(4) <u>Calculus</u>

בס״ד

13 march

Derivative $f'(x) = st\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$.

<u>Ex</u>. $f(x) = |x|$, find $f'$

<u>case 1</u> $x > 0$. Then $|x| = x$, and

$$y = x$$
$$\Delta y = \Delta x$$
$$\frac{\Delta y}{\Delta x} = 1, \quad f'(x) = 1.$$

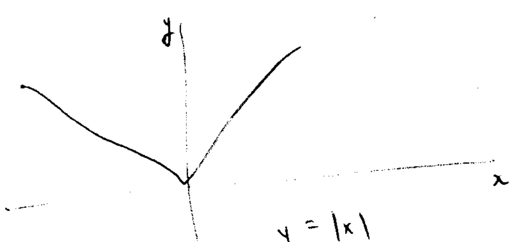
<u>Case 2</u> $x < 0$. Then $|x| = -x$. And

$$y = -x$$
$$\Delta y = -\Delta x$$
$$\frac{\Delta y}{\Delta x} = -1$$
$$f'(x) = -1$$

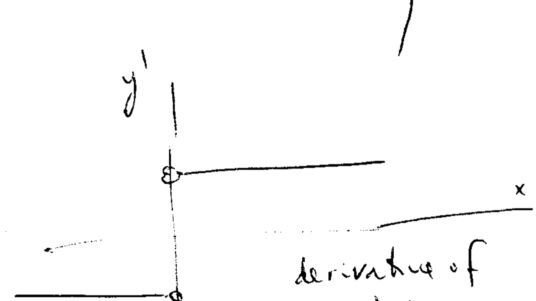
<u>Case 3</u>. $x = 0$. Then

$$y = 0$$
$$\Delta y = |\Delta x|$$
$$\frac{\Delta y}{\Delta x} = \frac{|\Delta x|}{\Delta x} = \begin{cases} +1 & \text{if } \Delta x > 0 \\ -1 & \text{if } \Delta x < 0 \end{cases}$$

Hence $f'(x)$ does not exist.




$y = |x|$




derivative of $y = |x|$

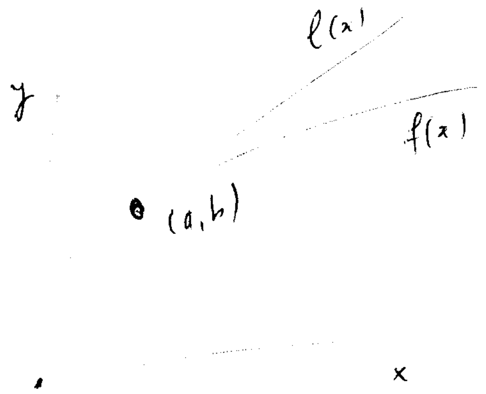
[ 29 oct '13

<u>Tangent line</u> of $y = f(x)$ at pt $(a, b)$ is given by

$$l(x) - b = f'(a)\,(x - a)$$

or

$$l(x) = f'(a)(x-a) + b$$

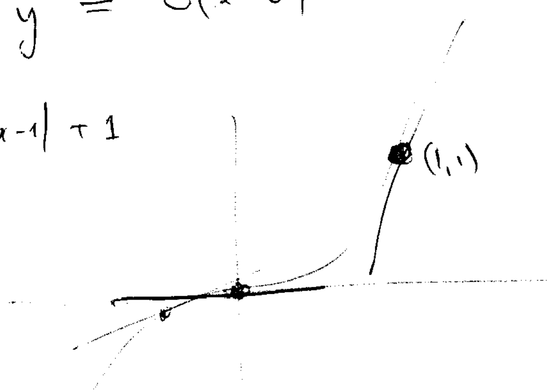



<u>Ex</u>. For curve $y = x^3$, find tangent lines at points $(0,0)$, $(1,1)$, and $\left(-\frac{1}{2}, -\frac{1}{8}\right)$

At $x = 0$, $f'(0) = 3 \cdot 0^2 = 0$. ~~t line~~

Tangent line is $y = 0(x-0) + 0 = 0$

At $x = 1$, $y = 3(x-1) + 1$




At $x = -\frac{1}{2}$, $y = \frac{3}{4}x + \frac{1}{4}$

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**Recall** $\underbrace{\Delta y}_{\text{dependent var.}} = f(x+\Delta x) - f(x)$

**Theorem** Let $y = f(x)$. Suppose $f'(x)$ exists at a certain point $x$, and $\Delta x$ is infinitesimal. Then $\Delta y$ is infinitesimal, and

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x$$

for some infinitesimal $\varepsilon$, which depends on $x$ and $\Delta x$.

