Multiplicity One Theorems for $GSp(k, 2n)$ and $O(k, n)$, where $k$ is a Finite Field

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Communicated by Walter Feit

Received June 26, 1983

INTRODUCTION

The aim of this paper is to give a result similar to the following result of Gelfand and Graev [1]: Let $U$ be a maximal unipotent subgroup of a finite Chevalley group $G$. For each nondegenerate $\gamma$, the induced representation $\text{Ind}_{U}^{G} \gamma$ is multiplicity-free. In part I we shall prove a theorem of that kind for $GSp(2n, k)$, and in Part II for $O(n, k)$. These theorems were conjectured and given to me by I. Piatetski-Shapiro.

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In Part I we shall consider the following conditions:

Let $Q$ be a parabolic subgroup of $GSp(2n, k)$ which preserves a one-dimensional subspace of $k^{2n}$. $Q = MU$ the Levi decomposition. Let $M_1$ be the centralizer in $M$ of the center of $U$. Let $D = M_1$. $U$ is isomorphic to the Heisenberg group of order $2n-2$ and $M_1$ is isomorphic to $sp(2n-2, k) \times k^*$. We look at the Weil representation of $D$. In Section 3, we prove that the induced representation of the Weil represemtation is multiplicity-free.

1. DEFINITION OF $D$

Let $k$ be a finite field. $\text{Char}(k) \neq 2$.

Let $T = T_n$ be the $n \times n$ matrix

* * Supported in part by the NSF under Grant MCS 81-08814.

0022-447X/87 $3.00$

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Let $J = J_{2n}$ be the $2n \times 2n$ matrix written in blocks by \(egin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\).

Let $G = GSp(2n, k) = \{ g \in GL(2n, k) \mid \exists \lambda(g) \in k^* \text{ s.t. } gJg = \lambda(g)J \}$.

Let $e_i$, $i = 1, \ldots, 2n$, be the standard basis in $k^{2n}$.

Denote by $\langle \cdot, \cdot \rangle$ the antisymmetric form induced by $J$.

Let $E = \text{Span}\{e_1, \ldots, e_{2n-1}\}$.

Let $Q = \{ g \in G \mid g(\text{Span}\{e_1\}) = \text{Span}\{e_1\} \}$.

Let $M = \{ g \in Q \mid gE = E, g(\text{Span}\{e_{2n}\}) = \text{Span}\{e_{2n}\} \}$.

Let $U = \{ g \in Q \mid \forall e \in E \exists \lambda \in k \text{ s.t. } g(e) = e + \lambda e_1, g(e_1) = e_1 \}$.

$Q = MU$ is the Levi decomposition of $Q$.

$M$ preserves $\text{Span}\{e_1\}$, $E$ and $\text{Span}\{e_{2n}\}$, $M \subset G$ so

$$M = \left\{ \begin{pmatrix} x & g \\ \beta & \gamma \end{pmatrix} \mid \begin{array}{l} g \in GSp(2n-2, k), \alpha, \beta \in k^*, x : \beta = \lambda(g) \end{array} \right\}.$$ 

Furthermore, $M \cong GSp(2n-2, k) \times k^*$.

$U$ preserves $\{e_1\}$, $\text{Span}\{e_1\} + E$, and acts trivially on $E \mod \text{Span}\{e_1\}$.

$U \subset G$ so $U$ is unipotent and $U$ actually equals

$$U = \begin{pmatrix} 1 & x_1 & \cdots & x_{n-1} & y_1 & \cdots & y_{n-1} & z \\ & I_{2n-2} & & & & & & \\ & & & & & & & \end{pmatrix} \begin{pmatrix} y_{n-1} \\ \vdots \\ y_1 \\ -x_{n-1} \\ \vdots \\ -x_1 \\ 1 \end{pmatrix} x_i, y_i, z \in k.$$ 

Furthermore, $U \cong$ Heisenberg group of order $2n-2 = k^{2n-2} \times k$, by
\[
\begin{pmatrix}
1 & x_1 & \cdots & x_{n-1} & y_1 & \cdots & y_{n-1} & z \\
& y_{n-1} & & & & & & \vdots \\
& & \ddots & & & & & * \\
& & & y_1 & & & & -x_{n-1} \\
& & & & \vdots & & & \\
& & & & & \vdots & & \\
& & & & & & -x_1 & 1 \\
\end{pmatrix}
\]

\[I_{3n-2} \leftrightarrow (x_1, \ldots, x_{n-1}, x_{n-1}, y_1, \cdots, y_{n-1}, z).
\]

In short, we shall denote the elements of \( U \) by the classical Heisenberg notations \((x, y, z)\). Note that \( M U = UM \) and the elements of \( Q \) are of the form

\[
\begin{pmatrix}
\alpha & \alpha x_1 & \cdots & \alpha x_{n-1} & \alpha y_1 & \cdots & \alpha y_{n-1} & \beta \\
& g & & & & & & \\
& & g & & & & & \\
\end{pmatrix}
\]

where \( g \in GSp(2n-2, k), \alpha \beta = \lambda(g) \).

Let \( M_1 = \{ m \in M \mid mu = um \ \forall u = (0 \ 0 \ z) \in Z(U) \} \). Then

\[M_1 = \{ m \in M \mid z = \beta \} = \left\{ x \begin{pmatrix} 1 & & & & \\
& g & & & \\
& & & & \\
& & & & \\
\end{pmatrix} \mid x \in k^*, g \in GSp(2n-2, k) \right\}.
\]

Define \( D = M_1 U \).

