(5) 3 april

Calculus

Recall $d(uv) = \ldots$ and $d\left(\frac{1}{v}\right) = \ldots$

Theorem (quotient rule)

Suppose u, v depend on x. Then for any value of x where $\frac{du}{dx}$ and $\frac{dv}{dx}$ exist and $v \neq 0$,

$$\frac{d(u/v)}{dx} = \frac{v\, du/dx - u\, dv/dx}{v^2}$$

or $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

Proof. Combine product rule and $d\left(\frac{1}{v}\right)$.

Write $y = \frac{1}{v} u$. Then

$$dy = d\left(\frac{1}{v} u\right) = \frac{1}{v} du + u\, d\left(\frac{1}{v}\right)$$

$$= \frac{1}{v} du + u(-v^{-2})\, dv$$

$$= \frac{v\,du - u\,dv}{v^2}$$

Theorem (power rule for negative exponents)

Suppose u depends on x, and n neg. integer. Then for any value of x where $\frac{du}{dx}$ exists and $u \neq 0$, $d(u^n)/dx$ exists and

$$\frac{d(u^n)}{dx} = n u^{n-1} \frac{du}{dx}, \quad d(u^n) = n u^{n-1} du$$

Proof Let $n = -m$, $n > 0$

Let $y = u^n = u^{-m}$

Then $y = \frac{1}{u^m}$. Hence

$$\frac{dy}{dx} = -\frac{1}{(u^m)^2}\,\frac{d(u^m)}{dx}$$

$$= -\frac{1}{u^{2m}}\, m u^{m-1}\,\frac{du}{dx}$$

$$= (-m)\, u^{-m-1}\,\frac{du}{dx}$$

$$= n u^{n-1}\,\frac{du}{dx}.$$

Example Find dy when $y = \frac{1}{x^2 - 3x + 1}$

Let $u = x^2 - 3x + 1$, Then $y = \frac{1}{u}$

$du = (2x - 3)dx$, Hence

$$dy = -\frac{1}{u^2}\,du = \frac{-(2x-3)}{(x^2-3x+1)^2}\,dx$$

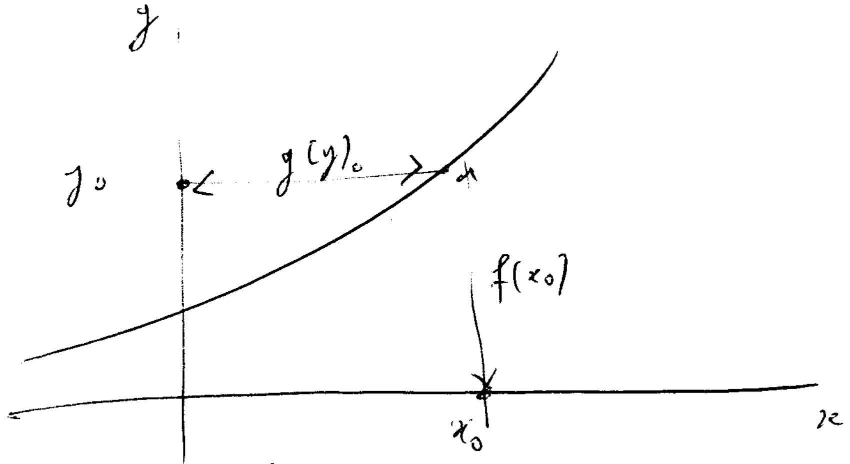
3 april

<u>Inverse functions</u>

Two functions f and g

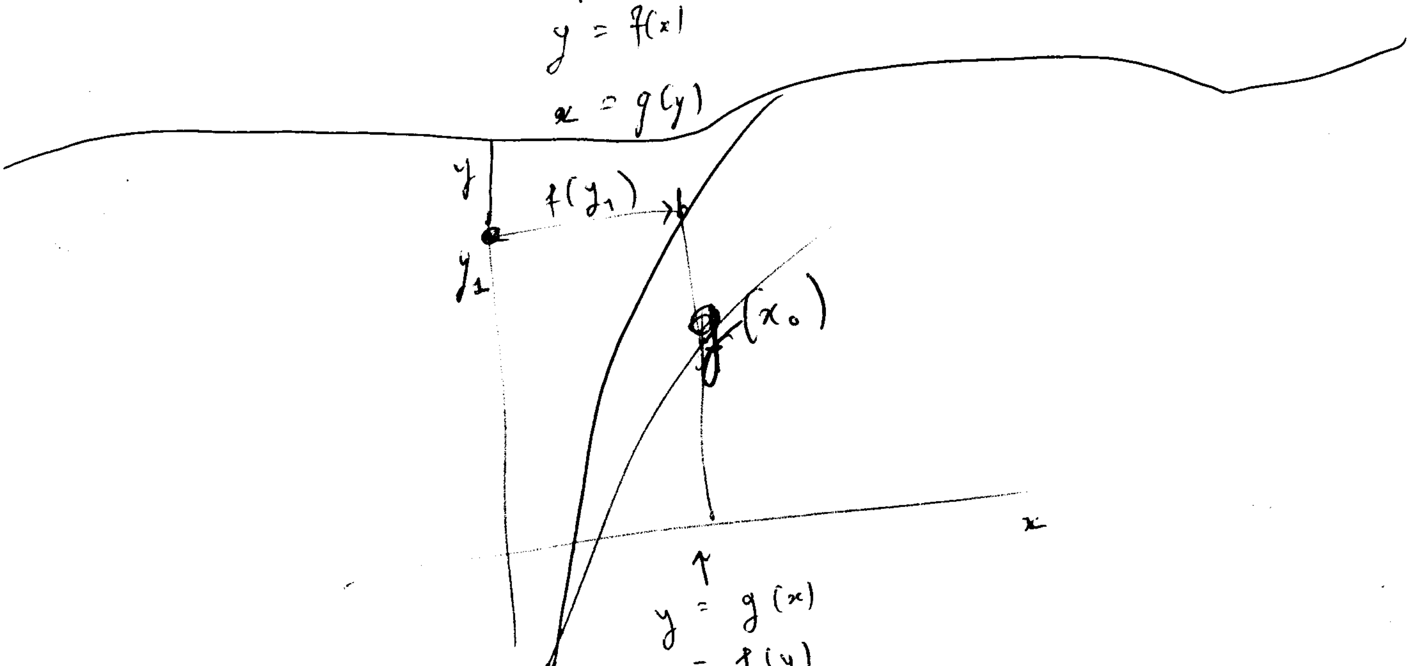
are called <u>inverse function</u> of the two equations $y = f(x)$, $x = g(y)$

Have the <u>same</u> graphs in the (x, y) - plane



$y = f(x)$

$x = g(y)$



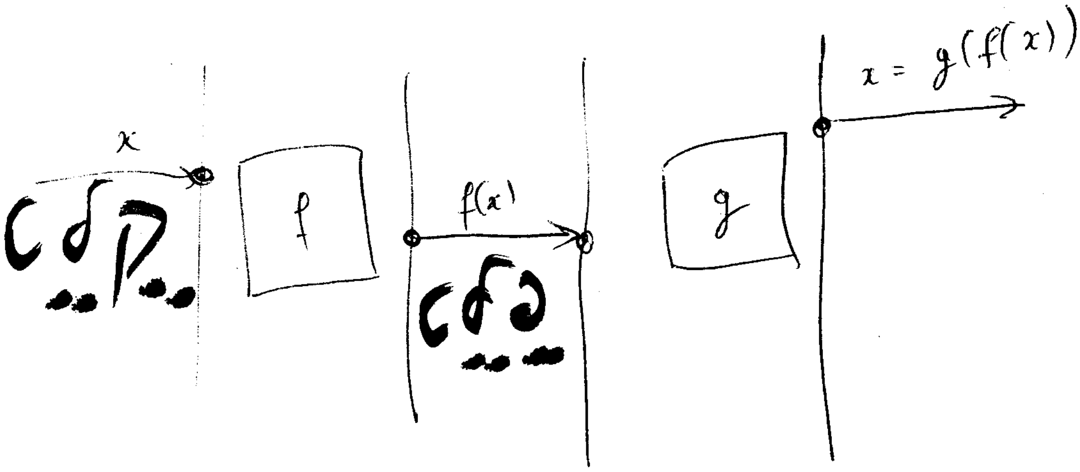
$y = g(x)$

$x = f(y)$

Example. Fun $y = x^2$, $x \geq 0$ has the inverse function $x = \sqrt{y}$.

