SYSTOLES OF HYPERBOLIC MANIFOLDS

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Abstract. We show that for every $n \geq 2$ and any $\epsilon > 0$ there exists a compact hyperbolic $n$-manifold with a closed geodesic of length less than $\epsilon$. When $\epsilon$ is sufficiently small these manifolds are non-arithmetic, and they are obtained by a generalised in-breeding construction which was first suggested by Agol for $n = 4$. We also show that for $n \geq 3$ the volumes of these manifolds grow at least as $1/\epsilon^{n-2}$ when $\epsilon \to 0$.

1. Introduction

Let $H^n$ denote the hyperbolic $n$-space. By a compact hyperbolic $n$-manifold we mean a quotient space $M = \Gamma \backslash H^n$ where $\Gamma$ is a cocompact torsion-free discrete subgroup of $\text{Isom}(H^n)$, the group of isometries of $H^n$. The systole of a compact Riemannian manifold $M$, denoted by $\text{Syst}_1(M)$, is the length of the shortest closed geodesic on $M$. We refer to a recent monograph by M. Katz [K] for more information about systoles and systolic geometry.

It is well-known that for any $\epsilon > 0$ there exist 2-dimensional compact hyperbolic manifolds having a systole of length less than $\epsilon$, and examples of such manifolds of any genus $g \geq 2$ can be easily constructed using Teichmüller theory. A similar result for $n = 3$ can be achieved using Thurston’s Dehn surgery. For a long time the existence of compact hyperbolic manifolds with arbitrarily short systoles in higher dimensions was an open problem. In a recent paper [A], Agol suggested a very interesting construction which solves the problem for $n = 4$. His paper was a starting point for our work.

Our main result is the following:

**Theorem 1.1.** For every $n \geq 2$ and any $\epsilon > 0$, there exist compact $n$-dimensional hyperbolic manifolds $M$ with $\text{Syst}_1(M) < \epsilon$.

The manifolds $M$ are obtained by a variant of an in-breeding construction which was first suggested by Agol for $n = 4$. We simplify the critical step in the argument of [A] which makes extensive use of geometrical finiteness and related properties, and hence limits his construction to some special examples. A principal ingredient in our proof is a lemma of Margulis and Vinberg [MV] which we generalise to cocompact discrete subgroups of $\text{Isom}(H^n)$. The proof of this generalised Margulis-Vinberg lemma is the main technical part of the proof of the theorem.

Systolic geometry studies relations between systole length and volume captured by isosystolic inequalities (cf. [K]). Our second result provides an inequality of this type for the manifolds from Theorem 1.1. To put it into context recall that for $n \geq 4$ (in contrast

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with \( n = 2 \) and 3 there exist only finitely many non-isometric hyperbolic \( n \)-manifolds of bounded volume \([W]\). Hence when \( \epsilon \to 0 \) for these dimensions we will necessarily have \( \text{Vol}(M) \to \infty \). It is natural to ask how fast the volume grows, and the following theorem gives the answer to this question for our manifolds.

**Theorem 1.2.** For every \( n \geq 3 \) there exists a positive constant \( C_n \) (which depends only on \( n \)), such that the systole length and volume of the manifolds obtained in the proof of Theorem 1.1 satisfy

\[
\text{Vol}(M) \geq \frac{C_n}{\text{Syst}_1(M)^{n-2}}.
\]

The proof of this theorem uses important recent work of Bridgeman and Kahn on orthospectra and volumes of hyperbolic \( n \)-manifolds [BK]. In fact, we can show that it is possible to achieve that \( \text{Vol}(M) \) grows exactly like a polynomial in \( 1/\text{Syst}_1(M) \) (see the discussion after the proof of Theorem 1.2 and Proposition 4.1). Therefore, Theorem 1.2 captures the growth rate of the volume in our construction. It is unknown if for \( n \geq 4 \) there exist hyperbolic \( n \)-manifolds \( M \) with \( \text{Syst}_1(M) \to 0 \) and \( \text{Vol}(M) \) growing slower than a polynomial in \( 1/\text{Syst}_1(M) \).

