Numerical Methods - Targil 12
Furrier Transform

1 Furrier series

1.1 Orthogonality

Let's look at these three equations:

$$\int_0^l \sin\left(\frac{2\pi n}{l}x\right)\cos\left(\frac{2\pi n}{l}x\right)dx = 0 \qquad \forall n$$
$$\int_0^l \sin\left(\frac{2\pi n}{l}x\right)\sin\left(\frac{2\pi k}{l}x\right)dx = 0 \qquad n \neq k$$
$$\int_0^l \cos\left(\frac{2\pi n}{l}x\right)\cos\left(\frac{2\pi k}{l}x\right)dx = 0 \qquad n \neq k$$

From these equations we can easily say that $\sin\left(\frac{2\pi n}{l}x\right)$ for different n's are orthogonal to each other. The same goes for $\cos\left(\frac{2\pi n}{l}x\right)$. And also $\sin\left(\frac{2\pi n}{l}x\right)$ are orthogonal to $\cos\left(\frac{2\pi n}{l}x\right)$.

1.2 Even and odd functions

The next function is even:

$$g(x) = \frac{f(x) + f(-x)}{2}$$

proof:

$$g(-x) = \frac{f(-x) + f(x)}{2} = g(x)$$

The next function is odd:

$$g(x) = \frac{f(x) - f(-x)}{2}$$

proof:

$$g(-x) = \frac{f(-x) - f(x)}{2} = -g(x)$$

Every function $f(x)$ can be written as a sum of an even function and an odd function:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

The proof of that is very easy, just calculate the R.H.S. and you'll get the L.H.S.

1.3 Furrier

According to Furrier sentence we know the following:

1. Every even and ciclyc function can be represented as a sum of Cosines.
2. Every odd and ciclyc function can be represented as a sum of Sines.

A ciclyc function can be defined this way:

$$f(x) = f(x + \tau) \qquad \tau \to \infty$$

So, for a ciclyc $f(x)$, instead of writing:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

we will write:

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{2\pi n}{l} x\right) + b_n \cos\left(\frac{2\pi n}{l} x\right)\right) \tag{1}$$

Now we would like to calculate the coefficients a_n and b_n. For that purpose, let's look at the integral

$$\frac{2}{l} \int_0^l f(x) \sin\left(\frac{2\pi n}{l} x\right) dx$$

Now use (1) and get

$$\frac{2}{l} \int_0^l f(x) \sin\left(\frac{2\pi n}{l} x\right) dx = \frac{2}{l} \int_0^l \sum_{n=0}^{\infty} \left(a_n \sin\left(\frac{2\pi n}{l} x\right) + b_n \cos\left(\frac{2\pi n}{l} x\right)\right) \sin\left(\frac{2\pi n}{l} x\right) dx$$

Because of the orthogonality, all that's left from the integral is

$$\frac{2}{l} \int_0^l a_n \sin^2\left(\frac{2\pi n}{l} x\right) dx = \frac{2}{l} a_n \int_0^l \sin^2\left(\frac{2\pi n}{l} x\right) dx \tag{2}$$

Now let's calculate the integral from (2), by defining

$$t = \frac{2\pi n x}{l} \qquad dt = \frac{2\pi n}{l} dx$$

we get

$$\begin{aligned}
\int_0^l \sin^2\left(\frac{2\pi n}{l}x\right) dx &= \int_0^{2\pi n} \frac{l}{2\pi n} \sin^2(t) dt \\
&= \frac{l}{2\pi n} \int_0^{2\pi n} \left(\frac{1}{2} - \frac{\cos(2t)}{2}\right) dt \\
&= \frac{l}{2\pi n} \left[\frac{1}{2}t - \frac{\sin(2t)}{4}\right]_0^{2\pi n} \\
&= \frac{l}{2\pi n}[\pi n - 0 - (0 - 0)] \\
&= \frac{l}{2}
\end{aligned}$$

and now (2) equals to

$$\frac{2}{l} a_n \frac{l}{2} = a_n$$

and the conclusion is:

$$a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{2\pi n}{l}x\right) dx \tag{3}$$

and similarly

$$b_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{2\pi n}{l}x\right) dx \tag{4}$$

and the series was introduced before in (1).

