5 february 2007

Differential geometry 88-526
Homework 6

1. This problem deals with curves in Euclidean space.
   (a) Define a regular curve in $\mathbb{R}^3$.
   (b) Define the arclength parameter.
   (c) Consider surfaces $M_1, M_2 \subset \mathbb{R}^3$ defined by
   $$M_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 169\},$$
   and
   $$M_2 = \{(x, y, z) \in \mathbb{R}^3 | x = 5\}.$$
   Consider the intersection $C = M_1 \cap M_2$. Find an arclength parametrisation of C.
   (d) Calculate the curvature of C.

2. This problem deals with surfaces in Euclidean space.
   (a) Define a regular surface.
   (b) Consider the surface $M_3 \subset \mathbb{R}^3$ defined by
   $$M_3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 4\}.$$
   Calculate the Weingarten map of M_3.
   (c) Calculate the Gaussian curvature function $K(u^1, u^2)$ of M_3.

3. Consider the surface $M_4 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 9\}$.
   (a) Find a parametrisation of M_4.
   (b) Consider a curve $\beta(s)$ on M_4 such that the vector $\beta''(s) \in \mathbb{R}^3$ is proportional to $\beta(s)$ for every value of parameter s. In other words, the pair of vectors $(\beta(s), \beta''(s))$ is linearly dependent. Find a differential equation satisfied by the curve.

4. In coordinates $(u^1, u^2) = (x, y)$, consider the metric $\lambda(y)(dx^2 + dy^2)$, where $\lambda(y) = y^{-2}$.
   (a) Calculate the symbol Γ_{11}^1 of the metric.
   (b) Calculate the Gaussian curvature function $K(x, y)$ of the metric.

5. Consider a lattice $L \subset \mathbb{R}^2$ defined by $L = \mathbb{Z} \oplus 2\mathbb{Z}$, in other words, $L = \{(n, 2m) \in \mathbb{R}^2 | n, m \in \mathbb{Z}\}$.
   (a) Calculate the successive minima λ_i of L.
   (b) Calculate the systole $\mathrm{sys}\pi_1(\mathbb{T}_{1,2})$ of the torus $\mathbb{T}_{1,2} = \mathbb{R}^2/L$.
   (c) Define conformal equivalence of metrics.

(d) Find the smallest constant $C > 0$ such that for every torus $\mathbb{T}$ conformally equivalent to $\mathbb{T}_{1,2}$, one has $\mathrm{sys}\pi_1(\mathbb{T})^2 \leq C\,\mathrm{area}(\mathbb{T})$.