

# On the Robustness of Preference Aggregation in Noisy Environments

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## ABSTRACT

In an election held in a noisy environment, agents may unintentionally perturb the outcome by communicating faulty preferences. We investigate this setting by introducing a theoretical model of noisy preference aggregation and formally defining the (worst-case) robustness of a voting rule. We use our model to analytically bound the robustness of various prominent rules. The results show that the robustness of voting rules is diverse, with different rules positioned at either end of the spectrum. These results allow selection of voting rules that support preference aggregation in the face of noise.

## Categories and Subject Descriptors

F.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity;  
I.2.11 [Artificial Intelligence]: Distributed Artificial Intelligence—*Multiagent Systems*;  
J.4 [Computer Applications]: Social and Behavioral Sciences—*Economics*

## General Terms

Algorithms, Theory, Economics

## Keywords

Computational complexity, Voting

## 1. INTRODUCTION

Preference aggregation mechanisms, and voting rules in particular, have been the object of scientific study for many years. Such mechanisms are used to aggregate the preferences of human or synthetic agents, over alternatives (or candidates). The alternatives in question may be entities such as joint plans for execution [5], schedules [8], or movie choices [6]. A voting rule generates an outcome that reflects the individual preferences over candidates, while striving to

satisfy different desiderata. Indeed, much of the research in social choice theory has focused on formally analyzing the properties of social choice mechanisms, with respect to these desiderata.

One important feature of study in preference aggregation mechanisms is their resistance to manipulation. Such manipulations are instances of adversarial worst-cases in the context of mechanisms: they consider self-interested voters that intentionally cast untruthful ballots in order to manipulate the outcome in their favor (often, under the assumption of prior knowledge of other voters' choices). An important theorem asserts that every voting rule (under certain minimal assumptions) is manipulable [7, 10]. More recent work in computer science suggests that computational complexity may help circumvent this impossibility result [1, 3].

However, little attention has been paid to a simpler—and arguably more common—form of voting distortion, where the truthful votes are *unintentionally* changed, as a result of uncertainty in the actions or perception of the agents casting the votes. For instance, agents may misunderstand the choices laid out for them, and may thus inadvertently cast a vote that is inconsistent with their true choice. Or, in the case of robots operating where communication is unreliable, true choices may be miscommunicated, resulting again in unintentional manipulation.

This paper takes first steps towards a formal analysis of the impact of errors in the preferences of voters. We define the  $k$ -robustness of a voting rule to be the resistance of the rule to  $k$  faults. In more detail, it is the probability that the outcome changes as a result of the faults, when each fault is chosen independently at random. We analyze the connection between 1-robustness (i.e., resistance to a single fault) and  $k$ -robustness, and conclude that it is sufficient to examine the 1-robustness of different rules. Most importantly, we use our definitions and tools to give tight upper and lower bounds on the robustness of several prominent voting rules. In fact, we show that the robustness of voting rules is extremely diverse, with different rules positioned at either end of the spectrum.

Given that voters rank the candidates (voters express *ordinal preferences*), we analyze a theoretical model where a fault is a switch in the rankings of two adjacent candidates (e.g., the fifth-ranked candidate is accidentally ranked sixth, and the sixth is ranked fifth). Such faults may easily be caused by confusion on the part of voters, or even by a single bit-flip when communicating the votes (see Section 3). Our goal is to understand the robustness of different voting rules to such faults; this understanding could aid system de-

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signers in selecting voting rules that can faithfully aggregate the preferences of agents in the system.

Previous work by Kalai [9] has investigated the issue of *noise-sensitivity* of social welfare functions in simple games; such functions give an entire social ranking of the candidates, instead of simply designating the winner of the election. The author engages in an asymptotic average-case analysis, where the basic assumption is that the voters' votes are distributed uniformly at random. Kalai presents a family of "chaotic" social welfare functions: a change in the preferences of a small fraction of the voters leads to social preferences that are asymptotically uncorrelated with the original preferences. In contrast, our model in this paper is quite different; in addition, we are interested in examining the robustness of prominent voting rules, as opposed to investigating extreme asymptotic phenomena.

This paper is organized as follows. In Section 2 we give an introduction to voting, and describe the voting rules we examine thereafter. In Section 3 we outline our model of preference profile errors, and give some general results regarding robustness. In Section 4 we bound the 1-robustness of some prominent voting rules, and in Section 5 we discuss our results and directions for future work.

## 2. PRELIMINARIES

In this section we give a brief introduction to classic social choice theory. The information here is sufficient to understand the paper, but readers who are interested in more details can consult [2].

Let  $V = \{v^1, v^2, \dots, v^n\}$  be the set of voters, and let  $C = \{c_1, c_2, \dots, c_m\}$  be the set of candidates,  $|C| = m$ . We usually use the index  $i$  (in superscript) to refer to voters, and the index  $j$  (in subscript) to refer to candidates.

Let  $\mathcal{L} = \mathcal{L}(C)$  be the set of all linear orders<sup>1</sup> on  $C$ . Each voter has ordinal preferences  $\succ^i \in \mathcal{L}$ , i.e., each voter  $v^i$  ranks the candidates:  $c_{j_1} \succ^i c_{j_2} \succ^i \dots \succ^i c_{j_m}$ . We refer to  $\succ^V = \langle \succ^1, \dots, \succ^n \rangle \in \mathcal{L}^N$  as a *preference profile*.

