quantity?

A related attitude on the part of Felscher is discussed in Section 3.

Like Felscher, Grabiner [34, p. 190] suppresses Cauchy's reference to an infinitesimal increment of the independent variable when citing Cauchy's first definition (see Table 1 in Section 3 above), thereby managing to avoid discussing Cauchy's infinitesimals altogether. We encounter the oft-repeated claim about "the same confusion between uniform and point-wise convergence"[27] [34, p. 191]. Her discussion of Cauchy's rigor does not mention that what rigor meant to Cauchy was the replacement of the principle of the generality of algebra, by geometry, including infinitesimals. Grabiner correctly points out [34, p. 193] that "Mathematicians are used to taking the rigorous foundations for calculus for granted." She concludes: "What I have tried to do as a historian is to reveal what went into making up that great achievement." What we have tried to do is to introduce a necessary correction to the modern understanding of Cauchy, influenced by an automated infinitesimal-to-limits translation originating no later than Boyer [7, p. 277].

## 7. Variable quantities and ultrapowers

On the subject of Robinson's theory, Grattan-Guinness comments as follows:

> I made no mention of non-standard analysis in my book, for it was obvious to me that this very beautiful piece of mathematics had nothing to tell us historically [35, p. 247].

When Grattan-Guinness announced that Robinson's construction of a non-Archimedean extension of the reals "bears no resemblance to past arguments in favor of infinitesimals" [35, p. 247], he was only telling part of the story. True, the construction favored by Robinson exploited powerful compactness theorems and eschewed the sequential approach. On the other hand, the ultrapower construction of the hyperreals, pioneered by E. Hewitt [37] in 1948 and popularized by Luxembourg [54] in 1962, is firmly rooted in the sequential approach, and hence connects well with the kinetic vision of Cauchy, shared by L. Carnot. Some details on the ultrapower construction appear in Appendix B.

## 8. Conclusion

Cauchy did not have the concept of an equivalence class. However, it would be too simplistic to dismiss the possibility that Cauchy may have intuited some kind of an encapsulation leading from the *process* of a null sequence tending to zero, to the concept/noun "infinitesimal". Newton, Leibniz, and Bernoulli did not have a completely coherent

[27]See footnote 1 for a detailed discussion of the controversy over the "sum theorem".

notion of either infinitesimal or limit, yet the historical founders of the calculus are credited with defining the derivative, nonetheless.

The historical sequence of events, as far as the notion of continuity is concerned, was as follows:[28]

(1) first came Cauchy's infinitesimal definition, namely "infinitesimal $x$-increment results in an infinitesimal $y$-increment";
(2) then came the Dirichlet/Weierstrass-style nominalistic reconstruction of the original definition in terms of real inequalities, dispensing with infinitesimals;
(3) as the result of the work of Hewitt [37], Łoś[52], and Robinson, a hyperreal version of the definition is crystallized.

The historical priority of the infinitesimal definition is clear; what is open to debate is the role of the modern definition in interpreting the historical definition. The common element here is the *null sequence*, a basis both for Cauchy infinitesimals, and for ultrapower-based infinitesimals.[29] Cauchy's elaborate theory of arbitrary orders of magnitude for his infinitesimals was a harbinger, not of Weierstrassian epsilontics, but of later theories of infinitesimal-enriched continua as developed by Stolz, du Bois-Reymond,[30] and others. du Bois-Raymond's investigations were in turned pursued further by such mathematicians as E. Borel. In 1902, Borel [5, p. 35-36] cites Cauchy's definition of such "order of infinitesimal" from Oeuvres de Cauchy, série 2, tome 4, p. 181, corresponding to Cauchy's *Leçons sur le calcul différentiel* from 1829, appearing in a section entitled "Préliminaires" (see G. Fisher [30, p. 144]).

[28]We leave out Zermelo's contribution which, while prior to Cauchy's, did not exert any influence until the 1860s.

[29]Modulo suitable foundational material, one can ensure that every infinitesimal is represented by a null sequence; an appropriate ultrafilter (called a P-point) will exist if one assumes the continuum hypothesis, or even the weaker Martin's axiom (see Cutland *et al* [16] for details).