1.4 The unit circle

Each point on the unit circle can be represented by the pair (a, b) on the complex axis $a + bi$, such that if α is the angle between the radius and the x axis, we'll get

$$\begin{aligned}
\cos x = \frac{a}{1} \quad & \sin x = \tfrac{b}{1} \\
a = \cos x \quad & b = \sin x
\end{aligned}$$

So, instead $a+bi$ we'll write $\cos x+i\sin x$. This is a trigonometric identity

$$\begin{aligned}
e^{ix} &= \cos x + i\sin x \\
e^{-ix} &= \cos(-x) + i\sin(-x) \\
&\Downarrow \\
e^{-ix} &= \cos x - i\sin x \\
&\Downarrow \\
\frac{e^{ix}+e^{-ix}}{2} &= \cos x \\
\frac{e^{ix}-e^{-ix}}{2i} &= \sin x
\end{aligned}$$

The last two lines are called 'Euler equations' and they are very usefull. From this day on, your life is changed (for good). No more remembering by heart trigonometric identity's. From these two equations you can develop any identity you want. For example, let's see the derivative of $\sin x$

$$(\sin x)' = \left(\frac{e^{ix}-e^{ix}}{2i}\right)' = \frac{ie^{ix}+ie^{-ix}}{2i} = \frac{e^{ix}+e^{-ix}}{2} = \cos x$$

2 Furrier Transform

Let f be a ciclyc function:

$$f(x) = f(x+\tau) \qquad \tau \to \infty$$

As seen before, a ciclyc function can also be writen as (converged) Furrier series:

$$f(x) = \sum_{n=0}^{\infty} a_n \sin\frac{2\pi n}{l}x + \sum_{n=0}^{\infty} b_n \cos\frac{2\pi n}{l}x$$

when l is the wave lengh, n is an integer.
Now we will define continuous $k = \frac{n}{l}$. Using this continuous k, the Furrier transform can be derived, when instead of fixed coefficients a_n, b_n we have continuous $a(k), b(k)$, and instead of the $\sum$ we have $\int_{k=0}^{\infty}$:

$$f(x) = \int_0^{\infty} a(k)\sin(2\pi kx) + b(k)\cos(2\pi kx)dk$$

Now, using Euler equations:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

we get:

$$f(x) = \int_0^\infty a(k)\frac{e^{i2\pi kx} - e^{-i2\pi kx}}{2i} + b(k)\frac{e^{i2\pi kx} + e^{-i2\pi kx}}{2}dk$$

because $\frac{1}{i} = -i$

$$f(x) = \int_0^\infty \frac{b(k) - ia(k)}{2}e^{i2\pi kx} + \frac{b(k) + ia(k)}{2}e^{-i2\pi kx}dk$$

Now let us define:

$$\begin{aligned} c(k) &= \frac{b(k) - ia(k)}{2} \\ c(-k) &:= c^*(k) = \frac{b(k) + ia(k)}{2} \end{aligned}$$

and now

$$f(x) = \int_0^\infty c(k)e^{i2\pi kx} + c(-k)e^{-i2\pi kx}dk$$

and we get the two fundamental equations of the Furrier transform:

$$f(x) = \int_{-\infty}^{\infty} c(k)e^{i2\pi kx}dk \tag{5}$$

$$c(k) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi kx}dx \tag{6}$$

Note that the lower limit of the integral is now $-\infty$.
By discretization of (5) we'll continue to the next section and develop the DFT. In a very similar way, we can develop (6) and get the inverse DFT.

3 DFT - Discrete Furrier Transform

Now let's suppose we have discrete x data, equally sparsed, when N is the total number of x values, and Δ is the wave lengh. We define:

$$x_n = \frac{n\Delta}{N} \tag{7}$$

$$k_z = z\frac{1}{\Delta} \tag{8}$$

both n and z are indexes. Discrete x is leading to discrete k.