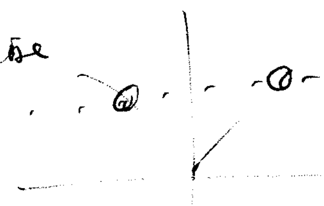
Ex. Fun $y = x^3$ has inverse fun $x = \sqrt[3]{y}$.

Black box:

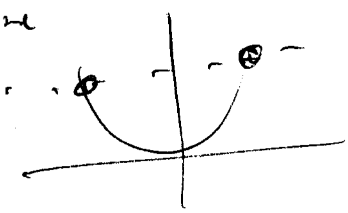


Horizontal line test: if line $y = c$ meets graph in more than one pt, $\nexists$ inverse.

Ex. $y = |x|$ has no inverse



Ex $y = x^2$ has no inverse



3 april

Table
~~Table~~ of inverses :

$\frac{dx}{dy} = \frac{1}{dy/dx}$	inverse $x = g(y)$	$\frac{dy}{dx}$	$y = f(x)$
1	$x = y - c$	1	$y = x + c$
$\frac{1}{k}$	$x = \frac{y}{k}$	k	$y = kx$
$\frac{1}{2\sqrt{y}} = \frac{1}{2x}$	$x = \sqrt{y}$	$2x$	$y = x^2,\ x \geq 0$
$-\frac{1}{2\sqrt{y}} = \frac{1}{2x}$	$x = -\sqrt{y}$	$2x$	$y = x^2,\ x \leq 0$
$-\frac{1}{y^2} = -x^2$	$x = \frac{1}{y}$	$-\frac{1}{x^2}$	$y = \frac{1}{x}$

Inverse function rule

Suppose f and g are inverse functions.

If both derivatives $f'(x)$ and $g'(y)$ are nonzero

then
$$f'(x) = \frac{1}{g'(y)} \;;$$

that is,
$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

<u>Proof</u>. Let Δx be a nonzero infinitesimal.

Let Δy be the corr. change in y.

Then Δy is also infinitesimal because $f'(x)$ exists,

and is nonzero because $\Delta y = f'(x)\,\Delta x + \underbrace{\varepsilon}_{\text{infinitesimal}}\Delta x$

By the rule of standard parts,

$$f'(x)\,g'(y) = st\left(\frac{\Delta y}{\Delta x}\right)\cdot st\left(\frac{\Delta x}{\Delta y}\right)$$

$$= st\left(\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y}\right)$$

$$= st(1)$$

$$= 1$$

Hence $f'(x) = \dfrac{1}{g'(y)}$.

↓ 12 nov

<u>Example</u>. Find $\dfrac{dy}{dx}$ when $x = 1 + y^{-3}$.

Note that $y = \dfrac{1}{\sqrt[3]{x-1}}$. Then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{\sqrt[3]{x-1}}\right)$ "hard"

Meanwhile, $\dfrac{dx}{dy} = \dfrac{d}{dy}\left(1 + y^{-3}\right) = -3y^{-4}$.

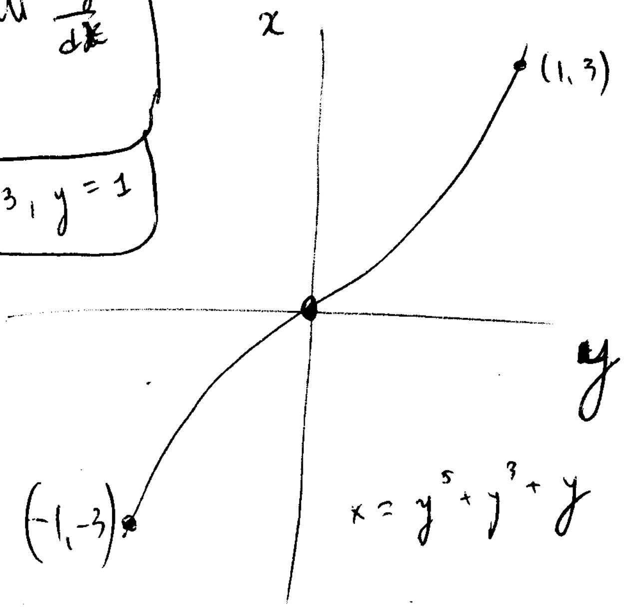
Hence $\dfrac{dy}{dx} = \dfrac{1}{-3y^{-4}} = -\dfrac{1}{3}y^{4}$