**Proof** **Case 1** $\Delta x = 0$. Then $\Delta y = 0$ and we set $\varepsilon = 0$

**Case 2** $\Delta x \neq 0$. Then

$$\frac{\Delta y}{\Delta x} \approx f'(x)$$

Thus for some infinitesimal $\varepsilon$,

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon.$$

Mult by $\Delta x$, get

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x.$$

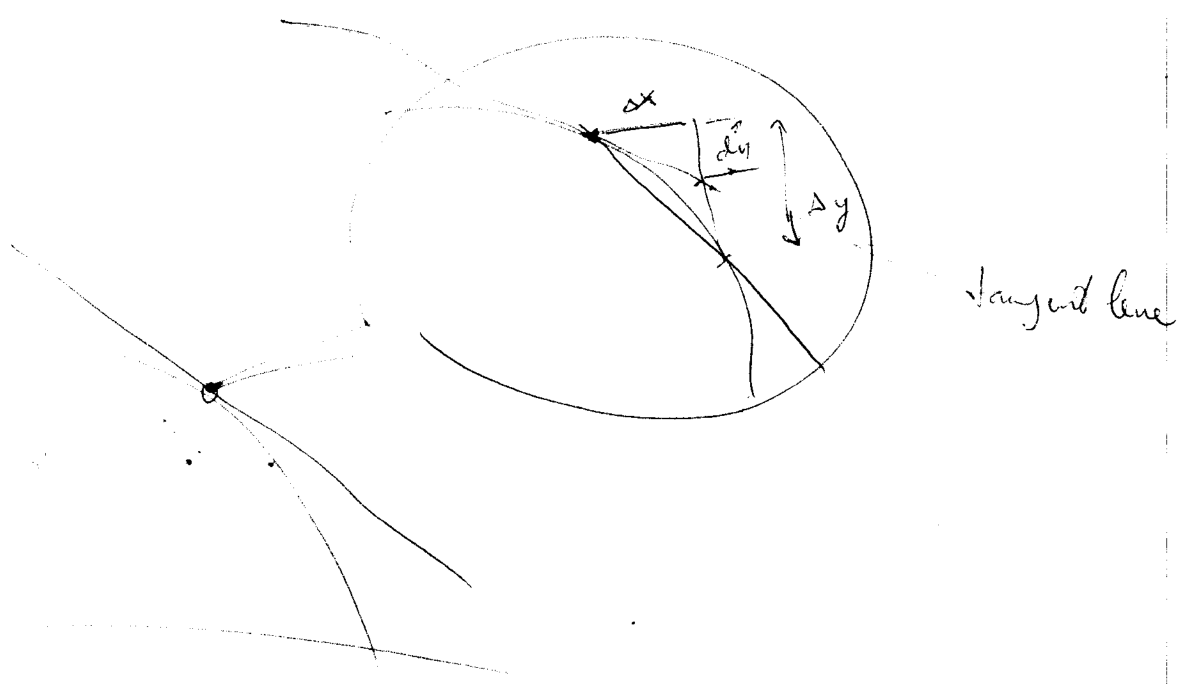
**Example** for $y = x^3$, $\Delta y = 3x^2\,\Delta x + \varepsilon\,\Delta x$,
where $\varepsilon = 3x\,\Delta x + (\Delta x)^2$.

Now introduce new dependent variable

~~Definition~~ The **differential** of $y$, denoted $dy$, is the dependent variable defined by

$$dy = f'(x)\,\Delta x.$$

Under the microscope:




To keep notation uniform, denote $\Delta x$ by $dx$

To summarize,

<u>Definition</u> ~~Sup~~ Let $y = f(x)$.

(i) the differential of $x$ is the indep var. $dx = \Delta x$

(ii) the differential of $y$ is the dep var $dy$ given by

$$dy = f'(x)\, dx.$$

When $dx \neq 0$, can write $\quad f'(x) = \dfrac{dy}{dx}$

$$\text{Compare} \quad \frac{\Delta y}{\Delta x} \approx f'(x).$$

The theorem above takes the short form $\underline{\Delta y = dy + \varepsilon dx}$

<u>Examples</u>
(a) $y = x^3$, $\quad dy = 3x^2\, dx$

(b) $y = \sqrt{x}$, $\quad dy = \dfrac{dx}{2\sqrt{x}}$ when $x > 0$.