Given  $\succ^i$ , let  $j_1, \dots, j_m$  be indices of candidates such that  $c_{j_1} \succ^i c_{j_2} \succ^i \dots \succ^i c_{j_m}$ ; we denote by  $\pi_l(\succ^i)$  the candidate that voter  $i$  ranks in the  $l$ 'th place, i.e.,  $\pi_l(\succ^i) = c_{j_l}$ . We also denote by  $l_j^i$  the ranking of  $c_j$  in  $\succ^i$ ; it holds that  $\pi_{l_j^i}(\succ^i) = c_j$ .

### 2.1 Voting rules

A *voting rule* is a function  $F : \mathcal{L}^V \rightarrow C$ , i.e., a mapping from preferences of voters to candidates, that designates the winning candidate. We shall consider the following voting rules:

- *Scoring rules* are defined by a vector  $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_m \rangle$ .<sup>2</sup> Given  $\succ \in \mathcal{L}^N$ , the score of candidate  $j$  is  $s_j = \sum_i \alpha_{l_j^i}$ . The winner of the election is  $F(\succ) = \operatorname{argmax}_j s_j$ . Some of the well-known scoring rules are:

- *Borda*:  $\vec{\alpha} = \langle m-1, m-2, \dots, 0 \rangle$ .
- *Plurality*:  $\vec{\alpha} = \langle 1, 0, \dots, 0 \rangle$ .
- *Veto*:  $\vec{\alpha} = \langle 1, \dots, 1, 0 \rangle$ .

<sup>1</sup>Binary relations that satisfy antisymmetry, transitivity, and totality.

<sup>2</sup>More formally, a scoring rule is defined by a sequence of such vectors, one for each value of  $m$ , but we abandon this formulation for clarity's sake.

- *Copeland*: we say that candidate  $j$  beats  $j'$  in a pairwise election if  $|\{i : l_j^i < l_{j'}^i\}| > n/2$ . The score  $s_j$  of candidate  $j$  is the number of candidates that  $j$  beats in pairwise elections, and  $\operatorname{Copeland}(\succ) = \operatorname{argmax}_j s_j$ .
- *Maximin*: the Maximin score of candidate  $j$  is the candidate's worst performance in a pairwise election:  $s_j = \min_{j'} |\{i : l_j^i < l_{j'}^i\}|$ , and  $\operatorname{Maximin}(\succ) = \operatorname{argmax}_j s_j$ .
- *Bucklin*: for any candidate  $c_j$  and  $l \in \{1, \dots, m\}$ , let  $B_{j,l} = \{i : l_j^i \leq l\}$ . It holds that  $\operatorname{Bucklin}(\succ) = \operatorname{argmin}_j (\min\{l : |B_{j,l}| > n/2\})$ .
- *Plurality with Runoff*: The election proceeds in two rounds. After the first round, only the two candidates that maximize  $|\{i \in N : l_j^i = 1\}|$  survive. In the second round, a pairwise election is held between these two candidates.

## 3. OUR MODEL OF FAULTS AND ROBUSTNESS

We consider situations where (for example) noisy communication leads to changes in voters' rankings of candidates. The exact manifestation of these faults largely depends on the representation of preferences. In order to obtain results that are as general as possible, we here simply regard a fault as an alteration of one voter's ordering of candidates, which nevertheless maintains the integrity of the voter's preferences as a linear ordering (other types of faults remain for future work).

**DEFINITION 1.** A preference profile  $\succ_1^V$  is obtained from a preference profile  $\succ^V$  by an *elementary transposition* (write:  $\succ^V \rightsquigarrow \succ_1^V$ ) if there exists a voter  $v^i$  and  $l \in \{2, \dots, m\}$  such that:

1. for all  $i' \neq i$ ,  $\succ^{i'} = \succ_1^{i'}$ .
2.  $\pi_l(\succ^i) = c = \pi_{l-1}(\succ_1^i)$ .
3.  $\pi_{l-1}(\succ^i) = c' = \pi_l(\succ_1^i)$ .
4.  $\succ^i \downarrow_{C \setminus \{c, c'\}} = \succ_1^i \downarrow_{C \setminus \{c, c'\}}$ .

We say that  $\pi_{l-1}(\succ^i)$  was *demoted* and  $\pi_l(\succ^i)$  was *promoted*.

**EXAMPLE 1.** The preference profile

$\succ^1$	$\succ^2$
$c_1$	$c_2$
$c_3$	$c_1$
$c_2$	$c_3$

is obtained from the preference profile

$\succ^1$	$\succ^2$
$c_1$	$c_1$
$c_3$	$c_2$
$c_2$	$c_3$

by an elementary transposition that promotes  $c_2$  and demotes  $c_1$  (in the notations of the definition,  $i = 2$  and  $l = 2$ ).

In other words, we focus here on faults where a switch has occurred between two adjacent candidates in a voter's ranking of candidates. Such faults are interesting from a theoretical perspective, but may also occur in practice. For instance, if voters build their preferences incrementally, they may easily be confused by spatial proximity of alternative candidates. In other cases, depending on the representation and

communication protocol, communication errors may cause a switch to occur. Below we describe a representation for preferences, in which a flip of a single bit either causes a switch between two adjacent candidates, or can easily be detected.