2. DEFINITION OF THE WEIL REPRESENTATION OF \( D \)

Let \( X \) be the following subgroup of \( U \):

\[X = \{(x, 0, z)\}.
\]

Let \( \psi \) be a nontrivial additive character of \( k \). Extend it to \( X \) by \( \psi(x, 0, z) = \hat{\nu}(z) \). Let \((\omega_\nu, Y)\) be the induced representation \( \text{Ind}_X^U \psi \). By the
definition of the induced representation and the multiplication laws in $U$
we can assume that $V = \{ f: k^{n-1} \rightarrow C \}$ and $U$ acts as follows:

\[
\omega_y(0 \ 0 \ 0) f(w) = f(w + y) \\
\omega_y(x \ 0 \ 0) f(w) = \psi(2(-x_1 w_{n-1} - \cdots - x_{n-1} w_1)) f(w) \\
\omega_y(0 \ 0 \ z) f(w) = \psi(z) f(w).
\]

Since $M_1 \sim k^* \times Sp(2n-2, k)$, the Weil $^*$ representation $\omega_y$ of $Sp(2n-2, k)$ induces the Weil representation of $M_1$ in $V$ (See [3, Theorem 2.4]). We take the following generators for $M_1$, written in blocks:

\[
\begin{pmatrix}
I_{n-1} & 0 \\
S & I_{n-1}
\end{pmatrix}, \quad S \in M(n-1, k), \quad T_{n-1} T_{n-1}^* = S
\]

\[
\begin{pmatrix}
1 \\
T_{n-1} A^{-1} T_{n-1}
\end{pmatrix}, \quad A \in GL(n-1, k)
\]

\[
\begin{pmatrix}
1 & 0 \\
-T_{n-1} & 0
\end{pmatrix}
\]

and

\[
\omega_y \begin{pmatrix}
1 \\
I_{n-1} & 0 \\
S & I_{n-1}
\end{pmatrix} f(w) = \psi \left( \frac{1}{2} S w \cdot w \right) f(w)
\]

\[
\omega_y \begin{pmatrix}
1 \\
T_{n-1} A^{-1} T_{n-1}
\end{pmatrix} f(w) = f(w A)
\]
$$\omega_k \left( \begin{array}{ccc} 1 & 0 & T_{n-1} \\ -T_{n-1} & 0 & 1 \end{array} \right) f(w) = \int_{x \in D} f(x) \psi(x \cdot w) \, dx.$$ 

\(\omega_k(\alpha I) < \eta(\alpha) \cdot I\) for some multiplicative character \(\eta\) of \(k^*\). The definitions of \(\omega_k\) on \(U\) and \(M_1\) are compatible with the multiplication in \(D\), so the equations glue to a representation of \(D\) [3], which we will denote also by \((\omega_k, V)\).

\(\omega_k\) is irreducible since its restriction to \(U\) is irreducible. We shall use the sign \(\omega_k\) as a representation of \(U, \text{Sp}(2n-2, k), M_1, D\) and there will be no confusion.

3. The Main Theorem

**Theorem I.** \(\tilde{\Pi} = \text{Ind}^{\mathbb{G}}_{\mathbb{G}} \omega_k\) is multiplicity-free.

**Proof.** To prove the statement, it is enough to prove that the Schur Algebra is commutative. The Schur Algebra is isomorphic to the algebra \(A\).

\[ A = \{ K: G \to \text{End}_k(V) | K(\delta g \delta_2) = \omega_k(\delta_1) K(g) \omega_k(\delta_2) \} \]

\[ \forall g \in G, \forall \delta_1, \delta_2 \in D \]

\[ K_{\delta_1} K_{\delta_2}(g) = \sum_{x \in \mathbb{H}} K_{\delta_1}(g x^{-1}) K_{\delta_2}(x) \]

So, we shall show that the ring structure of \(A\) is commutative. To do this, we shall use a method of Gelfand, first used in [1].

We shall introduce an involution \(\tau\) on \(A, \tau: A \to A, \tau(ab) = \tau(b) \tau(a)\), and prove that it is actually the identity. This will show that \(A\) is commutative.

To start, we shall introduce the following function on \(G\): \(g = h'gh\), where \(h\) is the following element of \(G\):

\[ h = T_{2n} = \left( \begin{array}{cc} 0 & T_n \\ T_n & 0 \end{array} \right) = \left( \begin{array}{ccccc} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \end{array} \right) \]

\(h\) satisfies \(h'h = -I, \quad h = h, \quad h^{-1} = h\), so \(g \in G, \quad \tau(g) = g\) and \(\tau(g_1 \cdot g_2) = \tau(g_1) \cdot \tau(g_2)\), and \(\tau\) is an involution.
We need to show that $\tau$ fixes $D$ and fixes $Z(U)$ elementwise. In fact,

$$
\begin{pmatrix}
1 \\
g
\end{pmatrix}
= \begin{pmatrix}
1 \\
T_{2n-2}^{-1} g T_{2n-2}^{-1}
\end{pmatrix},
$$

which is in $M_1$ since $g$, and $T_{2n-2} \in S(2n-2, k)$. Thus $\tau(g) = g$. Since $\omega_\delta$ is irreducible [2], there exists a basis of $V$ such that the transpose, $K$, with respect to this basis satisfies $\tau(\omega_\delta) = \omega_\delta$ for all $\delta \in D$.

Define an involution on $A$, $\tau: A \to A$ as follows: $\tau(K(g)) = K(\tau(g))$, $\tau(k) \in A$ since

$$
\tau(K(\delta_1, g \delta_2)) = K(\tau(\delta_1), g \delta_2)
= K(\tau(\delta_2), g \delta_1)
= \langle \omega_\delta(\tau(\delta_2), K(g) \omega_\delta(\delta_1))
= \langle \omega_\delta(\delta_1), K(g) \omega_\delta(\delta_2)
= \omega_\delta(\delta_1) \tau(K(g)) \omega_\delta(\delta_2).
$$

$\tau(K) = K$ since

$$
\tau(K(g)) = \langle (\tau(K)(g))
= \langle (K(g))
= K(g).
$$

$\tau(K_1 \cdot K_2) = \tau(K_2) \cdot \tau(K_1)$ since

$$
\tau(K_1 \cdot K_2)(g) = \tau((K_1 \cdot K_2)(g))
= \tau(K_1 \cdot \tau(K_2)(gh))
= \left( \sum K_1(h \cdot gh \cdot x^{-1}) K_2(x) \right) \text{ substitute } h \cdot x^{-1} = gh
= \left( \sum K_1(h \cdot gh^{-1} \cdot x \cdot (h) K_2(h \cdot x^{-1} \cdot gh) \right)
= \left( \sum K_1(h \cdot x \cdot (h) K_2(h \cdot (gx^{-1}) \cdot x) \right)
= \sum \tau(K_2)(gx) \tau(K_1(x))
\[ = \sum 'K_2(gx^{-1})^tK_1(x) \]
\[ = 'K_2 * 'K_1(g). \]

The rest of the section is devoted to the proof of 'K = K. This will finish the proof of the theorem.