The paper is organised as follows: In Sect. 2, we prove Theorem 1.1 modulo the generalised Margulis-Vinberg lemma. The proof of the lemma is given in Sect. 3. The next section is dedicated to Theorem 1.2. We end with remarks regarding arithmeticity, commensurability and related questions in Sect. 5.

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## 2. Proof of Theorem 1.1

Let \( \epsilon > 0 \). We first obtain a manifold \( M \) which contains a geodesic segment of length at most \( \epsilon/2 \) that is orthogonal to two hyperplanes.

Fix a totally real number field \( K \subset \mathbb{R} \), and let \( f \) be a non-singular \((n + 1)\)-ary quadratic form of signature \((n, 1)\) defined over \( K \). We assume that for Galois embeddings \( \sigma: K \hookrightarrow \mathbb{R} \) which are different from the original embedding, \( f^\sigma \) is positive definite. It is well known that \( \text{PO}_f(\mathbb{R}) \) is isomorphic to the group of isometries \( \text{Isom}(\mathbb{H}^n) \) of the hyperbolic \( n \)-space, and that \( \text{PO}_f(\mathcal{O}_K) = \text{O}_f(\mathcal{O}_K)/\{\pm 1\} \) is an arithmetic lattice in \( \text{PO}_f(\mathbb{R}) \), where \( \mathcal{O}_K \) denotes the ring of integers of \( K \). The lattices obtained this way (and subgroups of \( \text{PO}_f(\mathbb{R}) \) which are commensurable with them) are called arithmetic subgroups of the simplest type. From now on let us assume that the degree of the field \( K \) is at least 2; i.e., \( K \neq \mathbb{Q} \). Then by Godement’s compactness criterion \( \text{PO}_f(\mathcal{O}_K) \) is cocompact. Now, by Selberg’s Lemma, one can find a torsion-free subgroup \( \Gamma < \text{PO}_f(\mathcal{O}_K) \), of finite index. Thus \( \Gamma \backslash \mathbb{H}^n \) is a compact hyperbolic \( n \)-manifold.

Consider the hyperboloid model of hyperbolic \( n \)-space (cf. [R, Ch. 3]). Let us choose a vector \( e_0 \in K^{n+1} \) with \( f(e_0) > 0 \). The intersection \( H_0 = \langle e_0 \rangle^{\perp} \cap \mathbb{H}^n \) in the ambient space \( \mathbb{R}^{n+1} \) is a hyperplane in \( \mathbb{H}^n \) and, moreover, \( \Gamma_0 = \text{Isom}(H_0) \cap \Gamma \) is a cocompact discrete subgroup of \( \text{Isom}(H_0) \) (where we embed \( \text{Isom}(H_0) \hookrightarrow \text{Isom}(\mathbb{H}^n) \) in the natural way). The latter holds because \( \Gamma \) is defined over \( K \) and \( H_0 \) is a \( K \)-rational hyperplane. A well-known result of Margulis asserts that since the group \( \Gamma \) is arithmetic, its commensurator is dense in \( \text{PO}_f(\mathbb{R}) \) [M, Theorem 1, p. 2]. Indeed, this commensurator contains \( \text{PO}_f(K) \),
so we can find \( \gamma \in \text{PO}_f(K) \) with the property that \( H_1 \) — where \( H_1 = \langle e_1 \rangle^{±} \cap \mathcal{H}^n \) and \( e_1 = \gamma(e_0) \) — is disjoint, but at most distance \( \epsilon/2 \) away, from \( H_0 \). Note that since \( \gamma \in \text{PO}_f(K) \) both \( \Gamma_0 \setminus H_0 \) and \( \Gamma_1 \setminus H_1 \) (where \( \Gamma_1 = \text{Isom}(H_1) \cap \Gamma \)) embed in \( \Gamma \setminus \mathcal{H}^n \) as totally geodesic hypersurfaces.