### 3.1 The Pairwise Representation

We here describe a representation of preferences that is compatible with our fault model, and argue that it has certain advantages over other alternatives. One can represent preferences using a bit for each ordered pair of candidates (with  $\binom{m}{2}$  ordered pairs): the bit is 1 if the first candidate is preferred to the second, and 0 otherwise. We shall refer to this representation as the *pairwise representation*.

In this representation, a flip of a single bit corresponding to a pair of adjacent candidates in the ordering entails an elementary transposition. However, flipping a bit that does not correspond to adjacent candidates would create an ordering that is not transitive, and therefore not linear. Indeed, if (w.l.o.g.)  $c_1 \succ c_2 \succ c_3$ , and the bit corresponding to  $c_1$  and  $c_3$  is flipped, then we obtain the preferences  $c_1 \succ c_2$ ,  $c_2 \succ c_3$ , and  $c_3 \succ c_1$ —transitivity is not satisfied. It follows that faults which switch the ranking of two non-adjacent candidates can always be detected. So, when considering bit flips that may change the outcome without being detected, we can restrict our attention to faults that manifest themselves as elementary transpositions.

The pairwise representation is not the most compact possible. Consider the following elementary representation: each voter specifies the location of each candidate in their ranking; this requires  $m \log m$  bits. In the pairwise representation, each voter requires  $\binom{m}{2} = \frac{m(m-1)}{2}$  bits to express its preferences.

On the other hand, the pairwise representation allows us to test properties in constant time using bitmasks. For instance, say we want to test if a voter has ranked candidate  $c_1$  highest. We construct the ordered pairs in a way that candidate  $c_1$  is always first; we then examine the conjunction (a bitwise AND) of all pairs in which  $c_1$  participates— $c_1$  is ranked first iff this conjunction is 1. This can be done in constant time, while in the elementary representation, one would have to examine all  $\log m$  bits that represent  $c_1$ 's ranking in order to answer this question. Similarly, given that one knows (from polls, for example) which candidates are placed in the first  $k$  places, one can test in constant time whether a candidate is ranked in place  $k+1$ , using a bitmask on the pairwise representation.

### 3.2 The Definition of Robustness

So far, we have described our model of faults, and have argued that it has practical justification. For essentially any voting rule, there exist instances where even one adjacent switch changes the outcome of the election.

**THEOREM 1.** *Let  $F : \mathcal{L}^V \rightarrow C$  be a voting rule such that  $|\text{Ran}(F)| > 1$ . Then there exists a preference profile  $\succ^V$  and a profile  $\succ_1^V$  which is obtained from  $\succ^V$  by an elementary transposition, such that  $F(\succ^V) \neq F(\succ_1^V)$ .*

**PROOF.** Assume that for every preference profile  $\succ^V$  and any elementary transposition, the outcome does not change. Let  $\succ^V$  and  $\succ_1^V$  be any two preference profiles; we will derive a contradiction to the assumption on  $F$ 's range by showing that they necessarily have the same value under  $F$ .

Indeed, a preference profile is essentially a series of permutations on  $C$  (one for each voter); a basic result regarding permutation groups implies that  $\succ_1^V$  can be obtained from  $\succ^V$  by iterative elementary transpositions [4]. In other words, there are  $\succ_{i_1}^V, \dots, \succ_{i_t}^V$  such that  $\succ_{i_1}^V = \succ^V$ ,  $\succ_{i_t}^V = \succ_1^V$ , and each  $\succ_{i_{j+1}}^V$  can be obtained from  $\succ_{i_j}^V$  by an elementary transposition, for  $j = 1, \dots, t-1$ . By our assumption, all  $\succ_{i_j}^V$  have the same value under  $F$ , and in particular  $F(\succ^V) = F(\succ_1^V)$ —a contradiction.  $\square$

Given a voting rule, we wish to consider the implications of faults *in the worst-case*, i.e., in the worst instance. Theorem 1 motivates a probabilistic analysis: we will calculate the probability of the faults affecting the outcome in the worst-case.

Given a preference profile  $\succ^V$ , we define the probability distribution  $D_k(\succ^V)$  over preference profiles as follows: the probability of the preference profile  $\succ_1^V$  is the probability of obtaining  $\succ_1^V$  from  $\succ^V$  by  $k$  elementary transpositions chosen independently and randomly. In other words, in order to draw a profile  $\succ_1^V$  according to this distribution, we independently choose  $k$  values  $\{l_1, l_2, \dots, l_k\}$  and  $k$  values  $\{i_1, \dots, i_k\}$ , where each  $l_j$  is chosen according to the uniform distribution over  $\{2, \dots, m\}$ , and each  $i_j$  is chosen according to the uniform distribution over  $\{1, \dots, n\}$ . Now, starting with  $\succ^V$ , we perform  $k$  successive elementary transpositions—the  $j$ 'th transposition promotes candidate  $\pi_{l_j}(\succ^i)$  and demotes  $\pi_{l_j-1}(\succ^i)$ .