**Lemma.** There is the set of disjoint representatives of the double cosets \( D \backslash G / D \) written in blocks as follows:

\[
\begin{pmatrix}
0 & 0 & 0 \\
I_n & 0 & 0 \\
0 & L_n & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
I_n & 0 & 0 \\
0 & L_n & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
T_n & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\forall x \in k^*.
\]

**Proof of the Lemma.** We shall look first for disjoint representatives of \( Q \backslash G / Q \). Consider \( p^{2n-1} = p(k^{2n}) \). Denote its elements by \( l(x) \). \( G \) acts from the left transitively on \( p^{2n-1} \) as follows: \( g(l(x)) = l(g(x)) \). \( Q \) is the stabilizer of \( l(e_1) \). So by \( gQ \to l(g \cdot e_1) \), \( G / Q \cong p^{2n-1} \).

\( Q \) acts on \( p^{2n-1} \) as follows: \( q(l(x)) = l(q(x)) \). There are three orbits of \( p^{2n-1} \) under \( Q \):

1. \( \{ l(e_1) \} \).
2. \( \{ l(e) \mid e \in \text{Span}\{ e_1, \ldots, e_{2n-1} \} - \text{Span}\{ e_1 \} \} \).
3. \( \{ l(e) \mid e \notin \text{Span}\{ e_1, \ldots, e_{2n-1} \} \} \).

These are, indeed, three orbits:

1. The first set is a one element orbit since \( q(l(e_1)) = l(q(e_1)) = l(e_1) \).
2. The second set of lines is the full orbit of \( l(e_{2n-1}) \). In fact, it includes the orbit of \( l(e_{2n-1}) \), because \( Q \) preserves \( \text{Span}\{ e_1 \} \) and \( (\text{Span}\{ e_1 \})^\perp \) which equals \( \text{Span}\{ e_1, \ldots, e_{2n-1} \} \). In order to show that it is the full orbit, we have to find for each \( e \in \text{Span}\{ e_1, \ldots, e_{2n-1} \} - \text{Span}\{ e_1 \} \), \( q \in Q \) such that \( qe_{2n-1} = e \).

Let \( e = (x_1, \ldots, x_{2n-1}, 0) \), then

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & x_1 & 0 & \cdots \\
0 & \ddots & \ddots & \cdots & \vdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_n \\
x_{n+1} \\
x_n \\
\vdots \\
x_{2n-1} \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_n \\
x_{n+1} \\
x_n \\
\vdots \\
x_{2n-1} \\
0
\end{pmatrix}
\]
for \( g \in \text{Sp}(2n-2, k) \) such that \( g \cdot (0, \ldots, 0, 1^{n-1}, 0) = (x_2, \ldots, x_{2n-1}) \) and \( \beta = i(g) \). Such \( g \) exist since \((x_2, \ldots, x_{2n-1}) \neq 0 \).

3. The last set of lines is the full orbit of \( l(e_{2n}) \). In fact, \( q' \in \not\in \text{Span}\{e_1, \ldots, e_{2n-1}\} \), because \( Q \) preserves the last subspace, and to each \( e \in \text{Span}\{e_1, e_{2n-1}\}, e = (x_1, \ldots, x_{2n}), x_{2n} \neq 0 \), then

\[
\begin{pmatrix}
1 & -x_{2n-1} & \ldots & -x_2 & x_1 \\
-x_{2n-1} & x_{2n} & \ldots & x_2 & x_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-x_2 & x_2 & \ldots & x_{2n} & x_{2n} \\
-x_1 & x_1 & \ldots & x_1 & x_{2n}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
\vdots \\
x_{2n}
\end{pmatrix}.
\]

Every line in \( p^{2n-1} \) is in one of the three sets, so there are exactly three orbits.

The isomorphism \( G/Q \cong p^{2n-1} \) induces a left action of \( Q \) on \( G/Q \) whose orbits are \( Q(\bar{G}/Q) = G/Q \). The orbit of \( l(e) \) corresponds to the double class of a \( g \in G \) such that \( g(e_1) = e \). In particular, the orbits of \( l(e_1), l(e_{n+1}) \) and \( l(e_{2n}) \) correspond to

1. \( l \).
2. \( K = (1, 0) \).
3. \( J = (0, 1) \).

We shall now look for representatives for \( M \setminus M \) and \( M/M_1 \),

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} = \begin{pmatrix}
\beta I_n & 0 \\
0 & I_n
\end{pmatrix} \begin{pmatrix}
\beta & g' \\
\alpha & g''
\end{pmatrix}
= \begin{pmatrix}
I_n & 0 \\
0 & \beta I_n
\end{pmatrix} \begin{pmatrix}
\alpha & \beta \\
g'' & g'
\end{pmatrix}.
\]

where

\[
\lambda(g') = \frac{i(g)}{\beta} = \beta^2, \quad \lambda(g'') = \frac{i(g)}{\beta} = \alpha.
\]

So \( \alpha^{(2)}, \beta^{(2)} \) are sets of left representatives for \( M_1 \). The same sets are sets of right representatives.
Now, $D = M_1 U = U M_1$, $Q = M U = U M$, so there is the following set of representatives for $D \setminus G / D$:

$$
\begin{pmatrix}
  2I \\
  I
\end{pmatrix}_{s \circ k} \cdot 
\begin{pmatrix}
  2I \\
  I
\end{pmatrix}_{k \circ k} = 
\begin{pmatrix}
  2I \\
  -J
\end{pmatrix}_{s \circ k} \cdot 
\begin{pmatrix}
  2I \\
  I
\end{pmatrix}_{s \circ k}.
$$