Now, placing the discrete x_n, k_z in the continuous function (1) yields:

$$f(x_n) = \Delta \sum c(k_z) e^{i2\pi k_z x_n}$$

using (3) and (4) we obtain:

$$\begin{aligned} f(x_n) &= \Delta \sum_{z=0}^{N-1} c(k_z) e^{i2\pi k_z x_n} \\ &= \Delta \sum_{z=0}^{N-1} c(k_z) e^{\frac{i2\pi z \frac{1}{\Delta} n\Delta}{N}} \\ &= \Delta \sum_{z=0}^{N-1} c(k_z) e^{\frac{i2\pi z n}{N}} \\ &= \Delta \sum_{z=0}^{N-1} \vec{\vec{X}}_{nz} c(k_z) \end{aligned}$$

when

$$\sum_{z=0}^{N-1} e^{\frac{i2\pi z n}{N}} = \vec{\vec{X}}_{nz}$$

is a simmetric matrix. The assumption is that c becomes very small very fast. $f(x_n)$ and $c(k_z)$ are both vectors, so at the end we get a simple problem of matrix $\cdot$ vector multiplication:

$$f(\vec{x}_n) = \vec{\vec{X}}_{nz} \cdot c(\vec{k}_z)$$

which is a problem of $O(n^2)$ complexity.

The inverse DFT is

$$c(\vec{k}_z) = \vec{\vec{X}}_{nz} \cdot f(\vec{x}_n)$$

$$\sum_{z=0}^{N-1} e^{-\frac{i2\pi z n}{N}} = \vec{\vec{X}}_{nz}$$

Example: Given a sample of the function c:

$$c(k_0) = 1 \quad c(k_1) = 7 \quad c(k_2) = 3 \quad c(k_3) = 5$$

Find values of the function f at points $x_0, \ldots, x_3$ (find $f(x_i) \quad i = 0, 1, 2, 3$).

Solution: First we will calculate the matrice $\vec{\vec{X}}_{nz}$.

$$\begin{aligned}
X_{0z} &= e^{\frac{i2\pi 0z}{4}} && = 1 \qquad z = 0,1,2,3 \\
X_{n0} &= e^{\frac{i2\pi 0n}{4}} && = 1 \qquad n = 0,1,2,3 \\
X_{11} &= e^{\frac{i2\pi}{4}} && = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\
X_{12} &= e^{\frac{i2\pi 2}{4}} && = \cos\pi + i\sin\pi = -1 \\
X_{13} &= e^{\frac{i2\pi 3}{4}} && = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i \\
X_{22} &= e^{\frac{i2\pi 4}{4}} && = \cos(2\pi) + i\sin(2\pi) = 1 \\
X_{23} &= e^{\frac{i2\pi 6}{4}} && = \cos\pi + i\sin\pi = -1 \\
X_{33} &= e^{\frac{i2\pi 9}{4}} && = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i
\end{aligned}$$

All the other X_{ij}'s are known from the simetry of the matrice, and so

$$X_{nz} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Now we'll find f:

$$\begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 16 \\ -2+2i \\ -8 \\ -2-2i \end{pmatrix}$$

4 FFT - Fast Furrier Transform

Also called Radix-2-FFT. FFT is a method of computing the above DFT, but with $O(n \log n)$ complexity, which is a big improvement.
Let us continue developing the above equation, for our convenience, mark $c(k_z)$ as c_z.

$$f(x_n) \;=\; \Delta \sum_{z=0}^{N-1} e^{\frac{i2\pi zn}{N}} c_z$$

$$
\begin{aligned}
&= \Delta \left\{ \sum_{\text{even z}} e^{\frac{i2\pi zn}{N}} c_z + \sum_{\text{odd z}} e^{\frac{i2\pi zn}{N}} c_z \right\} \\
&= \Delta \left\{ \sum e^{\frac{i2\pi(2z_1)n}{N}} c_{z_1} + \sum e^{\frac{i2\pi(2z_1+1)n}{N}} c_{z_1+1} \right\} \\
&= \Delta \left\{ \underbrace{\sum e^{\frac{i2\pi z_1 n}{\frac{N}{2}}} c_{z_1}}_{D.F.T of even} + \underbrace{e^{\frac{i2\pi xn}{N}}}_{constant} \underbrace{\sum e^{\frac{i2\pi 2z_1 n}{\frac{N}{2}}} c_{z_1+1}}_{D.F.T. of odd} \right\}
\end{aligned}
$$

Now we have splitted the problem to two sub-problems, two D.F.T.. If the number of the x data is a power of 2, then we can continue splitting each sub-problem recursively, and that way reduce the complexity to $O(n \log n)$.