**DEFINITION 2.** The *k-robustness* of a preference profile  $\succ^V$  is:

$$\rho(F, \succ^V) = \Pr_{\succ_1^V \sim D_k(\succ^V)} [F(\succ^V) = F(\succ_1^V)].$$

The *k-robustness* of a profile reflects its immunity to  $k$  independent faults. As our analysis is worst-case, in order to define the robustness of a voting rule we take the minimum over all instances:

**DEFINITION 3.** The *k-robustness* of a voting rule  $F$  with  $n$  voters and  $m$  candidates is:

$$\rho_k^{n,m}(F) = \min_{\succ^V \in \mathcal{L}(C)^n} \rho(F, \succ^V).$$

**EXAMPLE 2.** Consider the Plurality rule with 3 voters and 2 candidates, and consider the preference profile  $\succ^V$  given by:

$\succ^1$	$\succ^2$	$\succ^3$
$c_1$	$c_1$	$c_2$
$c_2$	$c_2$	$c_1$

The outcome of this election is  $c_1$ . There are three possible profiles resulting from an elementary transposition:

$\succ_1^1$	$\succ_1^2$	$\succ_1^3$	$\succ_2^1$	$\succ_2^2$	$\succ_2^3$	$\succ_3^1$	$\succ_3^2$	$\succ_3^3$
$c_2$	$c_1$	$c_2$	$c_1$	$c_2$	$c_2$	$c_1$	$c_1$	$c_1$
$c_1$	$c_2$	$c_1$	$c_2$	$c_1$	$c_1$	$c_2$	$c_2$	$c_2$

In two of these profiles, the outcome is  $c_2$ . Therefore,

$$\rho(F, \succ^V) = \frac{1}{3}.$$

Repeating the same calculation for all preference profiles  $\succ^V \in \mathcal{L}(C)^n$  and taking the minimum, it is possible to conclude that  $\rho_1^{3,2}(\text{Plurality}) = \frac{1}{3}$ .

### 3.3 Bounding $k$ -robustness with 1-robustness

The definition of  $D_k(\succ^V)$  as sampling  $k$  independent elementary transpositions allows a very strong link between 1-robustness and  $k$ -robustness: a lower bound on the former entails a lower bound on the latter.

PROPOSITION 2.  $\rho_k^{n,m}(F) \geq (\rho_1^{n,m}(F))^k$ .

PROOF. Consider the preference profile  $\succ_1^V$ , and the preference profile  $\succ_2^V$  obtained by  $k$  independent and random elementary transpositions—we claim that the probability that  $F(\succ_1^V) = F(\succ_2^V)$  is at least  $(\rho_1^{n,m})^k$ .

Indeed, let  $\succ_{i_1}^V, \dots, \succ_{i_{k+1}}^V$  be the intermediate preference profiles obtained by the elementary transpositions, i.e., we have that  $\succ_{i_1}^V = \succ_1^V$ ,  $\succ_{i_{k+1}}^V = \succ_2^V$ , and each  $\succ_{i_{j+1}}^V$  is obtained from  $\succ_{i_j}^V$  by an independently and randomly chosen elementary transposition, for  $j = 1, \dots, k$ . By the definition of 1-robustness, we have that for every preference profile  $\succ^V$ , the probability that one randomly chosen elementary transposition does not change the outcome of the election under  $F$  is at least  $\rho_1^{n,m}(F)$ . Therefore, we have that for  $j = 1, \dots, k$ ,

$$\Pr[F(\succ_{i_j}^V) = F(\succ_{i_{j+1}}^V) \mid \succ_{i_j}^V] \geq \rho_1^{n,m}(F).$$

By analyzing the conditional probabilities we have that:

$$\begin{aligned} \Pr[F(\succ_1^V) = F(\succ_2^V)] &= \Pr[\forall j = 1, \dots, k, F(\succ_{i_j}^V) = F(\succ_{i_{j+1}}^V)] \\ &= \prod_{j=1}^k \Pr[F(\succ_{i_j}^V) = F(\succ_{i_{j+1}}^V) \mid \succ_{i_j}^V] \\ &\geq (\rho_1^{n,m})^k. \end{aligned}$$

□

The above proposition is very useful when the number of errors is constant. Otherwise, the bound on  $k$ -robustness which the proposition yields may not be very good, even if the voting rule seems 1-robust. Nevertheless, we have the following immediate corollary regarding  $k = m$ :

COROLLARY 3. Let  $F$  be a voting rule such that

$$\rho_1^{n,m}(F) \geq 1 - x/m$$

for some constant  $x$ , and let  $\epsilon > 0$ . Then

$$\rho_m^{n,m}(F) \geq \frac{1}{e^x} - \epsilon$$

for a large enough  $m$ .

## 4. RESULTS ON 1-ROBUSTNESS

Proposition 2 dictates the direction of the bulk of our results: we are satisfied with calculating the 1-robustness of voting rules. If we achieve a high lower bound, this also implies high  $k$ -robustness (at least for a constant  $k$ ). However, in case 1-robustness is low, there is no point in considering the rule's  $k$ -robustness.

REMARK 1. Given the number of voters and candidates, and a preference profile  $\succ^V$ , there are exactly  $n(m-1)$  possible elementary transpositions. Therefore:

$$\rho_1^{n,m}(F) = \frac{|\{\succ_1^V \in \mathcal{L}(C)^n : \succ^V \rightsquigarrow \succ_1^V \wedge F(\succ^V) = F(\succ_1^V)\}|}{n(m-1)}.$$

Before we deal with specific voting rules, we note that we cannot expect a rule's 1-robustness to be exactly 1.