Multiplying, we get the representatives

$$
\begin{pmatrix}
  xI_n \\
  0
\end{pmatrix}_{s \circ k} = 
\begin{pmatrix}
  0 \\
  xI_n
\end{pmatrix}_{s \circ k} = 
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}_{s \circ k} = 
\begin{pmatrix}
  0 \\
  xT_n
\end{pmatrix}_{s \circ k} = 
\begin{pmatrix}
  T_n \\
  0
\end{pmatrix}_{s \circ k}.
$$

We shall now prove that for all $K \in A$ and each representative $\eta$, $K(\eta) = K(\eta)$. Then we get the same result for any $g \in G$, $g = \delta_1 \eta \delta_2$, $K(g) = \omega_\phi(\delta_1) K(\eta) \omega_\phi(\delta_2) = \omega_\phi(\delta_1) K(\eta) \omega_\phi(\delta_2) = K(\eta) = K(g)$.

The method used will be the following: For each $\eta$ we shall look for

$$D_\eta = \eta D \eta^{-1} \cap D.$$  Then we shall look at the expressions of the kind $\eta d = d' \eta$ for $d, d' \in D$ (such $d, d'$ are in $D_\eta$), and we shall find conditions on $K(\eta)$ implied by those equations.

For $\eta = (1, z)$, $z \neq 1$, we use only one equation to get $K(\eta) = 0$. For $\eta = I$, we get $D_\eta = D$. For other two types, the situation is somewhat more difficult. For $\eta = (1, z)$ we get $K(\eta) = 0$ unless $n = 2$, $z = t_1$. There $K(\eta)$ vanishes on a subspace of codim 2. For $\eta$ the third type of representation we get that $K(\eta)$ is an automorphism.

((1.1)) $\eta = (1, z)$, $z \neq 1$, $z \in k^*$. Let $z \in k^*$ such that $\psi(z((x-1)/z)) \neq 1$. Such $z$ will give us $\psi(z) \neq \psi(z/z)$ as follows:

$$
\psi(z) \cdot \psi\left(\frac{z}{x}\right) = \psi\left(\frac{z-z}{x}\right) = \psi\left(\frac{z-1}{x}\right) \neq 1
$$

$$
\Rightarrow \psi(z) \neq \psi\left(\frac{z}{x}\right).
$$

We shall use that $z$ in the equation

$$
\eta
\begin{pmatrix}
  1 \\
  1 \\
  1
\end{pmatrix}
= 
\begin{pmatrix}
  1 \\
  z
\end{pmatrix}
\eta.
$$

So
\[ K(\eta) \omega_{\psi} \left( \begin{smallmatrix} 0 & 0 & z \\ 0 & 0 & z \\ \frac{z}{2} & \frac{z}{2} & \frac{z}{2} \end{smallmatrix} \right) = \omega_{\psi}(0 \ 0 \ z) \ K(\eta) \quad \forall K \in A \]

so

\[ \omega_{\psi}(0 \ 0 \ z) = \psi(z) \cdot I_{v} \]

\[ \psi \left( \frac{z}{2} \right) \cdot K(\eta) = \psi(z) \cdot K(\eta) \quad \forall K \in A. \]

* \]

Remembering that \( \phi(z) \neq \psi(z/z) \) we get

\[ K(\eta) = 0 \quad \forall K \in A. \]

In particular, \( 'K(\eta) = K(\eta) \).

(1.2) \quad \eta = I = I_{v}, \ (\delta I = I_{v} \quad \forall \delta \in D. \) Therefore \( \omega_{\psi}(\delta) \ K(I) = K(I) \omega_{\psi}(\delta) \quad \forall \delta \in D \quad \forall K \in A. \) By the Schur lemma \( K(I) = C_{k} I_{v}. \) So \( 'K(I) = 'K(I') = K(I) \).

(2) \quad \eta = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix} \right). \) We shall deal separately with the cases \( n = 2 \) and \( n \geq 3. \)

\[ n = 2. \] \( D_{4} \) then equals

\[ \begin{pmatrix} 1 & x & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x, z, c \in k. \]

We actually need only the next equation.

\[ K(\eta) \omega_{\psi} \left( \begin{smallmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & 1 \end{smallmatrix} \right) = \psi(z) \cdot K(\eta). \]

Substituting alternatively \( x = 0 \) and \( c = 0 \) and using the shorter Heisenberg notations, we get

(i) \( K(\eta) \omega_{\psi} \left( \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{smallmatrix} \right) = \omega_{\psi}(0 \ 0 \ c) \ K(\eta). \)

(ii) \( K(\eta) \omega_{\psi}(x, 0, 0) = \omega_{\psi}(-x, 0, 0) \ K(\eta). \)
In this case the space of representation is actually

\[ V = \{ f : Y \rightarrow C \}, \quad Y = k. \]

\( K(\eta) \) is determined by its values on a basis of \( V \). Take the following basis

\[ \{ \delta_i \mid i \in k \}, \quad \delta_i(x) = \delta_{ix}. \]

From the action of \( \omega_y \) we get that \( \omega_y(0 \circ \delta) f - \psi(z) f, \)

\[ \omega_y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & e & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \delta_i = \psi \left( \frac{ct^2}{2 \tau} \right) \delta_i. \]

Applying (i) on \( \delta_i \), we get 

\[ \psi(x) K(\eta) \delta_i = \psi(ct^2/2) K(\eta) \delta_i. \]

Therefore,

\[ K(\eta) \delta_i = 0 \quad \forall t, \quad t^2 \neq 2x. \]

**2x \neq \text{Square}.** Then 

\[ K(\eta) \delta_i = 0 \quad \forall \delta_i, \quad \forall K \in A \Rightarrow \]

\[ K(\eta) = 0 = \psi K(\eta) \]

\[ \Rightarrow K(\eta) = \psi K(\eta). \]