The generalised Margulis-Vinberg lemma (below) states that one can find a finite-index subgroup \( \Gamma' \prec \text{PO}_f(\mathcal{O}_K) \) with the property that for every \( h \in \Gamma' \),

\[
\text{either } h(H_0) = H_0 \quad \text{or} \quad h(H_0) \cap (H_0 \cup H_1) = \emptyset.
\]

Let \( \Lambda = \Gamma \cap \Gamma' \). Then the natural projections of \( H_0 \) and \( H_1 \) in the quotient \( \Lambda \setminus \mathcal{H}^n \) will not intersect. Thus the manifold \( L = \Lambda \setminus \mathcal{H}^n \) contains properly embedded totally geodesic hypersurfaces \( \Lambda_0 \setminus H_0 \) and \( \Lambda_1 \setminus H_1 \) (where \( \Lambda_i = \text{Isom}(H_i) \cap \Lambda, i = 1, 2 \)) which are \( \epsilon/2 \)-close.

Let \( g \) be a geodesic segment orthogonal to both of them, so as to fulfil our aim stated at the beginning of the proof.

To complete the proof, we ‘cut’ \( L \) along the hypersurfaces \( \Lambda_0 \setminus H_0 \) and \( \Lambda_1 \setminus H_1 \). Retaining the connected component containing \( g \) (if the cutting separates the manifold), we have a manifold \( M' \) with a geodesic boundary and \( g \) orthogonal to this boundary. Taking the double of \( M' \) results in a closed \( n \)-manifold \( M \), and the segment \( g \) becomes a closed geodesic of \( M \) of length at most \( \epsilon \). This concludes the construction. \( \square \)

3. Generalised Margulis-Vinberg Lemma

An earlier form of the result in this section appeared in a paper by Margulis and Vinberg [MV], and was also used by Kapovich, Potyagailo and Vinberg in a paper on non-coherence of lattices [KPV]. The generalised version here considers the case where a quadratic form is defined over a number field \( K/\mathbb{Q} \) rather than being purely rational. This generalisation is necessary for dealing with cocompact lattices.

As before, let \( K \subset \mathbb{R} \) be a totally real algebraic number field of degree \( d \), and \( f \) a quadratic form over \( K \) of signature \((n, 1)\) and such that \( f^\sigma \) is positive definite for all Galois embeddings \( \sigma_j: K \hookrightarrow \mathbb{R}, j = 2, \ldots, d \) which are different from the original embedding (denoted by \( \sigma_1 \)). Thus the group \( \text{PO}_f(\mathcal{O}_K) \) is an arithmetic lattice in \( \text{PO}_f(\mathbb{R}) \).

Let \( H_0, \ldots, H_k \) be pairwise disjoint hyperplanes in \( \mathcal{H}^n \) defined by \( H_i = \mathcal{H}^n \cap \langle e_i \rangle^{±} \), \( e_i \in K^{n+1}, i = 0, \ldots, k \).

**Lemma 3.1.** There exists a finite index subgroup \( \Gamma' \prec \text{PO}_f(\mathcal{O}_K) \) such that for every \( h \in \Gamma' \),

\[
either h(H_0) = H_0 \quad \text{or} \quad h(H_0) \cap (H_0 \cup \cdots \cup H_k) = \emptyset.
\]

**Proof.** Write \( \Gamma = \text{PO}_f(\mathcal{O}_K) \). Since multiplying the \( e_i \) by the denominators of their entries preserves their orthogonality with the \( H_i \), we can assume the \( e_i \) have entries in \( \mathcal{O}_K \). Thus if \( h \in \Gamma \), then \( (h(e_0), e_i) \in \mathcal{O}_K \). (Throughout this proof, inner products and orthogonal complements are understood to be with respect to \( f \).) Now assume \( \mathcal{P} \subset \mathcal{O}_K \) is an ideal with

\[
|N(\mathcal{P})| \geq 2\epsilon \max_{i=0,\ldots,k} |N((e_0, e_i))|
\]

(3.1)

where \( \epsilon \geq 1 \) is a constant to be determined later. The congruence subgroup \( \Gamma(\mathcal{P}) \prec \Gamma \) contains (by definition) precisely those elements \( h \in \Gamma \) with the property that

\[
h \equiv \text{Id} \pmod{\mathcal{P}}.
\]
For \( j \) whence (3.5) implies

Let us assume that (3.1) and general properties of norms in number fields; incidentally (3.1) also gives the conclusion of the lemma.