PROPOSITION 4. Let  $F : \mathcal{L}(C)^n \rightarrow C$  be a voting rule such that  $|\text{Ran}(F)| > 1$ . Then  $\rho_1^{n,m}(F) < 1$ .

PROOF. Follows directly from Theorem 1 and the definition of 1-robustness. □

### 4.1 Scoring rules

In this subsection we fully characterize the robustness of scoring rules as a function of their parameters. Our results imply that some common scoring rules are very robust, while others are extremely susceptible to faults.

Given a scoring rule  $F$  with parameters  $\vec{\alpha}$ , let  $A_F = |\{l \in \{2, \dots, m\} : \alpha_{l-1} > \alpha_l\}|$ ; denote  $|A_F| = a_F$ .

PROPOSITION 5. Let  $n$  and  $m$  be the number of voters and candidates, let  $F$  be a scoring rule. Then  $\rho_1^{n,m}(F) \geq \frac{m-1-a_F}{m-1}$ .

PROOF. For any preference profile  $\succ^V$ , the outcome can only be affected by elementary transpositions that promote  $\pi_l(\succ^i)$ , for some  $l \in A_F$  and  $i$ , and demote  $\pi_{l-1}(\succ^i)$ . For each voter  $v^i$ , there are exactly  $a_F$  such values of  $l$ , out of  $m-1$  possible elementary switches. Therefore, the number of elementary transpositions that are guaranteed *not* to change the outcome is at least  $n(m-1) - a_F n$ , and the 1-robustness of  $F$  is at least  $\frac{n(m-1) - a_F n}{n(m-1)} = \frac{m-1-a_F}{m-1}$ . □

We match this lower bound with a fairly tight upper bound. In this example, we require that the number of candidates evenly divide the number of voters. However, such a special case is sufficient, as it implies that the lower bound cannot be improved in general.

PROPOSITION 6. Let  $n$  and  $m$  be the number of voters and candidates such that  $m$  evenly divides  $n$ , and let  $F$  be a scoring rule. Then  $\rho_1^{n,m}(F) \leq \frac{m-a_F}{m}$ .

PROOF. By the assumption on  $n$  and  $m$ , it is possible to group the voters in  $m$  subsets of size  $d$ ,  $T_1, \dots, T_m$ . Consider the preference profile  $\succ^V$  where the subsets of voters vote cyclically:

$\succ^{T_1}$	$\succ^{T_2}$	$\dots$	$\succ^{T_m}$
$c_1$	$c_2$	$\dots$	$c_m$
$c_2$	$c_3$	$\dots$	$c_1$
$\dots$	$\dots$	$\dots$	$\dots$
$c_{m-1}$	$c_m$	$\dots$	$c_{m-2}$
$c_m$	$c_1$	$\dots$	$c_{m-1}$

Notice that under any scoring rule, all candidates have the same score; without loss of generality, candidate  $c_1$  is the winner of this election. How many profiles obtained by a single transposition necessarily have a different outcome? An elementary transposition between places  $l-1$  and  $l$ , where  $l \in A_F$ , strictly increases a candidate's score, and changes the outcome—given that the promoted candidate is not candidate 1. For every  $l \in A_F$ , exactly  $d$  voters rank candidate  $c_1$  in place  $l$ . Hence, there are  $da_F$  voters with  $a_F - 1$  possible elementary transpositions that change the outcome of the election (the voters that rank candidate  $c_1$  in place  $l \in A_F$ ), and  $n - da_F$  voters with  $a_F$  such transpositions. It follows that the probability that the outcome

changes, under the uniform distribution over instances such that  $\succ^V \rightsquigarrow \succ_1^V$ , is at least (substituting  $n = md$ ):

$$\frac{da_F(a_F - 1) + (dm - da_F)a_F}{dm(m - 1)} = \frac{a_F}{m}.$$

In other words, the probability that the outcome *does not change* is at most  $\frac{m - a_F}{m}$ . As the robustness is defined to be the minimum over all instances, we obtain the desired result.  $\square$

**COROLLARY 7.** *The Veto and Plurality rules, where  $a_F = 1$ , are extremely robust. On the other hand, the Borda rule, for which  $a_F = m - 1$ , is very susceptible to failures.*

## 4.2 Copeland

We give an upper bound that relies on an example where the number of voters is even. However, since the number of candidates is not restricted, this example implies that it is not possible to establish a good general lower bound. In addition, as the upper bound is very small, an exact lower bound is of no consequence.

