

DIFFERENTIAL GEOMETRY 88-526–FINAL EXAM–MOED B

24 february '09

PLEASE JUSTIFY ALL ANSWERS

Duration of exam: 2.5 hours (150 minutes)

1. This problem deals with curves in Euclidean space.
 - (a) Define the arclength parameter s of a curve;
 - (b) Consider the curve $\alpha(t) = (4 \cos t, 5 - 5 \sin t, -3 \cos t)$. Find an arclength parametrisation s of the curve;
 - (c) Calculate the curvature $k(s)$ of the curve in part (b).
2. This problem deals with surfaces in Euclidean space. Throughout this problem, assume that $g_{ij} = L_{ij} = 0$ if $i \neq j$.
 - (a) Define principal curvatures κ_1 and κ_2 ;
 - (b) Choose a suitable basis and express the Weingarten map of M as a matrix;
 - (c) Express the ratio κ_1/κ_2 as a function of the coefficients of the first and second fundamental forms;
 - (d) Calculate the ratio κ_1/κ_2 for the surface of revolution obtained by rotating the parabola $x = z^2 + \frac{1}{4}$ around the z -axis.

3. In coordinates $(u^1, u^2) = (x, y)$, let $f(x, y) = \frac{2}{y}$, and consider the metric

$$f(x, y)^2(dx^2 + dy^2).$$

- (a) Calculate the symbol Γ_{11}^1 of the metric;
 - (b) Calculate the Gaussian curvature function $K = K(x, y)$ of the metric.
4. This problem deals with surfaces.
 - (a) Derive an expression for Γ_{ij}^k in terms of the metric coefficients g_{ij} ;
 - (b) Prove that the expression $\frac{\partial}{\partial u^k} (\Gamma_{ij}^\ell x_\ell + L_{ij} n)$ is symmetric in j and k ;
Explain the relation between L_{ij} and $L_{\ell i}^k$;
 - (c) Express $L_{ij} L_{\ell i}^k$ in terms of the metric coefficients.

5. Consider the torus T_0^2 in \mathbb{R}^3 parametrized by

$$x(\theta, \phi) = ((5 + \cos \phi) \cos \theta, (5 + \cos \phi) \sin \theta, \sin \phi).$$

- (a) Define the stable norm $\| \cdot \|$ in 1-dimensional homology $H_1(T^2; \mathbb{Z})$ of a torus T^2 ;
- (b) Calculate all the successive minima λ_1 and λ_2 of $H_1(T_0^2; \mathbb{Z})$ with respect to the norm $\| \cdot \|$;
- (c) Express the conformal parameter $\tau_0 = \tau(T_0^2)$ in terms of an integral;
- (d) Does every Riemannian torus T^2 conformally equivalent to T_0^2 necessarily satisfy the inequality

$$\text{sys}_1(\mathbb{T})^2 \leq \text{area}(\mathbb{T})?$$

and provide an explanation.

GOOD LUCK!