$$= -\frac{1}{3}(x-1)^{-4/3}$$

3 april

Ex. Find $\frac{dy}{dx}$ when $x = y^5 + y^3 + y$.

$\boxed{\text{Compute } \frac{dy}{dx} \text{ at } x=3,\ y=1}$



Clear that $\exists$ inverse fn $y = f(x)$ but cannot compute it

$$\frac{dx}{dy} = 5y^4 + 3y^2 + 1$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 3y^2 + 1}$$

At $(1, 3)$, $\frac{dy}{dx} = \frac{1}{5\cdot(1)^4 + 3\cdot(1)^2 + 1} = \frac{1}{9}$.

<u>Theorem</u> If n is a positive integer and $y = x^{\frac{1}{n}}$, then

$$\frac{dy}{dx} = \frac{1}{n} x^{\left(\frac{1}{n}\right) - 1}$$

Proof. ~~by~~ $x = y^n$

$$\frac{dx}{dy} = n y^{n-1}$$

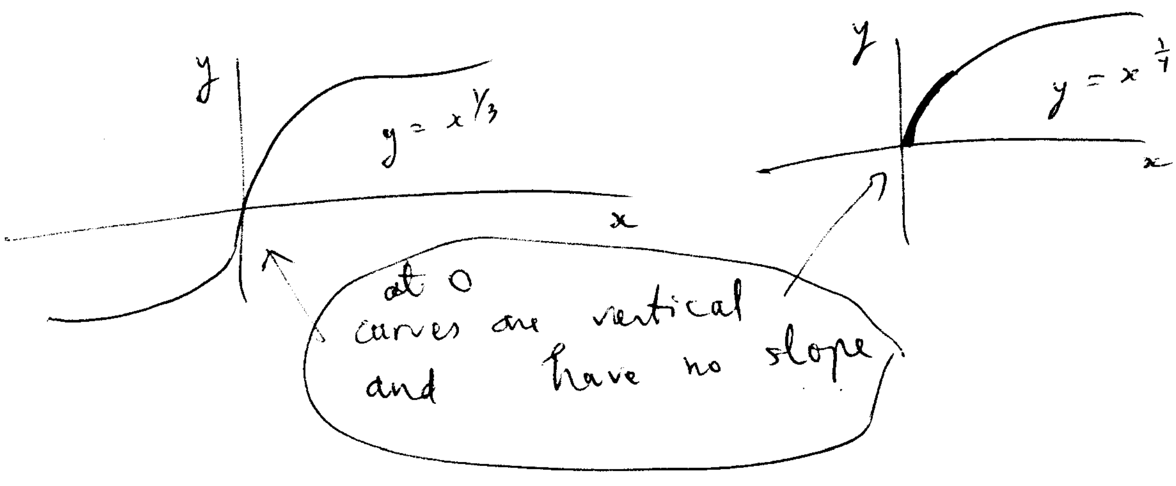
$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

$$= \frac{1}{n} y^{1-n}$$

$$= \frac{1}{n}\left(x^{\frac{1}{n}}\right)^{1-n}$$

$$= \frac{1}{n} x^{\left(\frac{1}{n}\right)-1} \qquad \square$$

Ex. graph of $y = x^{\frac{1}{3}}$, $y = x^{1/4}$



3 april

Power rule for rational exponents

Let $y = x^r$, r rational. Then

$$\frac{dy}{dx} = r\,x^{r-1}.$$

Proof Let $r = m/n$ m, n integers, $n > 0$.