**2x = \text{Square}.** 

\[ 2x = t_0^2. \]

\[ \omega_s(x, 0, 0) f(x) - \psi(-2x) f(x) \]

\[ \omega_s(x, 0, 0) \delta_i = 2\psi(-2x) \delta_i. \]

Applying (ii) on \( \delta_i \), in \( x \in k \) we get

\[ (K(\eta) (\psi(-2x) \delta_i))(x) = \psi(2sx)(K(\eta) \delta_i)(x) \]

\[ \psi(-2sx)(K(\eta) \delta_i)(x) = \psi(2sx)(K(\eta) \delta_i)(x). \]

Therefore, \( \forall x \neq -t, \quad K(\eta) \delta_i(x) = 0. \) So \( K(\eta) \delta_i = a_{K\eta} \delta_i. \) Therefore, \( \forall K \in A: \)

\[ K(\eta) \delta_i = 0 \quad \forall t \neq t_0. \]

\[ K(\eta) \delta_{-n} = a_{K\eta} \delta_{-n}. \]

\[ K(\eta) \delta_{-n} = a_{K\eta} \delta_{0}. \]

Let \( K \in A. \) Put \( K_1 = K - \psi K. \) We want to show that \( K_1(\eta) = 0. \) We shall prove the following lemma, and the proof for \( n = 2 \) will thus follow.
LEMMA. Let \( K \in A \) such that

\[
K(\tau) \delta_\tau = 0 \quad \forall \tau \neq t_0
\]
\[
K(\eta) \delta_\eta = a_1 \delta_{-a_1}
\]
\[
K(\eta) \delta_{-\eta} = a_2 \delta_{-\eta}.
\]

Assume that \( ^1K = -K \). Then \( K(\eta) = 0 \).

Proof. \( \forall g \in \text{End}(V) \), \( \dim V = k \) there exists an \( F: U \to C \) with the property \( F(uz) = g(z^{-1}) F(u) \) \( \forall z \in Z(u) \). \( G(v) = \sum_{z \in U} F(u) \omega_q(u) v \).

(Because different \( F's \) determine different \( G's \) and \( \dim \text{End}(V) = \dim(\text{End}(V)) \).

If \( K(\eta) \) is determined by \( F \) then \( ^1K(\eta) \) is determined by \( ^1F \). \( ^1F(u) = F(u) \).

and \( -K \) by \( -F \). In fact, \( ^1K(\eta) = \lambda K(\eta) = (K(\eta)) \) since \( \gamma = \eta \).

\[
^1K(\eta) = (K(\eta)) = \left( \sum_{z \in U \cup U'} F(u) \omega_q(u) \right)
\]
\[
= \sum_{z \in U \cup U'} F(u) \omega_q(u)
\]
\[
= \sum_{z \in U \cup U} F(u) \omega_q(u)
\]
\[
= \sum_{z \in U \cup U} (F(u) \omega_q(u)
\]
\[
- K(\eta) = \left( \sum_{z \in U \cup U} F(u) \omega_q(u) \right)
\]
\[
= \sum_{z \in U \cup U} (-F(u) \omega_q(u)
\]

By the conditions of the lemma \( ^1K(\eta) = -K(\eta) \), so \( ^1F = -F \).

\[
K(\eta)(v) = \sum_{x, y, 0} F(x, y, 0) \omega_q(x, y, 0) v.
\]

Since \( (x, y, 0) = (0, y, 0) \lor (x, 0, 0) \lor (0, 0, xy) \),

\[
K(\eta) = \sum_{x, y, 0} F(x, y, 0) \omega_q(x, y, 0) \omega_q(x, 0, 0) \omega_q(0, 0, xy).
\]

\( \omega_q(0, y, 0) f(w) = f(w + y) \). So \( \omega_q(0, y, 0) \delta_\tau = \delta_{\tau - \gamma} \).
Substituting \( \delta, \) for \( v \) in \( K(\eta)(v) \), we get

\[
K(\eta)(\delta, ) = \sum_{x,y \in K} F(x, y, 0) \psi(-2xy) \psi(xy) \delta_x \delta_y
\]

\[
K(\eta)(\delta, )|s = \sum_{x,y \in K} F(x, y, 0) \psi(-2xt) \psi(xy) \delta_x \delta_y(s)
\]

\[
= \sum_{x \in K} F(x, t-s, 0) \psi(-2xt) \psi(x(t-s)).
\]

We know the action of \( K(\eta) \) on \( \delta \), so we get

\[
\sum_{x \in K} F(x, t-s, 0) \psi(-x(t+s)) = 0 \quad \forall (t, s) \neq (t_0, -t_0)
\]

\[
\sum_{x \in K} F(x, 2t_0, 0) = a_1
\]

\[
\sum_{x \in K} F(x, -2t_0, 0) = a_2.
\]

We shall show that \( a_1 = a_2 = 0 \):

\[
-a_1 = -\left( \sum_{x \in K} F(x, 2t_0, 0) \right)
\]

\[
= \sum_{x \in K} -F(x, 2t_0, 0)
\]

\[
= \sum_{x \in K} f(x, 2t_0, 0)
\]

\[
= \sum_{x \in K} F(x, 2t_0, 0)
\]

\[
= \sum_{x \in K} F(x, 2t_0, 0)
\]

\[
= a_1
\]

\[
\Rightarrow a_1 = 0.
\]

So \( a_1 = 0 \). In the same way \( a_2 = 0 \). So \( K(\eta) = 0 \).

Now, \( K \in A \), \( \forall K \in A \) so \( K_1 \in A \) and therefore satisfies the first condition of the lemma. \( K_1 = -K_1 \), so by the lemma \( K(\eta) = 0 \). Therefore, \( K(\eta) = K(\eta) \).