To be able to examine the intersections of the hyperplanes we use the following inequality [R, Theorem 3.2.6]: two hyperplanes in \( H^n \), defined as above by normal vectors \( v_0 \) and \( v_1 \), intersect transversally if and only if

\[
\left| (v_0, v_1) \right| < \sqrt{(v_0, v_0)(v_1, v_1)}.
\]

So we find that the hyperplanes are disjoint or coincide completely if

\[
\left| (v_0, v_1) \right| \geq \sqrt{(v_0, v_0)(v_1, v_1)}.
\]  

(3.3)

Notice that if \( \alpha_i = 0 \) in (3.2) then

\[
\left| (h(e_0), e_i) \right| = \left| (e_0, e_i) \right| \geq \sqrt{(e_0, e_0)(e_i, e_i)} = \sqrt{(h(e_0), h(e_0))(e_i, e_i)},
\]

and hence the hyperplanes \( h(H_0) \) and \( H_i \) are either disjoint or equal. We will eliminate the possibility of equality when \( i \geq 1 \) later in the proof.

If on the other hand \( \alpha_i \neq 0 \), then

\[
\left| N((h(e_0), e_i)) \right| = \left| N((e_0, e_i) + \alpha_i P) \right| \geq \frac{1}{2} \left| N(\alpha_i P) \right| \geq C \left| N(e_0, e_i) \right|.
\]

For the first inequality we must assume that \( \left| N(\alpha_i P) \right| \geq 2 \left| N(e_0, e_i) \right| \) which follows from (3.1) and general properties of norms in number fields; incidentally (3.1) also gives the second inequality.

By definition of the norm, we can rewrite it as

\[
\left| (h(e_0), e_i) \cdot (h(e_0), e_i)^{\sigma_2} \cdots (h(e_0), e_i)^{\sigma_d} \right| \geq C \left| (e_0, e_i) \cdot (e_0, e_i)^{\sigma_2} \cdots (e_0, e_i)^{\sigma_d} \right|.
\]  

(3.5)

For \( j \geq 2 \), the forms \( \langle \cdot, \cdot \rangle^{\sigma_j} \) are positive-definite, so the Cauchy-Schwartz inequality gives

\[
\left| (h(e_0), e_i)^{\sigma_j} \right| \leq \sqrt{(h(e_0), h(e_0))^{\sigma_j}(e_i, e_i)^{\sigma_j}} = \sqrt{(e_0, e_0)^{\sigma_j}(e_i, e_i)^{\sigma_j}},
\]

whence (3.5) implies

\[
\left| (h(e_0), e_i) \right| \geq C \left| (e_0, e_i) \right| \prod_{j=2}^{d} \frac{\left| (e_0, e_i)^{\sigma_j} \right|}{\left| (h(e_0), e_i)^{\sigma_j} \right|}
\]

\[
\geq C \left| (e_0, e_i) \right| \prod_{j=2}^{d} \frac{\left| (e_0, e_i)^{\sigma_j} \right|}{\sqrt{(e_0, e_0)^{\sigma_j}(e_i, e_i)^{\sigma_j}}}
\]

Let us assume that \( C \) satisfies the inequality

\[
C \geq \prod_{j=2}^{d} \frac{\sqrt{(e_0, e_0)^{\sigma_j}(e_i, e_i)^{\sigma_j}}}{\left| (e_0, e_i)^{\sigma_j} \right|}.
\]  