[30]In 1966, Robinson wrote: "Following Cauchy's idea that an infinitely small or infinitely large quantity is associated with the behavior of a function $f(x)$, as $x$ tends to a finite value or to infinity, du Bois-Raymond produced an elaborate theory of orders of magnitude for the asymptotic behavior of functions ... Stolz tried to develop also a theory of arithmetical operations for such entities" [59, p. 277-278]; "It seems likely that Skolem's idea to represent infinitely large natural numbers by number-theoretic functions which tend to infinity (Skolem [1934]), also is related to the earlier ideas of Cauchy and du Bois-Raymond" [59, p. 278]. The reference is to Skolem [66].

## Acknowledgments

We are grateful to Hourya Benis Sinaceur, David Sherry, and Detlef Spalt for a careful reading of an earlier version of the manuscript and for their helpful comments.

## Appendix A. Five common misconceptions in the Cauchy literature

Both Laugwitz and Hurya Benis Sinaceur [65] have exposed a number of misconceptions in the literature concerning Cauchy's foundational work. Some of the most common ones are reproduced below in italics.

1. *Bolzano, Cauchy, and Weierstrass were all gardeners who contributed to the ripening of the fruit of the notion of limit.*

Here the implicit assumption is that the Weierstrassian epsilontic notion of "limit" in the context of an Archimedean continuum is the centerpiece of any possible edifice of analysis. Such an assumption is questionable on two counts. First, as Felix Klein pointed out in 1908, there are two parallel threads in the development of analysis, one based on an Archimedean continuum, and the other exploiting an infinitesimal-enriched continuum.[31] One risks pre-judging the outcome of any analysis of Cauchy by postulating that he is working in the Archimedean thread. The second implicit assumption is that the Weierstrassian notion of limit is central in Cauchy. Thus, Boyer [7, p. 277] postulates that Cauchy is working with a notion of limit similar to the Weierstrassian one. This requires further argument, and at least at first glance is incorrect: Cauchy emphasizes infinitesimals as a foundational notion, but he never emphasizes limits as a foundational notion. Thus, in Cauchy's definition of continuity the word "limit" does occur, but only in the sense of the "endpoint" of the interval of definition of the function, rather than the behavior of its values.

2. *Cauchy, along with other mathematicians, abandoned infinitesimals in favor of other more rigorous notions.*

During the period 1815-1820, Cauchy appears to have been ambivalent about infinitesimals. Starting in about 1820, he uses them with increasing frequency both in his textbooks and his research publications, and insists on the centrality of infinitesimals as a foundational notion.

3. *Cauchy was forced to teach infinitesimals at the Ecole.*

[31] See footnote 12.


FIGURE 1. Thick-to-thin: taking standard part (the thickness of the top line is merely conventional)

The intended implication appears to be that Cauchy only used infinitesimals because of the pressure from the Ecole administration. During the period 1815-1820 there were some tensions with the administration over the delayed appearance of infinitesimals in the syllabus. At any rate, Cauchy continued using infinitesimals thoughout his career and long after completing his teaching stint at the Ecole. Thus he reproduces his 1821 definition of continuity (in terms of infinitesimals) as late as 1853, in his text on the sum theorem [13].

4. *Cauchy based his infinitesimals on the notion of limit.*
This is an ambiguous claim, and essentially a play on words on the term "limit". The modern audience understands "limit" as a Weierstrassian epsilontic notion. If this is what is claimed, then the claim is false. As far as the kinetic notion of limit that Cauchy does mention in discussing a variable quantity approaching a limit, it is conspicuously absent in Cauchy's discussion of infinitesimals. Thus, rather than infinitesimals being based on the notion of limit, it is the notion of a variable quantity that's primitive, and both infinitesimals and limits are defined in terms of it. Note again that the term "limit" does appear in Cauchy's definition of continuity, but in an entirely different sense, namely endpoint of the interval where the function is defined.