For \( n \geq 3 \) we get that \( K(\eta) \) is simply 0, \( \forall K \in A \), which is not true, in general, for \( n = 2 \).
We do not need to find the whole of $D_\eta$, it is enough to consider the following relation:

\[
\begin{pmatrix}
I_n & 0 & \cdots & 0 \\
0 & I_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_n
\end{pmatrix}
\begin{pmatrix}
0 \\
y \\
\vdots \\
y
\end{pmatrix}
= 
\begin{pmatrix}
I_n \\
y \\
\vdots \\
y
\end{pmatrix}
\begin{pmatrix}
0 \\
y \\
\vdots \\
y
\end{pmatrix}
\begin{pmatrix}
0 \\
y \\
\vdots \\
y
\end{pmatrix}

So,

(i) $K(\eta) \omega_{\eta} \begin{pmatrix}
I_n \\
0 \\
\vdots \\
0
\end{pmatrix} = \omega_{\eta}(0, 0, z) K(\eta)$

(ii) $K(\eta) \omega_{\eta}(0, \ldots, 0, y, 0) = \omega_{\eta} \begin{pmatrix}
I_n \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
y \\
\vdots \\
y
\end{pmatrix}
\begin{pmatrix}
0 \\
y \\
\vdots \\
y
\end{pmatrix} K(\eta)$.

Let $\delta_{n-n-1} \in V$ be the function $\delta_{n-n-1}(w_1, \ldots, w_{n-1}) = 1$ if $w_i = t_i$ and 0 otherwise. $\{\delta_{n-n-1}\}$ is a basis for $V$. We shall show that $K(\eta)(\delta_{n-n-1}) = 0$ for each $t_1, \ldots, t_{n-1}$. From the action of $Sp(2n-2, k)$ on $V$, we get:

I. $\omega_{\eta} \begin{pmatrix}
I_n \\
0 \\
\vdots \\
0
\end{pmatrix} \delta_{n-n-1} = \psi \left( \frac{1}{2} t_1, \ldots, t_{n-1} \right) \delta_{n-n-1}$.

II. $\omega_{\eta}(0, 0, z) f = \psi(z) f$. 
III. \( \omega(f, 0, 0, \ldots, 0, 0, 0) \delta_{n-\ldots-n} = \delta_{n-\ldots-n-\ldots-\ldots-n} \)

IV. \( \omega \psi \begin{pmatrix} 1 & 0 \\ S & \end{pmatrix} \begin{pmatrix} l_{n-1} \\ 0 \\ \end{pmatrix} \begin{pmatrix} w_{1\ldots n} \\ w_{n-1} \\ \end{pmatrix} = \psi(s) f(w_{1\ldots n}, w_{n-1}) \)

From the first relation we get for \( \delta_{n-\ldots-n} \)
\( \psi(A) K(\eta)(\delta_{n-\ldots-n}) = \psi(xz) K(\eta)(\delta_{n-\ldots-n}) \)

\( t_1 \cdot t_{n-1} \neq 2z \)

\( K(\eta)(\delta_{n-\ldots-n}) = 0 \)

\( t_1 \cdot t_{n-1} = 2z \) Substituting \( \delta_{n-\ldots-n} \) in the second relation with \( y \neq 0 \) and arbitrary \( w_{1\ldots n}, w_{n-1} \),

\( K(\eta)(\delta_{n-\ldots-n})(w_{1\ldots n}, w_{n-1}) = \psi(s)(K(\eta)(\delta_{n-\ldots-n}))(w_{1\ldots n}, w_{n-1}) \)

\( t_1 \cdot (t_{n-1} - y) \neq 2z \Rightarrow K(\eta)(\delta_{n-\ldots-n}) = 0 \).

\( \psi(s) \neq 0 \) and, therefore, \( K(\eta)(\delta_{n-\ldots-n})(w_{1\ldots n}, w_{n-1}) = 0 \). \( w_{1\ldots n}, w_{n-1} \) are arbitrary so \( K(\eta)(\delta_{n-\ldots-n}) = 0 \).

(3) \( \eta = \begin{pmatrix} T_n & 0 \\ 0 & T_n \end{pmatrix} \)

\( t_\eta \begin{pmatrix} 0 & 0 \\ T_n & 0 \end{pmatrix} \begin{pmatrix} 0 & T_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & xT_n \\ T_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & xT_n \\ T_n & 0 \end{pmatrix} = \eta. \)

We want to show \( 'K(\eta) = K(\eta) \), but \( 'K(\eta) = 'L(K(\eta)) = 'K(\eta) \). So we need to show \( 'K(\eta) = K(\eta) \).

Let \( m \in Sp(2n-2, k) \) denote \( m^\eta = \eta \eta^{-1} \). Then \( m^\eta \in Sp(2n-2, k) \) and, therefore,

\( \eta = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \eta^{-1} = \begin{pmatrix} 1 & m^\eta \\ 0 & 1 \end{pmatrix} \in M_1. \)

So \( M_1 \subset D \) and

\( K(\eta) \omega \psi \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \omega \psi \begin{pmatrix} 1 & m^\eta \\ 0 & 1 \end{pmatrix} K(\eta). \)
We want to find under what assumptions \( \omega_{\phi} \mid_M \) and \( \omega_{\psi} \mid_M \) are equivalent. 

\( \phi(\eta) = -\eta \), so on \( \text{Sp}(2n - 2, k) \omega_{\phi}^{s} \) and \( \omega_{\psi} \mid_M \) are equivalent with the equivalence operator \( \omega_{\psi_{s}}(t_{-1}^{-1}, -t_{-1}) \). In fact,

\[
\begin{pmatrix}
0 & -T_{s-1} & 0 \\
0 & 0 & -t_{0}s_{-1} \\
T_{s-1} & 0 & 0
\end{pmatrix} = \eta_{s-1}.
\]

\(-2 \neq \text{Square} \). \( \omega_{\psi_{s}} \mid_{M} \) is not equivalent to \( \omega_{\psi} \) as representations of \( \text{Sp}(2n - 2, k) \) [3, 2, 4, d]. So \( \omega_{\psi} \mid_M \) and \( \omega_{\psi} \mid_M \) are not equivalent, and therefore, \( K(\eta) = 0 \), \( \forall K \in A \). In particular, \( K(\eta) = K(\eta) \).