(3.6)
Note that the right-hand side of (3.6) depends only on the vectors \( e_i \) but not on \( h \), and hence it can be used as a condition for determining \( C \) in (3.1). From this we get

\[
\left| (h(e_0), e_i) \right| \geq \left| (e_0, e_i) \right| \geq \sqrt{(e_0, e_0)(e_i, e_i)} = \sqrt{(h(e_0), h(e_0))(e_i, e_i)},
\]

where in the second inequality we used again the condition that initial hyperplanes are pairwise disjoint, and the last equality follows from \( h \) being an isometry of \( \mathbb{H}^n \). Hence by (3.3) the hyperplanes \( h(H_0) \) and \( H_i \) are either disjoint or equal.

To avoid the possibility of \( h(H_0) \) coinciding with \( H_i \) for \( i \geq 1 \), we have to ensure that \( h(e_0) \neq \beta_i e_i \) for some \( \beta_i \in \mathbb{R} \). If it exists, this \( \beta_i \) would be given by

\[
\beta_i = \pm \sqrt{(e_0, 0)/(e_i, e_i)}
\]

and there are two possible cases: (a) \( \beta_i \notin \mathcal{O}_K \), in which case \( h(e_0) = \beta_i e_i \) is impossible; or (b) \( \beta_i \in \mathcal{O}_K \), and in order to avoid the coincidence we can assume that \( \mathcal{P} \) does not divide some of the nonzero entries of the vectors \( e_0 + \sqrt{(e_0, 0)/(e_i, e_i)} e_i \) and \( e_0 - \sqrt{(e_0, 0)/(e_i, e_i)} e_i \). This may give some additional constraints on \( \mathcal{P} \) which are all satisfied for the ideals of sufficiently large norm. \( \square \)

4. Proof of Theorem 1.2

Let \( M' \) be a compact hyperbolic \( n \)-manifold as in Sect. 2, for which \( M \) is a double. By the construction, \( M' \) has a totally geodesic arc of length \( \ell = \frac{1}{2} \text{Syst}_1(M) \) with endpoints in \( \partial M' \). This value \( \ell \) appears in the orthospectrum of \( M' \) as defined in [BK]. In order to bound the volume of \( M' \), and hence of \( M \), we can apply the result of Bridgeman and Kahn which relates the volume and orthospectrum of a compact hyperbolic \( n \)-manifold with non-empty totally geodesic boundary.

Assuming \( n \geq 3 \), by [BK, Theorem 1] we have \( \text{Vol}(M') \geq F_n(\ell) \), and by Lemma 9(3) (loc. cit.), \( \lim_{\ell \to 0} \ell^{n-2} F_n(\ell) = K_n \), where \( F_n \) is a continuous monotonically decreasing function \( \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) and \( K_n \) is an explicit positive constant given there (loc. cit.). It follows that there exists \( K'_n > 0 \) such that if \( \ell < 1 \), then

\[
F_n(\ell) \geq \frac{K'_n}{\ell^{n-2}}.
\]

Therefore,

\[
\text{Vol}(M) = 2 \text{Vol}(M') \geq \frac{2^{n-1} K'_n}{\text{Syst}_1(M)^{n-2}},
\]

if \( \text{Syst}_1(M) < 2 \).

When the systole of \( M \) is large, say \( \text{Syst}_1(M) \geq 2 \), the desired inequality is peripheral to the general Gromov inequality [K, Theorem 12.2.2]

\[
\text{Vol}(M) \geq A_n \text{Syst}_1(M)^n,
\]

where \( A_n \) is a positive constant which depends only on \( n \).