5. *Cauchy introduced rigor into calculus that anticipates the rigor of Weierstrass.*
While Cauchy certainly emphasizes rigor, postulating a continuity between Cauchy's rigor and Weierstrassian rigor is a methodological error; again see Klein's comment on the two strands. To Cauchy, rigor meant abandoning the principle of the "generality of algebra" as practiced by Euler and others, and its replacement by geometry–and by infinitesimals.

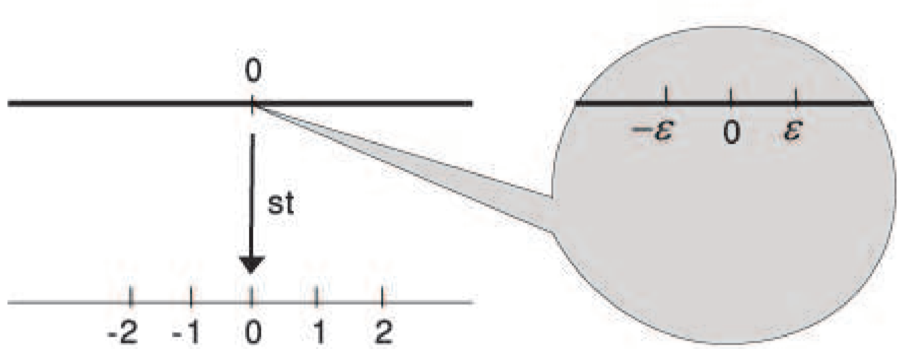


FIGURE 2. Zooming in on infinitesimal $\epsilon$

## APPENDIX B. RIVAL CONTINUA

A Leibnizian definition of the derivative as the infinitesimal quotient

$$\frac{\Delta y}{\Delta x},$$

whose logical weakness was criticized by Berkeley, was modified by A. Robinson by exploiting a map called *the standard part*, denoted "st", from the finite part of a B-continuum (for "Bernoullian"), to the A-continuum (for "Archimedean"), as illustrated in Figure 1.[32]

We illustrate the construction by means of an infinite-resolution microscope in Figure 2.

We will denote such a B-continuum by a new symbol $\mathrm{I\!I\!R}$. We will also denote its finite part, by

$$\mathrm{I\!I\!R}_{<\infty} = \{x \in \mathrm{I\!I\!R} : \ |x| < \infty\}.$$

The map "st" sends each finite point $x \in \mathrm{I\!I\!R}$, to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to $x$:

$$\begin{array}{c} \mathrm{I\!I\!R}_{<\infty} \\ \downarrow{\scriptstyle \mathrm{st}} \\ \mathbb{R} \end{array}$$

[32]In the context of the hyperreal extension of the real numbers, the map "st" sends each finite point $x$ to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to $x$. In other words, the map "st" collapses the cluster of points infinitely close to a real number $x$, back to $x$. A comparative study of continua from a predicative angle is in Feferman [28].

Robinson's answer to Berkeley's *logical criticism* (see D. Sherry [64]) is to define the derivative as

$$\mathrm{st}\left(\frac{\Delta y}{\Delta x}\right),$$

instead of $\Delta y/\Delta x$. For an accessible introduction to the hyperreals, see H. J. Keisler [40, 41].

Note that both the term "hyper-real field", and an ultrapower construction thereof, are due to E. Hewitt in 1948, see [37, p. 74]. The transfer principle allowing one to extend every first-order real statement to the hyperreals, is due to J. Łoś in 1955, see [52]. Thus, the Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle. To elaborate on the ultrapower construction of the hyperreals, let $\mathbb{Q}^{\mathbb{N}}$ denote the space of sequences of rational numbers. Let $\left(\mathbb{Q}^{\mathbb{N}}\right)_C$ denote the subspace consisting of Cauchy sequences. The reals are by definition the quotient field

$$\mathbb{R} := \left(\mathbb{Q}^{\mathbb{N}}\right)_C \big/ \mathcal{F}_{null},$$

where the ideal $\mathcal{F}_{null}$ contains all the null sequences. Meanwhile, an infinitesimal-enriched field extension of $\mathbb{Q}$ may be obtained by forming the quotient

$$\mathbb{Q}^{\mathbb{N}} \big/ \mathcal{F}_u,$$

see Figure 3. Here a sequence $\langle u_n \rangle$ is in $\mathcal{F}_u$ if and only if the set

