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LEIBNIZ'S LAWS OF CONTINUITY AND HOMOGENEITY

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LEIBNIZ'S LAWS OF CONTINUITY AND HOMOGENEITY

MIKHAIL G. KATZ AND DAVID M. SHERRY

Abstract. The foundations of the historical infinitesimal calculus of Newton and Leibniz have been the object of numerous criticisms. Some of the critics believed to have found logical contradictions in its foundations. We argue that Leibniz's system was free of contradictions.

Contents

1. From the *characteristica universalis* to ideal entities

Leibniz envisioned the creation of a universal language, *characteristica universalis*, ambitiously designed to serve as the vehicle of expression in all fields of knowledge. Couturat [2] pointed out in 1901 that in Leibniz's vision, the infinitesimal calculus was but the first salvo, or sample, of his *characteristica universalis*. Leibniz introduced the term "infinitesimal" in 1673; his infinitesimal calculus dates from 1675. His vision of the ideal nature of mathematical entities was remarkably modern. Leibniz's description of infinitesimals as fictional entities shocked his disciples J. Bernoulli, l'Hôpital, and Varignon. And his infinitesimals certainly shocked critics such as Berkeley, whose empiricist philosophy tolerated no theoretical entities, like infinitesimals, without an empirical counterpart or referent.

While today we are puzzled by Berkeley's rigid rejection of the idea of an infinitely divisible continuum, he also articulated a logical criticism of the calculus (see Sherry [21]), alleging that the system suffered from fatal logical flaws and even contradictions. Even today, many historians believe Berkeley's criticism to have been on target. Not even Robinson escaped this trend, praising Berkeley's criticism of the foundations of the calculus as "a brilliant exposure of their logical inconsistencies" [20, p. 280].

We will argue that, contrary to Berkeley's view, Leibniz's system of the calculus was robust, free of contradiction, and incorporated versatile heuristic principles such as his law of continuity and laws of homogeneity, which were amenable, in the ripeness of time, to mathematical implementation as general principles governing the manipulation of modern infinitesimal and infinitely large quantities. We

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will draw on Leibniz's work, as well as that of Leibniz historians Bos, Horváth, Laugwitz, Ferraro, and Guicciardini so as to clarify the nature and consistency of Leibniz's theory.

In a seminal 1974 study of Leibnizian methodology, Bos described a pair of distinct approaches to justifying the calculus:

> Leibniz considered two different approaches to the foundations of the calculus; one connected with the classical methods of proof by "exhaustion", the other in connection with a law of continuity [1, item 4.2, p. 55].

The first approach relies on an Archimedean "exhaustion" methodology. We will therefore refer to it as the A-methodology. The second methodology relies more directly on infinitesimals and the law of continuity. We will refer to it as the B-methodology, in an allusion to Johannes Bernoulli, who, having learned an infinitesimal methodology from Leibniz, never wavered from it.

One of the cornerstones of the B-methodology, Leibniz's *law of continuity*, was clarified by Knobloch in the following terms:

> In his treatise Leibniz used a dozen rules which constitute his arithmetic of the infinite. He just applied them without demonstrating them, only relying on the *law of continuity*: The rules of the finite remains valid in the domain of the infinite. [10, p. 67].

As Laugwitz perceptively pointed out, Leibniz's law of continuity

> contains an *a priori* assumption: our mathematical universe of discourse contains both finite objects and infinite ones [11, p. 145].

What is the ontological status of such infinitary (infinitesimal or infinite) objects in Leibniz's theory? Leibniz's remarkably modern insight that mathematical entities need not have a *referent*, or empirical counterpart, was clarified by Guicciardini in the following terms:

> The calculus, according to Leibniz, should also be seen as an *ars inveniendi* (an art of discovery): as such it should be valued by its fruitfulness, rather than by its referential content. We can calculate, according to Leibniz, with symbols devoid of referential content (for instance, with $\sqrt{-1}$), provided the calculus is structured in such a way as to lead to correct results [6, p. 1737].

The nature of Leibniz's infinitesimals is further clarified by Ferraro, who analyzes a lengthy quotation from Leibniz's famous letter [16] to Varignon of 1702 (which we do not reproduce here to save space), in the following terms:

> According to Leibniz, imaginary numbers, infinite numbers, infinitesimals, the powers whose exponents were not "ordinary" numbers and other mathematical notions are not mere inventions; they are auxiliary and ideal quantities that [...] serve to shorten the path of thought (Ferraro [3, p. 35]).

On Ferraro's view, Leibniz's infinitesimals enjoy an *ideal* ontological status similar to that of the complex numbers, *surd* (irrational) exponents, and other ideal quantities. One significant addition to this list is such a simple geometric figure as a *circle*, which is similarly described by Leibniz as a fictional entity, see (Leibniz [12], [13]).

2. Assignable and inassignable quantities

How did Leibniz view the relation of assignable and inassignable quantities? The rule governing infinitesimal calculation that Knobloch represents as Leibniz's rule 2.2, states:

> 2.2 x, y finite, $x = (y+$ infinitely small$) \iff x - y \approx 0$ (not assignable difference) (Knobloch [10, p. 67]).