Hence we can take \( C_n = \min(2^{n-1} K'_n, A_n) \) and the theorem is proven. \( \square \)

It is interesting to see how close the inequality of Theorem 1.2 approximates the actual growth of volume in our construction. In Sect. 2 the desired manifold \( M \) is obtained as a double of \( M' \), which in turn appears as a part of the quotient manifold \( L = \Lambda \backslash \mathbb{H}^n \). Hence

\[
\text{Vol}(M) = 2 \text{Vol}(M') \leq 2 \text{Vol}(L).
\]
The volume of $L$ depends on the index of $\Lambda$ in the initial group $\Gamma$ which can be estimated from the proof of the generalised Margulis-Vinberg lemma in Sect. 3. Let us note that by the argument in Sect. 3 we can always make the index $|\Gamma : \Lambda|$ arbitrarily large, and hence cannot bound the volume of $L$ from above. However, we are rather interested in understanding how small the volume can get when $\epsilon$ in Theorem 1.1 is close to zero and here we can say more.

**Proposition 4.1.** For every $n \geq 2$ there exists a sequence of manifolds $\{M_i\}$ from Theorem 1.1, such that when $i \to \infty$, $\text{Syst}_1(M_i) \to 0$ and

$$\text{Vol}(M_i) \leq \frac{B_n}{\text{Syst}_1(M_i)^\gamma_n},$$

where $B_n$ and $\gamma_n$ are positive constants depending on $n$.

**Proof.** Consider a sequence of inbred manifolds $M_i$ with $\text{Syst}_1(M_i) = \epsilon_i \to 0$ when $i \to \infty$. For each $M_i$ we have an associated vector $e_i^1$ defined in Sect. 2 and an ideal $P_i \in \mathcal{O}_K$ defined in Sect. 3. By (3.1), $|N(P_i)| \geq 2C_i|N((e_0, e_1^i))|$, where $C_i$ satisfies inequality (3.6). Recall that the vectors $e_0$ and $e_1^i$ are normalised so that they have coordinates in $\mathcal{O}_K$. This implies that when the angle between $e_0$ and $e_1^i$ tends to 0, either $|N((e_0, e_1^i))|$ or the lower bound for $C_i$ will grow, and hence the absolute value of the norm $|N(P_i)| \to \infty$.

To make this more concrete we consider an example. Let $K = \mathbb{Q}[\sqrt{5}]$ and $f_n = -\frac{1}{2}(1 + \sqrt{5})x_0^2 + x_1^2 + \cdots + x_n^2$. Following the proof of Theorem 1.1, consider the vectors

$$e_0 = (0, 0, \ldots, 0, 1) \text{ and } e_1^i = ((3 - \sqrt{5})^i, 0, \ldots, 0, 1) \quad (i = 2, 3, \ldots)$$

so that

$$(e_0, e_0) = (e_0, e_1^i) = 1 \quad \text{and} \quad (e_1^i, e_1^i) = \frac{1 + \sqrt{5}}{2}((3 - \sqrt{5})^{2i} + 1),$$

where the inner products are with respect to the form $f_n$. Since $(e_0, e_0)$ and $(e_1^i, e_1^i)$ are positive, these vectors define hyperplanes $H_0$ and $H_1^i$ in $\mathbb{H}^n$; and moreover the hyperplane $H_0$ does not intersect any of the $H_1^i$ since

$$|(e_0, e_1^i)| \geq \sqrt{(e_0, e_0)(e_1^i, e_1^i)} \quad (\text{cf. (3.3)}).$$

The hyperbolic distance between $H_0$ and $H_1^i$ is given by the following formula [R, Theorem 3.2.8]:

$$\cosh \rho(H_0, H_1^i) = \frac{|(e_0, e_1^i)|}{\|e_0\|\|e_1^i\|} = \frac{1}{1 - \frac{1}{2}(1 + \sqrt{5})(3 - \sqrt{5})^{2i}}.$$

From this we conclude that as $i \to \infty$,

$$\epsilon_i = 2\rho(H_0, H_1^i) \sim \frac{C_0}{\exp(i)} \quad (4.1)$$

for some constant $C_0 > 0$. Now recall that by (3.1), we have

$$|N(P_i)| \geq 2C_i|N((e_0, e_1^i))| \geq 2C_i,$$
and by (3.6),
\[ C_i \geq \frac{\sqrt{(e_0, e_0)(e_1, e_1)}}{|(e_0, e_1)|}, \]
where the Galois embedding \( \sigma : K \hookrightarrow \mathbb{R} \) is obtained from the initial embedding by applying the automorphism \( a + b\sqrt{5} \mapsto a - b\sqrt{5} \) of \( \mathbb{Q}[\sqrt{5}] \).