$$\{n \in \mathbb{N} : u_n = 0\}$$

is a member of a fixed ultrafilter.[33] To give an example, the sequence $\left\langle \frac{(-1)^n}{n} \right\rangle$ represents a nonzero infinitesimal, whose sign depends on whether or not the set $2\mathbb{N}$ is a member of the ultrafilter. To obtain a full hyperreal field, we replace $\mathbb{Q}$ by $\mathbb{R}$ in the construction, and form a similar quotient

$$\mathrm{I\!I\!R} := \mathbb{R}^{\mathbb{N}} \big/ \mathcal{F}_u.$$

A more detailed discussion of the ultrapower construction can be found in M. Davis [20]. See also Błaszczyk [4] for some philosophical implications. More advanced properties of the hyperreals such as saturation were proved later, see Keisler [41] for a historical outline. A

[33] An ultrafilter on $\mathbb{N}$ can be thought of as a way of making a systematic choice, between each pair of complementary infinite subsets of $\mathbb{N}$, so as to prescribe which one is "dominant" and which one is "negligible". Such choices have to be made in a coherent manner, e.g., if a subset $A \subset \mathbb{N}$ is negligible then any subset of $A$ is negligible, as well. The existence of ultrafilters was proved by Tarski [72], see Keisler [42, Theorem 2.2].

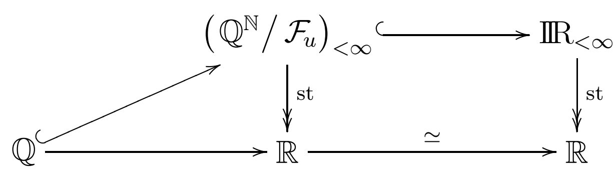


FIGURE 3. An intermediate field $\mathbb{Q}^{\mathbb{N}}/\mathcal{F}_u$ is built directly out of $\mathbb{Q}$

helpful "semicolon" notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [51]. A discussion of infinitesimal optics is in Magnani and Dossena [55, 21].

In this connection, the following items should be mentioned:

(1) Nelson's internal set theory is a re-thinking of the foundational material with a view to allow a more stratified (hierarchical) number line. Thus, the canonical set theory, namely ZFC, is modified by the introduction of a "standard" predicate. Then what is known as the usual construction of the "real" line produces a line that bears a striking resemblance to the Hewitt-Łoś-Robinson hyperreals.
(2) Hewitt's construction of hyper-real fields has roots in functional analysis, including works by Gelfand and Kolmogorov [32].[34]
(3) An interesting example is an almost trivial and degenerate case of "double" numbers, that is, numbers of the form $a + b\delta$,

[34] In 1990, Hewitt reminisced about his

> "efforts to understand the ring of all real-valued continuous [not necessarily bounded] functions on a completely regular $T_0$-space. I was guided in part by a casual remark made by Gel'fand and Kolmogorov (Doklady Akad. Nauk SSSR 22 [1939], 11-15). Along the way I found a novel class of real-closed fields that superficially resemble the real number field and have since become the building blocks for nonstandard analysis. I had no luck in talking to Artin about these hyperreal fields, though he had done interesting work on real-closed fields in the 1920s. (My published "proof" that hyperreal fields are real-closed is false: John Isbell earned my gratitude by giving a correct proof some years later.) [...] My ultrafilters also struck no responsive chords. Only Irving Kaplansky seemed to think my ideas had merit. My first paper on the subject was published only in 1948" [38]

(Hewitt goes on to detail the eventual success and influence of his 1948 text). Here Hewitt is referring to Isbell's 1954 paper [39], proving that Hewitt's hyper-real fields are real closed. Note that a year later, Łoś [52] proved the general transfer principle for such fields, implying in particular the property of being real closed, the latter being first-order.

where $\delta^2 = 0$. A recent short crash course on classical Lie groups used double numbers to compute Lie algebras of groups like $SO(3, \mathbb{R})$ with great effectiveness.

(4) Related to (1) is an issue of actual/potential infinity and how it is resolved in modern computer science.

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