**PROPOSITION 8.** *Let  $m$  be the number of candidates, and let the number of voters  $n$  be even. Then  $\rho_1^{n,m}(\text{Copeland}) \leq 1/(m - 1)$ .*

**PROOF.** Consider the preference profile where for  $i = 1, 3, 5, \dots, n - 1$ , voters  $v^i$  and  $v^{i+1}$  vote as follows:

$\succ^i$	$\succ^{i+1}$
$c_1$	$c_m$
$c_2$	$c_{m-1}$
$\cdot$	$\cdot$
$\cdot$	$\cdot$
$c_m$	$c_1$

Under the above profile, for every two candidates  $c$  and  $c'$ , exactly  $n/2$  voters prefer  $c$  over  $c'$ . Thus, the Copeland score of all candidates is 0, and the winner is some candidate  $c \in C$ . Any elementary transposition that promotes candidate  $c' \neq c$  would raise the score of  $c'$  to 1, making  $c'$  the new winner. This implies that for every voter, there are at least  $m - 2$  elementary transpositions that change the outcome of the election, and thus the probability that the outcome does not change is at most  $1 - \frac{n(m-2)}{n(m-1)} = \frac{1}{m-1}$ .  $\square$

## 4.3 Maximin

**PROPOSITION 9.** *Let  $n$  and  $m$  be the number of voters and candidates such that  $m$  evenly divides  $n$ . Then  $\rho_1^{n,m}(\text{Maximin}) \leq 1/(m - 1)$ .*

**PROOF.** Our adversarial preference profile is identical to the one in the proof of Proposition 6. However, we are going to construct the profile algorithmically, as this is going to aid us in establishing some of the profile's properties. We iteratively expand the list of candidates; initially, it contains only  $c_1$ , so each voter's linear preferences are in fact the empty set. In the second stage, we add to the slate the candidate  $c_2$ ; for  $\frac{1}{m}n$  voters, candidate  $c_2$  is ranked at the top (above  $c_1$ ), but the other  $\frac{m-1}{m}n$  voters rank  $c_2$  below  $c_1$ . Now,  $c_3$  is added as follows:  $\frac{1}{m}n$  voters that ranked  $c_2$  last (i.e., previously voted  $c_1 \succ c_2$ ), now rank  $c_3$  first (i.e., vote  $c_3 \succ c_1 \succ c_2$ ); the other  $\frac{m-1}{m}n$  voters rank  $c_3$  immediately below  $c_2$  (e.g., if the ranking was  $c_2 \succ c_1$ , it is now  $c_2 \succ c_3 \succ c_1$ ). In general, when adding candidate  $c_j$ ,

$\frac{1}{m}n$  voters that ranked  $c_{j-1}$  last now rank  $c_j$  first,<sup>3</sup> and the rest rank  $c_j$  just below  $c_{j-1}$ .

For example, for 8 voters and 4 candidates, initially we have: (in each stage  $j$ , the  $\frac{1}{m}n = 2$  grayed voters are the ones that rank candidate  $c_j$  first instead of just under  $c_{j-1}$ )

$\succ^1$	$\succ^2$	$\succ^3$	$\succ^4$	$\succ^5$	$\succ^6$	$\succ^7$	$\succ^8$
$c_1$							

In the second stage we have:

$\succ^1$	$\succ^2$	$\succ^3$	$\succ^4$	$\succ^5$	$\succ^6$	$\succ^7$	$\succ^8$
$c_2$	$c_2$	$c_1$	$c_1$	$c_1$	$c_1$	$c_1$	$c_1$
$c_1$	$c_1$	$c_2$	$c_2$	$c_2$	$c_2$	$c_2$	$c_2$

In the third stage we have:

$\succ^1$	$\succ^2$	$\succ^3$	$\succ^4$	$\succ^5$	$\succ^6$	$\succ^7$	$\succ^8$
$c_2$	$c_2$	$c_3$	$c_3$	$c_1$	$c_1$	$c_1$	$c_1$
$c_3$	$c_3$	$c_1$	$c_1$	$c_2$	$c_2$	$c_2$	$c_2$
$c_1$	$c_1$	$c_2$	$c_2$	$c_3$	$c_3$	$c_3$	$c_3$

Ultimately, the preference profile that the algorithm constructs is:

$\succ^1$	$\succ^2$	$\succ^3$	$\succ^4$	$\succ^5$	$\succ^6$	$\succ^7$	$\succ^8$
$c_2$	$c_2$	$c_3$	$c_3$	$c_4$	$c_4$	$c_1$	$c_1$
$c_3$	$c_3$	$c_4$	$c_4$	$c_1$	$c_1$	$c_2$	$c_2$
$c_4$	$c_4$	$c_1$	$c_1$	$c_2$	$c_2$	$c_3$	$c_3$
$c_1$	$c_1$	$c_2$	$c_2$	$c_3$	$c_3$	$c_4$	$c_4$

**LEMMA 1.** *In stage  $j$  (after candidate  $c_j$  is added to the slate), it holds that for every  $i < j$ , the number of voters that prefer  $c_i$  to  $c_j$  is  $\frac{m-(j-i)}{m}n$ .*

**PROOF.** By induction on  $j$ . The basis of the induction ( $j = 1$ ) is trivial. Now, assume the claim holds for  $j - 1$ ; we shall prove it for  $j$ . Let  $i < j$ ; if  $i = j - 1$ , notice that  $c_j$  is ranked under  $c_i$ , except in  $\frac{1}{m}n$  cases. In other words, the number of voters that prefer  $c_i = c_{j-1}$  to  $c_j$  is  $\frac{m-1}{m}n$ , as desired.