Let $u = x^{\frac{1}{n}}$, $y = u^m$

Then $$\frac{du}{dx} = \frac{1}{n}\,x^{\left(\frac{1}{n}\right)-1}$$

and

$$\frac{dy}{dx} = m\,u^{m-1}\,\frac{du}{dx}$$

$$= m\left(x^{\frac{1}{n}}\right)^{m-1}\frac{1}{n}\,x^{\left(\frac{1}{n}\right)-1}$$

$$= \frac{m}{n}\,x^{\frac{m}{n}-1}$$

$$= r\,x^{r-1}$$

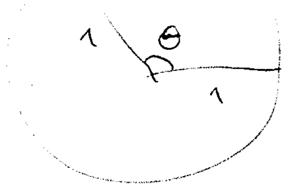
<u>Functions</u> $\underline{\sin x,\ \cos x,\ \tan x,\ e^x,\ \log x}$.

<u>Notation</u>: θ and ϕ used for <u>angles</u>.

in <u>radians</u>

360 degrees = 2π radians

θ

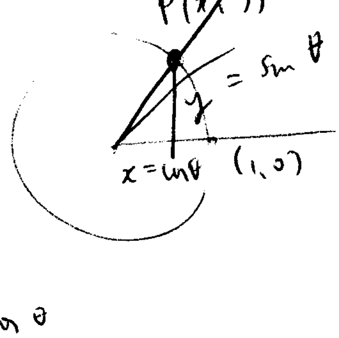


right angle 90 degrees $= \frac{\pi}{2}$ radians.

Consider unit circle $x^2 + y^2 = 1$

$P(x, y)$ point on circle.

Let θ be angle measured <u>counter clockwise</u> from $(1, 0)$ to $P(x, y)$.



$x = \cos\theta$

$y = \sin\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

בס״ד

-11-

3 april

Alternative def

$$\sin\theta = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{a}{c}$$

יתר

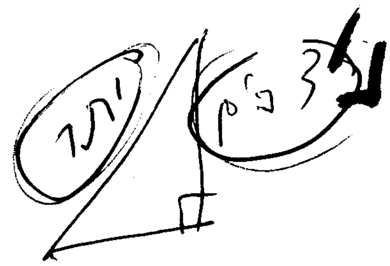
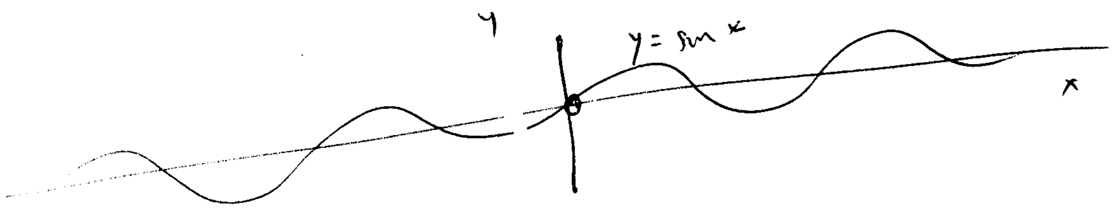


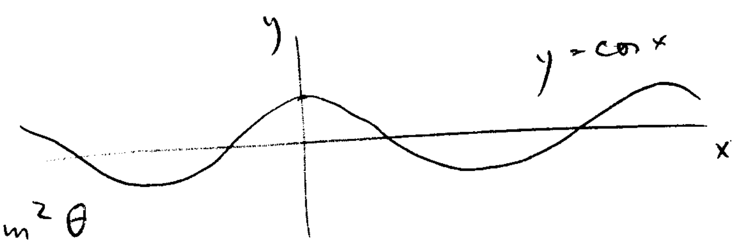
Table $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$$\sin^2\theta + \cos^2\theta = 1$$

Derivatives $$\frac{d(\sin\theta)}{dx} = \cos\theta$$

$$\frac{d(\cos\theta)}{dx} = -\sin\theta$$





$y = \sin^2\theta$

Ex. Find $\frac{dy}{d\theta}$ Let $u = \sin\theta$