We have
\[ (e_0, e_0)^\sigma = (e_0, e_1)^\sigma = 1 \quad \text{and} \quad (e_1, e_1)^\sigma = \frac{\sqrt{5} - 1}{2} (3 + \sqrt{5})^i + 1. \]

Note that when \( i \to \infty \), \( (e_1, e_1)^\sigma \to \infty \); and moreover, recalling (4.1), we can check that \((e_i, e_i)^\sigma)^{1/2} \sim C(1/e_i)^\gamma\) for some constants \( C, \gamma > 0 \) when \( e_i \to 0 \). This establishes the growth rate of \( C_i \). The norm of \((e_0, e_1)\) is bounded (for it equals 1), and \( \beta_i = \sqrt{(e_0, e_0)/(e_1, e_1)} \notin \mathcal{O}_K \); hence the (minimal possible) growth rate of \( |N(P_i)| \) is the same as that of \( C_i \).

Note that if we had chosen \( e_1^i = (0, 1, 0, \ldots, 0, (3 - \sqrt{5})^i) \), then the \( C_i \)'s would be bounded but \( |N((e_0, e_1))| \) would grow, providing a lower bound for \( |N(P_i)| \) of the same rate of growth with respect to \( \epsilon_i \). We leave the details of this case to the interested reader.

For a given \( \mathcal{P}_i \), we have \(|\Gamma : \Lambda_i| < |N(\mathcal{P}_i)|^{(n+1)^2}\) which comes from the order of PO\(_f\) over the residue field of \( \mathcal{P}_i \). Hence
\[ \frac{1}{2} \operatorname{Vol}(M_i) \leq \operatorname{Vol}(L_i) \lesssim (C(1/e_i)^\gamma)^{(n+1)^2}, \]
which is a polynomial in \( 1/\text{Syst}_1(M_i) \) of degree \( \gamma_n = \gamma(n + 1)^2 \). This finishes the proof of the proposition. 

5. Remarks

5.1. Arithmeticity. If \( \epsilon \) in Theorem 1.1 is less than some \( \epsilon_0 > 0 \), which depends only on the degree \( d \) of the field \( K \) in the construction, and on the dimension \( n \), then the manifolds \( M \) produced by the theorem are non-arithmetic. This can be seen as follows: Assume that the manifold \( M \) is arithmetic. The fundamental group \( \pi_1(M') \) (of the compact manifold \( M' \) with boundary, of which \( M \) is a double) injects into \( \pi_1(M) \) and both are Zariski dense in \( \text{PO}(n,1)^0 \). This is shown in [GPS, Sects 0.2 and 1.7]. By the commensurability criterion [GPS, Sect. 1.6] we conclude that \( \Gamma_M = \pi_1(M) \) is commensurable with \( \text{PO}_f(\mathcal{O}_K) \). Now we can follow a known argument relating the lengths of geodesics of an arithmetic manifold \( M \) to eigenvalues of integral matrices (see [G, Sect. 10]). This implies in our case that \( \text{Syst}_1(M) \geq C_{n,d} \), since the integral polynomials which arise have their degree bounded above by \( d(n+1) \). Hence if \( \epsilon < C_{n,d} \), then \( M \) has to be non-arithmetic.

