Common families of orthogonal polynomials.

Legendre Polynomials, $P_n(x)$

Interval: $[-1,1]$ Weight function: 1

Three-term recursion: $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

Orthogonality relations:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \ , \qquad n \neq m$$

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}$$

First few:

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= x \\
P_2 &= \frac{3}{2}x^2 - \frac{1}{2} \\
P_3 &= \frac{5}{2}x^3 - \frac{3}{2}x \\
P_4 &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \\
P_5 &= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x \\
P_6 &= \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16} \\
P_7 &= \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x \\
P_8 &= \frac{6435}{128}x^8 - \frac{3003}{32}x^6 + \frac{3465}{64}x^4 - \frac{315}{32}x^2 + \frac{35}{128}
\end{aligned}$$

Note: $P_n(x)$ even if n even, odd if n odd.

Tchebyshev Polynomials, $T_n(x)$

Interval: $[-1,1]$ Weight function: $(1-x^2)^{-\frac{1}{2}}$

Three-term recursion: $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$

Orthogonality relations:

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{(1-x^2)^{\frac{1}{2}}}\,dx \qquad n \neq m$$

$$\int_{-1}^{1} \frac{(T_n(x))^2}{(1-x^2)^{\frac{1}{2}}}\,dx = \begin{cases} \pi & n=0 \\ \frac{\pi}{2} & n>0 \end{cases}$$

First few:

$$\begin{aligned}
T_0 &= 1 \\
T_1 &= x \\
T_2 &= 2x^2 - 1 \\
T_3 &= 4x^3 - 3x \\
T_4 &= 8x^4 - 8x^2 + 1 \\
T_5 &= 16x^5 - 20x^3 + 5x \\
T_6 &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7 &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8 &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{aligned}$$

Note: $T_n(x)$ even if n even, odd if n odd.

Laguerre Polynomials, $L_n(x)$

Interval: $[0,\infty)$ Weight function: e^{-x}

Three-term recursion: $L_{n+1}(x) - (2n+1-x)L_n(x) + n^2 L_{n-1}(x) = 0$

Orthogonality relations:

$$\int_0^\infty L_n(x)L_m(x)e^{-x}\ dx = 0 \qquad n \neq m$$

$$\int_0^\infty (L_n(x))^2 e^{-x}\ dx = (n!)^2$$

First few:

$$\begin{aligned}
L_0 &= 1\\
L_1 &= 1 - x\\
L_2 &= x^2 - 4\,x + 2\\
L_3 &= -x^3 + 9\,x^2 - 18\,x + 6\\
L_4 &= x^4 - 16\,x^3 + 72\,x^2 - 96\,x + 24\\
L_5 &= -x^5 + 25\,x^4 - 200\,x^3 + 600\,x^2 - 600\,x + 120\\
L_6 &= x^6 - 36\,x^5 + 450\,x^4 - 2400\,x^3 + 5400\,x^2 - 4320\,x + 720\\
L_7 &= -x^7 + 49\,x^6 - 882\,x^5 + 7350\,x^4 - 29400\,x^3 + 52920\,x^2 - 35280\,x + 5040\\
L_8 &= x^8 - 64\,x^7 + 1568\,x^6 - 18816\,x^5 + 117600\,x^4 - 376320\,x^3 + 564480\,x^2 - 322560\,x + 40320
\end{aligned}$$

Hermite Polynomials, $H_n(x)$

Interval: $(-\infty,\infty)$ Weight function: e^{-x^2}

Three-term recursion: $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$

Orthogonality relations:

$$\int_{-\infty}^\infty H_n(x)H_m(x)e^{-x^2}\ dx = 0 \qquad n \neq m$$

$$\int_{-\infty}^\infty (H_n(x))^2 e^{-x^2}\ dx = 2^n n!\sqrt{\pi}$$

First few:

$$\begin{aligned}
H_0 &= 1\\
H_1 &= 2\,x\\
H_2 &= 4\,x^2 - 2\\
H_3 &= 8\,x^3 - 12\,x\\
H_4 &= 16\,x^4 - 48\,x^2 + 12\\
H_5 &= 32\,x^5 - 160\,x^3 + 120\,x\\
H_6 &= 64\,x^6 - 480\,x^4 + 720\,x^2 - 120\\
H_7 &= 128\,x^7 - 1344\,x^5 + 3360\,x^3 - 1680\,x\\
H_8 &= 256\,x^8 - 3584\,x^6 + 13440\,x^4 - 13440\,x^2 + 1680
\end{aligned}$$

Note: $H_n(x)$ even if n even, odd if n odd.