(c) $y = \dfrac{1}{x}$ $\quad dy = -\dfrac{dx}{x^2}$ when $x \neq 0$.

(d) $y = |x|$, $\quad dy = \begin{cases} dx & x > 0 \\ -dx & x < 0 \\ \text{undefined} & x = 0 \end{cases}$

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Applying notation to $\boxed{\text{terms}}$ ביטוי ~~[illegible]~~

$\tau(x) = f(x), \quad d(\tau(x)) = f'(x)\,dx$

Ex $\quad d(x^3) = 3x^2\,dx$

$d(\sqrt{x}) = \dfrac{dx}{2\sqrt{x}}$ if $x > 0$

Recall if dependent var $y = f(x)$ then $\Delta y = f(x+\Delta x) - f(x)$ and $dy = f'(x)dx$ are new dependent variables where $dy = \Delta y + \varepsilon\Delta x$ ($\varepsilon$ infinitesimal). Similarly if we have dep. var $u = g(x)$ have $\Delta u$, $du$, etc. Draw figure!

etc

↓ 30 oct

## Derivatives of rational functions

A linear fun is $a_1 x + a_0$ where $a_i \in \mathbb{R}$.
poly deg $n$ in $x$ is a term

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \quad a_n \neq 0.$$

A rational term obtained from $x$ and real numbers using 4 arithmetic operations
פעולות

$$\frac{(3x^2 - 5)(x-2)^3}{5x - 11} \qquad \frac{\left(1 + \frac{1}{x}\right)^9}{x^3 + \frac{1}{2-x}}$$

**Theorem** The derivative of a linear fun is the coff of $x$

$$\frac{d(bx + c)}{dx} = b$$

**Proof.**
$$\Delta y = b(x + \Delta x) + c - (bx + c)$$
$$= b\Delta x$$

$$\frac{\Delta y}{\Delta x} = \frac{b\,\Delta x}{\Delta x} = b$$

$$\frac{dy}{dx} = st(b) = b$$

**Theorem** (sum rule)

Let $u$ and ~~v~~ be dep on indep var $x$. Then ~~and~~ we have for any value of $x$ where $du$ and $dv$ exist,

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad \text{or} \quad d(u+v) = du + dv$$

**Proof** Let $y = u+v$, and let $\Delta x \neq 0$ infinitesimal. Then $y + \Delta y = (u+\Delta u) + (v+\Delta v)$
Hence $\Delta y = (u+\Delta u) + (v+\Delta v) - [u+v] = \Delta u + \Delta v$. In other words, the quotients
~~Similarly~~, $\frac{\Delta y}{\Delta x} = \frac{\Delta u + \Delta v}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$. Taking standard parts obtains $st\left(\frac{\Delta y}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right) = st\left(\frac{\Delta u}{\Delta x}\right) + st\left(\frac{\Delta v}{\Delta x}\right) \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Similarly,
$$\frac{d(u_1 + \dots + u_n)}{dx} = \frac{du_1}{dx} + \dots + \frac{du_n}{dx}, \quad \text{or} \quad d(u_1 + \dots + u_n) = du_1 + \dots d$$

**Example** Let $y = \sqrt{x}$ and $z = 3x$. Find $d(y+z)$ and $d(y/z)$.

**Theorem** Suppose $u$ depends on $x$, and $c$ ~~…~~ a real num
Then for any value of $x$ where $\frac{du}{dx}$ exists,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}, \quad \text{or} \quad d(cu) = c\,du$$

**Proof** Let $y = cu$. Let $\Delta x \neq 0$ be infinitesimal. Then $y + \Delta y = c(u+\Delta u)$
and $\Delta y = c(u+\Delta u) - cu = c\Delta u$. Therefore $\frac{\Delta y}{\Delta x} = \frac{c\Delta u}{\Delta x} = c\frac{\Delta u}{\Delta x}$. Taking standard parts, $st\left(\frac{\Delta y}{\Delta x}\right) = st\left(c\frac{\Delta u}{\Delta x}\right) = c\frac{du}{dx}$ whence $\frac{dy}{dx} = c\frac{du}{dx}$. Example. $u = x^2$, $y = 7u$

**Thm** (product rule of "u,v" of SDS) Suppose $u$ and $v$ dep on $x$. Then for any value of $x$ where $\frac{du}{dx}$ and $\frac{dv}{dx}$ exist,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \quad \text{or} \quad d(uv) = u\,dv + v\,du$$