A conjecture of Lehmer from number theory claims that there exists a constant \( m > 1 \) such that the Mahler measure \( M(P) \) of any non-cyclotomic integral monic polynomial \( P \) satisfies \( M(P) \geq m \). (Recall that the Mahler measure of an integral monic polynomial \( P \) of degree \( d \) with roots \( \theta_1, \theta_2, \ldots, \theta_d \) is defined by \( M(P) = \prod_{i=1}^{d} \max(1, |\theta_i|) \).) Our argument shows that if Lehmer’s conjecture is true then \( \epsilon_0 \) in the statement above is an
absolute constant which does not depend on $K$ or $n$. We refer to [M, p. 322] and [G, Sect. 10] for a related discussion addressing arithmetic manifolds.

We showed that our method provides a new construction of non-arithmetic hyperbolic $n$-manifolds for every dimension $n$. It has some similarities with the interbreeding construction of Gromov and Piatetski-Shapiro [GPS] but at the same time is different from the former. Following Agol [A] we call it an inbreeding construction.

5.2. Commensurability. If $\epsilon \to 0$ then at most finitely many of the manifolds $M$ provided by Theorem 1.1 will be commensurable to each other. Indeed, assume that we have an infinite sequence of commensurable non-arithmetic manifolds $M_1 = \Lambda_1 \backslash \mathcal{H}^n$, $M_2 = \Lambda_2 \backslash \mathcal{H}^n$, $\ldots$, such that $\text{Syst}_1(M_i) = \epsilon_i \to 0$ when $i \to \infty$. By Margulis' Theorem [M, Theorem 1, p. 2], the commensurability group $\Gamma$ of $\Lambda_i$ will be a lattice in $\text{Isom}(\mathcal{H}^n)$ and hence $\Gamma \backslash \mathcal{H}^n$ is a compact hyperbolic $n$-orbifold. We have

$$\Gamma \supset \Lambda_1, \Lambda_2, \ldots,$$

so the orbifold $\Gamma \backslash \mathcal{H}^n$ has systoles of arbitrarily short length, which is impossible. Note that this argument works for any non-arithmetic manifolds with short geodesics, not only those provided by our theorem.

5.3. Non-compact manifolds. For a finite volume hyperbolic $n$-manifold which is not necessarily compact we can define the systole to be the smallest positive length of a closed geodesic on $M$. This way we consider only geodesics which correspond to the hyperbolic elements from $\pi_1(M) \hookrightarrow \text{Isom}(\mathcal{H}^n)$ and discard those corresponding to parabolics. With this definition at hand the results of the paper can be generalised to finite volume non-compact hyperbolic $n$-manifolds.

5.4. Some applications. Our non-arithmetic manifolds $M$ contain properly embedded separating totally geodesic hypersurfaces, and hence the fundamental group $\pi_1(M)$ has the structure of a free product with amalgamated subgroup similar to the fundamental groups of the Gromov-Piatetski-Shapiro manifolds [GPS]. This enables one to use our manifolds for the construction of Belolipetsky and Lubotzky [BL], which proves that every finite group can be realised as the full isometry group of some compact hyperbolic $n$-manifold. Another immediate application is to the construction of some new non-coherent lattices in $\text{Isom}(\mathcal{H}^n)$, which can be achieved by applying the argument of Kapovich-Potyagailo-Vinberg [KPV, Sect. 4] to the lattices provided by our construction.

5.5. Other locally symmetric spaces. It is natural to ask what can be said about systoles of other locally symmetric manifolds. This question pertains to the quotients of symmetric spaces $X = H/K$ by torsion-free lattices, where $H$ is now a semisimple Lie group and $K$ its maximal compact subgroup.

If all lattices in $H$ are arithmetic and Lehmer’s conjecture (or its weaker version by Margulis [M, Ch. IX, Sect. 4.21]) holds, then the systoles of the $X$-locally symmetric manifolds will be bounded below by a constant which depends only on $X$ (cf. [G, Sect. 10]). The arithmeticity of lattices is known for groups of real rank at least 2 by Margulis [M, Theorem 1, p. 2], and for $H = \text{Sp}(n,1)$ by Gromov and Schoen [GS]. Hence the only case for which one may hope to have a version of our result is when $H = \text{PU}(n,1)$ and $X$ is complex hyperbolic $n$-space.
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References


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