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Katz, Mikhail G. (IL-BILN); **Sherry, David** **[Sherry, David M.]** (1-NAZ-PH)
Leibniz's infinitesimals: their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond. (English summary)
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This long and polemical but important article is written as a symphony in which three leading theses on the history of infinitesimal calculus (IC) for which the authors provide evidence recur with interesting subject matter variations. Sparing the reader a discussion on variations, what follows focuses mainly on the key arguments given in support of these theses. First, the authors recall and firmly establish why and how diverse paths through history have led from the IC of the seventeenth century to its version implemented in non-standard analysis (NSA) by A. Robinson and his followers during the twentieth century (mainly Sections 8 to 10). Second, the authors show convincingly that the final version of Leibniz's theory of infinitesimals was free of logical fallacies (Sections 4 and 5), just owing to its implementation in ZFC via NSA (in fact, this shows only that this theory is as consistent as ZFC, owing to Gödel's theorem). Third, they detail how and why Berkeley, as a philosopher of mathematics, and the strength of his criticisms of Leibniz's infinitesimals have been overestimated by many historians and mathematicians until now (Sections 6 and 7), whereas these criticisms stand on shakier ground than the underestimated mathematical and philosophical resources available to Leibniz for defending his theory (it seems that the authors themselves underestimate the weight of the long time during which the publication and careful readings of most of Leibniz's writings about IC have been delayed; a historical study, intertwining in a nearly chronological order the dates of publication of these Leibniz writings with the progress of works aiming to reintroduce Leibniz's infinitesimals and methods of calculation, would no doubt be useful).

As a prelude to the defense of their theses, the authors warn against the misleading use of the terms "indivisibles" and "infinitesimals" (Section 2) that often occurs in texts about IC. Both notions were used long before Leibniz, who in 1699 mentioned the use of the latter by Mercator; it is recalled in Section 3 that Leibniz himself resorted to using both in his work. The significance of this distinction lies in the fact that as limits of division the indivisibles are one dimension less than the objects they compose, while the infinitesimals are dimensionally homogeneous with the objects they compose; the IC could not even exist in absence of this homogeneity, which is indispensable for performing on the aforesaid objects, and on the infinitesimals of which they consist, the fundamental set-theoretical operations on which the algebra of numbers which measure them is based—in a fiction for the "inassignable quantities" (i.e. deprived of any ratio expressible by a real number to whatever unity) so introduced.

Berkeley's attacks on Leibniz's IC were launched in his book entitled *The analyst* and published in 1734. In conformity with an empiricist philosophy which was current in his time and with which he agreed, Berkeley neither allowed the infinite divisibility of a continuum nor admitted non-referential concepts (Section 6). Leibniz, on the contrary, admitted such concepts, as those of infinitely small or infinitely great magnitudes which he took as mental fictions (Section 1), while he did not admit them philosophically (Robinson, among others, recognized a kindred connection between this conception and that of Hilbert's ideal notions). On such grounds, Berkeley put the blame on Leibniz's acceptance of infinite divisibility of the continuum. There is no answer in such a philosophical clash between two irreconcilable metaphysics (Section 13). Rather, the authors note that Berkeley's knowledge of mathematics was not updated, for infinite divisibility had then been commonly accepted by advanced mathematicians for nearly a century. They note also that the rejection of infinite divisibility has the absurd consequence that nearly all of traditional geometry must be abandoned, and that Berkeley's philosophical positions were lacking in consistency, for they varied with branches of mathematics.

Yet about infinitesimals Berkeley had launched another criticism, in the realm of logic. This criticism is the one which has made the greatest impression on mathematicians who learned more or less directly of Berkeley's objections to the infinitesimals. Even Robinson regarded it "as aptly demonstrating the inconsistency of reasoning with historical infinitesimal magnitudes", quote the authors, who seem to see the content of such logical objections of Berkeley as reducible to asking "how can a quantity (dx) possess a 'ghost' ($dx \neq 0$), and at the same time be 'departed' ($dx = 0$)?" The authors contend that he misunderstood Leibniz's theory of infinitesimals, and find the answer in the study of this theory.