\(-2 = \text{Square} = t_{0} \). \( \omega_{\psi_{s}} \mid_{M} \) is equivalent to \( \omega_{\psi} \) as representations of \( \text{Sp}(2n - 2, k) \) with

\[
\omega_{\psi}(t_{0}I_{n-1}^{-1}, 0 \ 0 \ t_{0}s_{-1})
\]

as an equivalence operator. In fact,

\[
\begin{pmatrix}
1 & 0 \\
0 & -t_{0}s_{-1}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & t_{0}s_{-1}^{-1}
\end{pmatrix} \begin{pmatrix}
0 & -t_{0}s_{-1}^{-1} \\
0 & 0
\end{pmatrix}.
\]

The multiplication of the two operators gives

\[
\omega_{\psi}(t_{0}I_{n-1}^{-1} T_{n-1}, 0)
\]

As a representation of \( \text{Sp}(2n - 2, k) \), \( \omega_{\psi} \) has two irreducible components \( \omega_{\psi}^{s} \) and \( \omega_{\psi}^{s} \) (see [3]). So by Schur lemma, and using the \( M_{\tau} \) notations,

\[
K(\eta) = C_{1, k} \omega_{\psi}^{s} \begin{pmatrix} 1 & -t_{0}T_{n-1} \ 0 & t_{0}I_{n-1}^{-1} \end{pmatrix} + C_{2, k} \omega_{\psi}^{s} \begin{pmatrix} 1 & t_{0}I_{n-1}^{-1} \ 0 & -t_{0}T_{n-1} \end{pmatrix}.
\]

But the involution \( \tau \) fixes
\[
\begin{pmatrix}
1 & -t_0 T_{n-1} \\
-t_0^{-1} T_{n-1} & 1
\end{pmatrix}
\]

So \( K(\eta) = K(\eta) \).

II

Part II of this paper deals with the following situation:
Let \( k \) be a finite field. We take the algebraic group \( G = O(n, k) \), \( n \geq 6 \). Let \( O(n, k) \) act on \( V \); the appropriate symmetric form is denoted by \( ( \ , \ ) \). In \( O(n, k) \) we look at \( Q \) a parabolic subgroup which preserves a one-dimensional subspace. \( Q \) has a Levi decomposition. \( Q = MU \), \( U \) is isomorphic as groups to \( L \), an \( n-2 \) dimensional subvector space. Fix \( \psi \) a non-trivial additive character of \( k \). Fix \( t \in L \). We have a character of \( U \); \( \chi(u) = \psi(\langle u, t \rangle) \) (\( \psi \) regarded as an element of \( L \)). Let \( M_1 = M \) be the connected component of the subgroup that stabilizes \( \chi \). Take \( D = M_1 U \). Extend \( \chi \) to \( D \) by the identity or by \( \text{det} \). We consider \( \text{Ind}^D_{M_1} \chi \) and its irreducible subrepresentations. Unlike the representation we consider in Part I, this representation is not always multiplicity-free, but "almost" multiplicity-free.

We shall start by presenting the groups \( G \) and \( D \).
Let \( k \) be a finite field \( \text{char}(k) \neq 2 \). Let \( T = T_n \) be the \( n \times n \) matrix
\[
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
\]

Let \( G = O(n, k) = \{ g \mid GL(n, k), \ gTg^T = T \} \). Let \( e_i, i = 1, n \), be the standard basis in \( k^n \) written in columns. Denote by \( ( \ , \ ) \) the symmetric form induced by \( T \). \( e_1, e_n \) are isotropic vectors. Let \( Q \) be the subgroup that stabilizes \( e_1 \) up to scalar, i.e., \( Q = \{ g \in G \mid g(\text{Span}(e_1)) = \text{Span}(e_1) \} \). Let \( U = \{ g \in Q \mid g(e_1) = e_1, \forall e \in L, g(e) = e + xe, \text{ for some } x \in k \} \). \( Q = MU \) is a Levi decomposition of \( Q \). In fact
\[
M = \begin{pmatrix}
x \\
g \\
\beta
\end{pmatrix}
\]

\[
\begin{pmatrix}
x T_{n-2} g = T_{n-2} \beta = 1
\end{pmatrix}
\]
and
\[
U = \begin{pmatrix}
1 & x_{n-1} & \cdots & x_2 & 1 \\
x_2 & 1 & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_n & \cdots & x_2 & 1 & 0
\end{pmatrix}
\]
\[
z = \sum_{i=2}^{n-2} x_i x_{n+1-i}
\]

As groups \(M \sim O(L) \times k^*\) and \(U \sim L\) by
\[
u \rightarrow \begin{pmatrix}
0 \\
x_2 \\
\vdots \\
x_{n-1} \\
0
\end{pmatrix}
\]

We identify
\[
\begin{pmatrix}
0 \\
x_2 \\
\vdots \\
x_{n-1} \\
0
\end{pmatrix}
\]
with
\[
\begin{pmatrix}
x_2 \\
\vdots \\
x_{n-1}
\end{pmatrix}
\]
and call the last vector \(\nu\). For every \(m\) in \(M\) we have \(mUm^{-1} = U\). If
\[
m = \begin{pmatrix}
m \\
n \\
x^{-1}
\end{pmatrix}
\]
and \(u \rightarrow \nu\) under the above correspondence then \(mum^{-1} \rightarrow xn(\nu)\).