It remains to deal with the case where  $i < j - 1$ . Recall that  $c_j$  is always ranked directly under  $c_{j-1}$ , except for  $\frac{1}{m}n$  voters that rank  $c_j$  first. As for the rest of the voters,  $c_i$  is ranked above  $c_j$  iff  $c_i$  was ranked above  $c_{j-1}$  in stage  $j - 1$ . By the induction assumption, we had  $\frac{m - ((j-1) - i)}{m}n$  ranking  $c_i$  above  $c_{j-1}$  in stage  $j - 1$ , and thus the number of voters ranking  $c_i$  above  $c_j$  is:

$$\frac{m - ((j - 1) - i)}{m}n - \frac{1}{m}n = \frac{m - (j - i)}{m}n,$$

as desired.  $\square$

Lemma 1 implies that candidate  $c_j$ 's *unique* worst pairwise election is against  $c_{j-1}$  for  $j > 1$ : the number of voters that prefer  $c_j$  to  $c_{j-1}$  is exactly  $\frac{1}{m}n$ ; notice that this is also true for  $c_1$  versus  $c_m$ : only  $\frac{1}{m}n$  rank  $c_1$  above  $c_m$ . In addition, for  $j > 1$ , clearly  $c_j$  is ranked just under  $c_{j-1}$  by  $\frac{m-1}{m}n$  voters—but this, too, is also true for  $c_1$  versus  $c_m$ ; indeed,  $c_1$  is ranked just under  $c_m$  by all  $\frac{m-1}{m}n$  voters that do not rank  $c_1$  first.

So, the candidates are all tied with respect to their maximin scores, and each candidate  $c_j$  is ranked just below its “worst pairwise” candidate by all voters that do not rank  $c_j$  first. Therefore, any elementary transposition that promotes a candidate that is not the current winner of the election must change the outcome of the election. As before,

<sup>3</sup>It is easy to verify that there always are  $\frac{1}{m}n$  such voters.

we have that the probability of the outcome changing as a result of a single transposition, under our adversarial preference profile, is at least  $\frac{m-2}{m-1}$ , and thus robustness of this preference profile is at most  $\frac{1}{m-1}$ .

#### 4.4 Bucklin

PROPOSITION 10.  $\rho_1^{n,m}(\text{Bucklin}) \geq \frac{m-2}{m-1}$  for any values of the number of voters  $n$  and the number of candidates  $m$ .

PROOF. Consider a preference profile  $\succ^V$ , and assume that the winner  $c_j$  of the election satisfies:

$$l_0 = \min_j B(j, l) > n/2.$$

We argue that any elementary transposition that switches the candidates in places  $l$  and  $l-1$ , for  $l \neq l_0, l_0+1$ , cannot change the outcome of the election. Indeed, we consider two cases:

Case 1:  $l > l_0+1$ . In this case, if some candidate  $c_k \neq c_j$  is promoted, the switch increases  $B(k, l-1)$ —but this is irrelevant to the outcome of the election, since  $B(k, l_1)$  remains unchanged for  $l_1 \leq l_0$ .

Case 2:  $l < l_0$ . If candidate  $c_k$  is promoted, this might increase  $B(k, l_0-2)$ . However, the value of  $B(k, l_0-2)$  after the switch took place is bounded from above by the value of  $B(k, l_0-1)$  before the switch. We know that  $B(k, l_0-1) \leq n/2$  before the switch—so this transposition is not going to affect the value of  $\min_k(B(k, l) > n/2)$ .

If so, it remains to consider the case where  $l = l_0$  or  $l = l_0+1$ . When  $l = l_0$ , promoting  $\pi_{l_0}(\succ^i)$  may affect the outcome only if  $\pi_{l_0}(\succ^i) \neq c_j$ , where  $c_j$  is the winner of the election. However, when  $l = l_0+1$ , promoting  $\pi_{l_0+1}(\succ^i)$  and demoting  $\pi_{l_0}(\succ^i)$  might affect the outcome only if  $\pi_{l_0}(\succ^i) = c_j$ . Otherwise, if  $\pi_{l_0+1}(\succ^i) = c_k \neq c_j$ , then  $B(k, l_0)$  might be affected, but since  $c_j$  already has a majority of voters ranking it in the top  $l_0$  places, the outcome of the election is indifferent to this perturbation.

As these two last subcases are mutually exclusive, it follows that for every voter there is at most one transposition that may affect the outcome of the election. Thus  $\rho_1^{n,m}(\text{Bucklin}) \geq \frac{m-2}{m-1}$ .  $\square$

#### 4.5 Plurality with Runoff

The rules we have discussed in the previous subsections all have in common some concept of score. Since Plurality with Runoff is a bit different, we require an additional assumption regarding tie-breaking. Consider a situation where, say,  $c_{j_1}$  and  $c_{j_2}$  survive the first round, and exactly half the voters prefer  $c_{j_1}$  to  $c_{j_2}$ , but  $c_{j_1}$  is the winner of the election. We assume that if a fault makes  $c_{j_1}$  and  $c_{j_3}$  survive the first round, and again these two candidates are tied in the second round, then  $c_{j_1}$  loses the election. This assumption is consistent with our worst-case analysis throughout.