**Proof.**
$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)$$
$$= u\,st\left(\frac{\Delta v}{\Delta x}\right) + v\,st\left(\frac{\Delta u}{\Delta x}\right) + 0\cdot st\left(\frac{\Delta v}{\Delta x}\right)$$

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**Theorem** (power rule)

Let $u$ ~~depend~~ depend on $x$, and let $n$ positive integer. For any value of $x$ where $\frac{du}{dx}$ exists,

$$\frac{d(u^n)}{dx} = n u^{n-1} \frac{du}{dx}, \text{ or}$$

$$d(u^n) = n u^{n-1} du$$

**Proof.** By induction (exploiting the product rule). For $n=1$, $\frac{d(u^1)}{dx} = 1 \cdot u^0 \frac{du}{dx}$, correct.

**Assume** proved for $m$, i.e. $\frac{d(u^m)}{dx} = m u^{m-1} \frac{du}{dx}$.

Then prove it for $m+1$.

$$\frac{d(u^{m+1})}{dx} = \frac{d(u \cdot u^m)}{dx} \underset{\text{rule}}{\overset{\text{product}}{=}} u \frac{du^m}{dx} + u^m \frac{du}{dx}$$

$$= u\, m\, u^{m-1} + u^m \frac{du}{dx} = (m+1) u^m \frac{du}{dx}$$

**Corollary** Derivative of polynomial of degree $n$ is a polynomial of degree $n-1$.

**Lemma** Suppose $v$ depends on $x$. Then for any value of $x$ where $v \neq 0$ and $\frac{dv}{dx}$ exists,

$$\frac{d\left(\frac{1}{v}\right)}{dx} = -\frac{1}{v^2} \frac{dv}{dx}, \text{ or } d\left(\frac{1}{v}\right) = -\frac{1}{v^2} dv$$

Proof. Let $y = \frac{1}{v}$. Let $\Delta x \neq 0$ infinitesimal.

$$\Delta y = \frac{1}{v+\Delta v} - \frac{1}{v}$$

$$\frac{\Delta y}{\Delta x} = \frac{\frac{1}{v+\Delta v} - \frac{1}{v}}{\Delta x}$$

$$= -\frac{1}{v(v+\Delta v)} \frac{\Delta v}{\Delta x}$$

Taking standard parts,

$$st\left(\frac{\Delta y}{\Delta x}\right) = st\left(-\frac{1}{v(v+\Delta v)}\frac{\Delta v}{\Delta x}\right)$$

$$= st\left(-\frac{1}{v(v+\Delta v)}\right) st\left(\frac{\Delta v}{\Delta x}\right)$$

to here

$$= -\frac{1}{v^2}\frac{dv}{dx}$$

Theorem (quotient rule). Suppose $u, v$ depend on $x$. Then for any value of $x$ where $\frac{du}{dx}$, $\frac{dv}{dx}$ exist and $v \neq 0$,

$$\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \quad \text{or} \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

Proof. Combine product rule and lemma.

Theorem (Power rule for negative exponents). Suppose $u$ depends on $x$ and $n$ is a negative integer. Then for any value of $x$, where $\frac{du}{dx}$ exists and $u \neq 0$,

$$\frac{d(u^n)}{dx} = n u^{n-1}\frac{du}{dx}, \quad \text{or} \quad d(u^n) = n u^{n-1} du.$$

Proof. Let $n = -m$. Let $y = u^n = u^{-m}$, i.e. $y = \frac{1}{u^m}$.

By the lemma and the power $\frac{dy}{dx} = -\frac{1}{(u^m)^2}\frac{d(u^m)}{dx} = \ldots n u^{n-1}\frac{du}{dx}$

<u>Ex</u>. find $dy$ when

$$y = \frac{1}{x^2 - x + 1}$$

<u>Ex</u> let $y = \dfrac{(x^4-2)^3}{5x-1}$, find $dy$

<u>Ex</u>. Let $y = \dfrac{1}{x^3} + \dfrac{3}{x^2} + \dfrac{4}{x} + 5$.

$$dy = \left(-\frac{3}{x^4} - \frac{6}{x^3} - \frac{4}{x^2}\right) dx$$

<u>Ex</u>. Find $dy$ when $y = \left(\dfrac{1}{x^2+x} + 1\right)^2$.

use substitution $u = x^2 + x$

$$v = \frac{1}{u} + 1$$

Then $y = v^2$, etc