Leibniz's system for the calculus incorporated two heuristic principles. The first mentioned in the paper, at the end of the abstract, is called the transcendental law of homogeneity (TLH) by Leibniz, and has the effect of eliminating higher-order infinitesimal terms; it is implemented in Robinson's NSA under the form which consists in applying the standard part function. The second principle is called the law of continuity (LC) by Leibniz, and appears in several forms in Leibniz's writings. The most enlightening form is extracted by the authors from a work in Latin which is designated by its two first words, *Cum prodiisset*; this work was supposedly written in 1701 and published by Gerhardt in 1846. The formulation of the LC in this text is translated by the authors as follows: "In any supposed continuous transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included." In this phrase, the word "continuous" refers to a continuous transformation (passing implicitly through an infinity of states, so that each of them may be characterized by the value of a suitable parameter) of some geometrical figure, and the word "terminus" refers to a "catastrophic" state (in Thom's sense) attained when this parameter takes an infinite value. However, this catastrophic state, the authors insist much on this point, is not a limit in today's sense, for it is inassignable when a limit would be assignable. In effect, it turns out that Leibniz designates the terminus also as the *status transitus*, the "state of transition" of the deformation, i.e. the last state until which the characteristics of the figure at the start are preserved through all states met since the start; these characteristics will be lost and replaced by those of a new figure born through the sudden transition which leads to the assignable result of the deformation, calculated by applying the TLH at this step.

Then, in order to finish clarifying the sense of the unusual terms in the statement of the LC above, the authors develop their "implementation" in NSA of three examples given by Leibniz. The most discussed example is that which presents the continuous transformation of the ellipse with apex at $(0, -1)$ and foci at the origin and at $(0, H)$ in a parabola, by removing this last focus at infinity. After having expressed analytically those geometrical data by using the definition of an ellipse in terms of the constant sum of the radii vectors of each of its points, one obtains in a well-known way the equation

$$(y + 2 + 2/H)^2 - (x^2 + y^2)(1 + 4/H + 4/H^2) = 0. \tag{4.4}$$

What the above expression "general reasoning" means is illustrated on that example where our "in a well-known way" covers habits of general reasoning such as undoing a radical by squaring, using the binomial formula, transferring terms of an equation to the other side, etc. Now, the status transitus of the deformation of the ellipse is attained when the parameter H becomes infinite, so the terms $2/H$, $4/H$ and $4/H^2$ become infinitesimals while remaining not equal to 0; nevertheless, the equation (4.4) remains valid, for "the validity of transfering such general reasoning originally instituted in the finite realm, to the realm of the infinite, is postulated by Leibniz's law of continuity", say the authors. And they add in a footnote that by extending this proposition from (fuzzy) general reasoning to the (precisely determined) first-order properties, one obtains a formulation of the transfer principle of NSA, which consists in the modern implementation of the LC; but now this principle has been proved in ZFC by Łoś and Robinson, and so it is no longer heuristic.

Now, the purpose of calculations is to end with assignable quantities. Leibniz certainly knew the current practice of the physicists which consisted in neglecting the terms, even of finite value, when their joint effect on a result of experiences is significantly smaller than the known margin of error proper to the related measures. Leibniz's TLH lifts this idea to the height of pure reason by taking only infinitesimals as negligible in order to pass to the value of the desired issue of the instantaneous transition from the status transitus into its infinitely close assignable standard part. Now, if x and y are finite, applying the TLH to (4.4) for H infinite allows one to neglect (but not to equalize to 0) the infinitesimal terms; and then, after having simplified the equation obtained, Leibniz finds the equation $y = (x^2/4) - 1$ of the parabola onto which the deformation leads; the authors think it better to take first the standard parts x_0 of x and y_0 of y and then to calculate the standard parts of both sides of (4.4) after having replaced x by x_0 and y by y_0, for eradicating the misleading impression that the equalization to 0 of the first side in the final equation is rooted in an equalization of the infinitesimals to 0: here stands the point that Berkeley misunderstood. He could not understand that the Democritian dialectic about things submitted to continual changes is explained by the infinite proximity between the instant in which a change starts and the next infinitely close instant of the issue of this change, and that it is so reintegrated in logic without inconsistency. So Leibniz can claim on one hand that (4.4) is still the equation of an ellipse with a focus at infinity when H is taken to be infinite, and on the other hand that "a parabola is not an ellipse at all" and that "it is really true" that this parabola has no focus at infinity.

This article presents a rigorous work, manifestly grounded on extensive and minutely detailed readings of many required sources; it deserves to find many readers ready to learn, question, and dispute, if the case arises, the arguments of its authors.

Marcel Guillaume