PROPOSITION 11. For all values of  $n$  and  $m$ ,

$$\rho_1^{n,m}(\text{Plurality with Runoff}) \geq \frac{m-5/2}{m-1}.$$

PROOF. Consider some preference profile  $\succ^V$ , and assume w.l.o.g. that candidates  $c_1$  and  $c_2$  survive the first round, and  $c_1$  wins the election. Only two types of elementary transposition can potentially affect the outcome of the election. The

first is promoting the candidate  $\pi_2(\succ^i)$  for some  $i$ , i.e., making this candidate voter  $v^i$ 's favorite—this might affect the list of candidates that are eliminated in the first round. A second transposition which might have an effect is one that promotes candidate  $c_2$  and demotes  $c_1$ —this might change the outcome of the second round, but only if exactly half the voters prefer  $c_1$  to  $c_2$  in  $\succ^V$  (it cannot be the case that more voters prefer  $c_2$ , as then  $c_2$  would have prevailed in the second round). To conclude, at most  $n/2$  voters have two transpositions that may affect the outcome, and at least  $n/2$  voters have only one. We have that

$$\rho(F, \succ^V) \geq \frac{n(m-1) - (n/2 \cdot 1 + n/2 \cdot 2)}{n(m-1)} = \frac{m-5/2}{m-1}.$$

$\square$

PROPOSITION 12.

$$\rho_1^{2m,m}(\text{Plurality with Runoff}) \leq \frac{m-5/2}{m-1} + \frac{5/2}{m(m-1)}.$$

PROOF. Consider a preference profile  $\succ^V$ , where the voters are divided into two subsets of size  $m$ , each voting cyclically, in a way that voter  $v^i$ 's preferences in one subset are mirrored by voter  $v^{m+i}$ 's preferences in the other subset (the grayed voters are the second subset of size  $m$ ):

$\succ^1$	$\succ^2$	$\dots$	$\succ^m$	$\succ^{m+1}$	$\succ^{m+2}$	$\dots$	$\succ^{2m}$
$c_1$	$c_2$	$\dots$	$c_m$	$c_m$	$c_1$	$\dots$	$c_{m-1}$
$c_2$	$c_3$	$\dots$	$c_1$	$c_{m-1}$	$c_m$	$\dots$	$c_{m-2}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$c_{m-1}$	$c_m$	$\dots$	$c_{m-2}$	$c_2$	$c_3$	$\dots$	$c_1$
$c_m$	$c_1$	$\dots$	$c_{m-1}$	$c_1$	$c_2$	$\dots$	$c_m$

Notice that in the first round, every candidate is ranked first by exactly two voters. In addition, for every two candidates  $c, c' \in C$ , exactly  $n/2$  voters prefer  $c$  to  $c'$ . Assume that candidates  $c_1$  and  $c_2$  survive the first round, and candidate  $c_1$  wins the election. Now, for any voter  $v^i$  such that  $\pi_2^i \neq c_1, c_2$ , promoting  $\pi_2^i$  affects the list of candidates surviving the first round, and ultimately (by our new assumption) changes the outcome of the election; there are  $2m-4 = n-4$  such voters. In addition, voters  $v^1$  and  $v^3, \dots, v^m$  rank  $c_2$  directly below  $c_1$ , so promoting  $c_2$  and demoting  $c_1$  changes the outcome of the second round. Therefore, the total number of elementary transpositions that influence the outcome of the election is  $(n-4) + (n/2-1)$ . We have that

$$\begin{aligned} \rho(\text{Plurality with Runoff}, \succ^V) &\leq \frac{n(m-1) - [(n-4) + (n/2-1)]}{n(m-1)} \\ &= \frac{m-5/2}{m-1} + \frac{5}{n(m-1)} \\ &= \frac{m-5/2}{m-1} + \frac{5/2}{m(m-1)}. \end{aligned}$$

$\square$

## 5. DISCUSSION

We have defined the  $k$ -robustness of a voting rule as the worst-case probability that  $k$  independent switches in the preferences of voters change the outcome of the election. We have shown that high 1-robustness implies high  $k$ -robustness,

Function	Lower Bound	Upper Bound
Scoring	$\frac{m-1-a_F}{m-1}$	$\frac{m-a_F}{m}$
Copeland	0	$\frac{1}{m-1}$
Maximin	0	$\frac{1}{m-1}$
Bucklin	$\frac{m-2}{m-1}$	1
Plurality w. Runoff	$\frac{m-5/2}{m-1}$	$\frac{m-5/2}{m-1} + \frac{5/2}{m(m-1)}$

**Table 1: Some upper and lower bounds on the 1-robustness of several prominent voting rules.**

at least for a constant  $k$ . Inversely, low 1-robustness clearly suggests that the rule is not robust in general. Accordingly, we have presented bounds on the 1-robustness of different voting rules; these bounds are summarized in Table 1.

We consider our results to be a useful tool for designers of multiagent systems. When dealing with noisy environments, successful aggregation of preferences can only be expected when a robust voting rule is applied. In particular, among the prominent voting rules, our results imply that Plurality, Plurality with Runoff, Veto, and Bucklin are robust to faults, whereas Borda, Copeland, and Maximin are more susceptible to faults.

The model of errors we have introduced is a theoretical one, but we have also shown it is grounded in a reasonable representation of preferences. Nevertheless, future work should include an investigation of different error models.

In addition, our analysis was worst-case—an approach which leads to the conclusion that when the number of errors is large, voting rules are bound to fail. It would be interesting to complement our results with an asymptotic *average-case* analysis.

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