Here the pair of parallel wavy lines represents the relation of being infinitely close. Leibnizian assignable quantities mark locations in what would be called today an Archimedean continuum, or A-continuum for short. Such a continuum stems from the 16th century work of Simon Stevin [22], [23]. Simon Stevin (1548-1620) initiated a systematic approach to decimal representation of measuring numbers, marking a transition from a discrete arithmetic as practiced by the Greeks, to the arithmetic of the continuum taken for granted today, see Malet [18] and Naets [19].

Closely related to the distinction between the A- and B-methodologies is a distinction between two types of continua, which could be called an A-continuum and a B-continuum. The latter encompasses inassignable entities such as infinitesimals (in addition to assignable ones), and can be described as a "thick" continuum. On occasion, Leibniz describes such entities as "incomparable quantities", and defines them in terms of the violation of what today is called the Archimedean property. Thus, Leibniz writes in a letter to l'Hôpital:

> I call incomparable quantities of which the one can not become larger than the other if multiplied by any finite number. This conception is in accordance with the fifth definition of the fifth book of Euclid (Leibniz [14, p. 288], cited in Horváth [8, p. 63]).[1]

To mediate between assignable and inassignable quantities, Leibniz developed an additional principle called the *law of homogeneity*. Leibniz's law of homogeneity governs equations involving differentials, as follows:

> A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,
>
> $$a + dx = a \tag{2.1}$$
>
> $$dx + ddy = dx$$
>
> etc. The resulting equations satisfy this [...] requirement of homogeneity (cited in Bos [1, p. 33]).

How did Leibniz use the law of homogeneity in developing the calculus? In Section 3, we will illustrate an application of the law of homogeneity in the particular example of the derivation of the product rule.

3. Justification of the product rule

The issue is the justification of the last step in the following calculation:

$$\begin{aligned} d(uv) &= (u + du)(v + dv) - uv = udv + vdu + du\,dv \\ &= udv + vdu. \end{aligned} \tag{3.1}$$

[1] Horváth notes that Leibniz is actually referring to the *fourth* definition of the fifth book.

The last step in the calculation (3.1), namely

$$udv + vdu + du\,dv = udv + vdu$$

is an application of Leibniz's law of homogeneity.[2]

In his 1701 text *Cum Prodiisset* [15, p. 46-47], Leibniz presents an alternative justification of the product rule (see Bos [1, p. 58]). Here he divides by dx and argues with differential quotients rather than differentials. Adjusting Leibniz's notation to fit with (3.1), we obtain an equivalent calculation[3]

$$\begin{aligned} \frac{d(uv)}{dx} &= \frac{(u+du)(v+dv)-uv}{dx} \\ &= \frac{udv + vdu + du\,dv}{dx} \\ &= \frac{udv + vdu}{dx} + \frac{du\,dv}{dx} \\ &= \frac{udv + vdu}{dx}. \end{aligned}$$

Under suitable conditions the term $\frac{du\,dv}{dx}$ is infinitesimal, and therefore the last step

$$\frac{udv + vdu}{dx} + \frac{du\,dv}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

is legitimized as a special case of the law of homogeneity. We see that the law of homogeneity interprets the equality sign in (2.1) as the relation of being infinitely close, i.e., an equality up to infinitesimal error.

4. Was Leibniz's system consistent?

Berkeley's *logical criticism* of the calculus is that the evanescent increment is first assumed to be non-zero to set up an algebraic expression, and then *treated as zero* in *discarding* the terms that contained that increment when the increment is said to vanish. The rebuttal of the criticism is that the evanescent increment need *not* be "treated as zero", but, rather, are merely *discarded* through an application of the law of homogeneity by Leibniz, as illustrated in the previous section in the case of the product rule.

While consistent, Leibniz's system unquestionably relied on heuristic principles such as the laws of continuity and homogeneity, and thus fell short of a standard of rigor if measured by today's criteria. On the other hand, the consistency of Leibniz's system is confirmed through the development of modern implementations of Leibniz's heuristic principles. Thus, in the 1940s, Hewitt [7] developed a modern implementation of a B-continuum extending $\mathbb{R}$, by means of a technique referred to today as the ultrapower construction. We will denote such a B-continuum by the new symbol IIR ("thick-R"). In the next decade, Łoś [17] proved his celebrated theorem on ultraproducts, implying in particular that elementary statements over $\mathbb{R}$ are true if and only if they are true over IIR, yielding a modern implementation of the Leibnizian law of *continuity*. Such a result is equivalent to what is known in

[2] Leibniz had two laws of homogeneity, one for dimension and the other for the order of infinitesimalness. Bos notes that they 'disappeared from later developments' [1, p. 35], referring to Euler and Lagrange.

[3] The special case treated by Leibniz is $u(x) = x$. This limitation does not affect the conceptual structure of the argument.

the literature as the *transfer principle*, see Keisler [9]. The map that associates to every finite element of $\mathrm{I\!I\!R}$, the real number infinitely close to it, is known in the literature as the *standard part function* (alternatively, the *shadow*). Such a map is a mathematical implementation of the Leibnizian law of *homogeneity*.

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Sir,

The article “Leibniz’s laws of continuity and homogeneity” attached is hereby submitted for publication in the American Mathematical Monthly.

Sincerely,

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