Fix \(\psi: k^* \rightarrow C^*, \psi \neq 1\). Fix \(t \in L\) such that \((t, t) \neq 0\). Let \(u \rightarrow (u, t)\) be a \(k\)-linear form on \(L\). Let \(\chi: U \rightarrow C^*\) be \(\chi(u) = \psi((u, t))\). Let us look at \(S_\chi(M) = \{m \in M \mid \chi(mum^{-1}) = \chi(u) \forall u \in U\}\). To describe it more explicitly we have to go back to the isomorphism between \(U\) and \(L\). Recall that for
\[
m = \begin{pmatrix}
x \\
n \\
x^{-1}
\end{pmatrix}, \quad mum^{-1} \rightarrow xn(\nu),
\]
So:

\[ St_\chi(M) = \left\{ m \in M \mid m = \begin{pmatrix} x & n \\ \chi^{-1} \end{pmatrix}, \psi((xn(y), t)) = \psi(y t) \forall y \in L \right\} \]

\[ = \left\{ m \in M \mid m = \begin{pmatrix} x & n \\ \chi^{-1} \end{pmatrix}, \psi((xn^{-1}(t) - t)) = 0 \forall y \in L \right\}. \]

Since \((\cdot, \cdot)\) is nondegenerate and \(\psi\) is non-trivial we must have

\[ St_\chi(M) = \left\{ m \in M \mid m = \begin{pmatrix} x & n \\ \chi^{-1} \end{pmatrix}, \psi(t) = 2t \right\}. \]

Since \(n \in O(L)\), \(x\) must be \(\pm 1\). Let \(M_1\) be the connected component of \(St_\chi(M)\). Then

\[ M_1 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in O(L), \psi(t) = t \right\} = \text{Stab}_{O(L)}(t). \]

We consider the following representations of \(D \cdot \chi_{\mu, \lambda}(m)\mu) = \chi_{\lambda}(u)\) \(\chi_{\lambda}(mu) = \det m \cdot \chi(u)\) \(\chi_{\lambda}(um) = \chi(u)\) \(\det m\) are of the form \(mu\) in a unique form and thus \(\chi_{\lambda}\) are well defined. Moreover: If \(d \in D\) is of the form \(um\) then \(\chi_{\lambda}(um) = \chi(u)\) \(\det m\) because \(um = umu^{-1}u\) and \(umu^{-1}\in M\) of the same determinant as \(m\).

We are interested in \(\text{Ind}_{\lambda}^{G} \chi_{\lambda}\). A representation is multiplicity-free if and only if the Schur Algebra is commutative. For the induced representation the Schur Algebra is isomorphic to the algebra \(A\):

\[ A = \{ K : G \to C \mid K(d_1 d_2) = K(d_1) K(d_2), d_1, d_2 \in D \}

K_{\lambda} \star K_{\mu}(g) = \sum_{x \in G} K_{\lambda}(gx^{-1}) K_{\nu}(x). \]

It is clear that if \(A\) admits an involution \(\tau : A \to A\) (i.e., \(\tau \cdot \tau = \text{Id}\), \(\tau(K_{\lambda} \star K_{\mu}) = \tau(K_{\lambda}) \cdot \tau(K_{\mu})\)) which can be proven to be trivial then \(A\) is commutative. This is the method we used in Part 1. Here, we introduce an involution and prove that it is "almost always" trivial. Thus we get that \(\text{Ind}_{\lambda}^{G} \chi_{\lambda}\) is "almost" multiplicity-free.
Let $SO(n)$ satisfy $\tilde{x}(t) = -t$ and $\tilde{x}|_{(1)} = \text{Id}$. Since $t \in L$ we can write

$$\tilde{x} = \begin{pmatrix} 1 & \mathbf{z} \\ \mathbf{z}^T & 1 \end{pmatrix}.$$ 

We regarded elements of $L$ as $(n-2)$-vectors and thus we can assume $x \in O(L)$, $x^2 = \text{Id}$, det $x = -1$. We take the following involution on $G$, $\sigma = k g^{-1} k^{-1}$ $g$. It induces the following involution on $A: K(g) = K(g)$ $K$. We omit the proof that $\sigma$ is an involution because we have a similar proof in Part 1. \( \tau: A \rightarrow A \) is trivial if $K = K$ for every $K$ in $A$. It is easy to see that $K = K$ if and only if $K(g) = K(g)$ for a set of representatives of $D \setminus G/D$. So we shall choose a set of representatives for $D \setminus G/D$ and prove that $K = K$ on "almost all" of them.

**Proposition 1.** Then are three types of representatives for the double coset of $D \setminus G/D$: (I) $m$, (II) $m \eta_2 m'$, (III) $m \eta_3 m'$, where

$$\eta_2 = \begin{pmatrix} 1 \\ \mathbf{1} \\ \mathbf{I}_{n-4} \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 1 \\ \mathbf{1} \\ \mathbf{I}_{n-3} \end{pmatrix},$$

$m \in M_1 \setminus M$ and $m' \in M \setminus M_1$.

**Proof.** We shall first look for a set of representatives for $Q \setminus G/Q$. $G$ acts transitively on $X = \{ e \in k^* | (r, e) = 0 \}$ (as a consequence of Witt's theorem). This induces an action of $G$ and $Q \setminus P^*(X)$. $G$ acts on $P^*(X)$ transitively and $Q$ is then the stabilizer of the line of $e_i$. Thus $P^*(X)$, $G$ acts on $P^*(X)$ does not act transitively on $P^*(X)$, and there are three orbits of this actions: (I) line of $e_i$, (II) $\{ e | (r, e) = 0, r \not\in (e_i) \}$, (III) $\{ e | (r, e) \neq 0, \exists (e_i) \}$. Since $P^*(X)$ is isomorphic to $G/Q$ these orbits are in 1-1 correspondence with $Q \setminus G/Q$. So we have the following three representatives for $Q \setminus G/Q$: (I) $I_n$, (II) $\eta_2$, (III) $\eta_3$, where

$$\eta_2 = \begin{pmatrix} 1 \\ \mathbf{1} \\ \mathbf{I}_{n-4} \end{pmatrix}$$

and

$$\eta_3 = \begin{pmatrix} 1 \\ \mathbf{1} \\ \mathbf{I}_{n-3} \end{pmatrix}.$$
A set of representatives for $D \setminus G/D$ can be written in terms of the above three matrices and the chosen representatives for $M_1 \setminus M$ and $M_1 \setminus M$. Let us take $g \in G$, $g = q_n q'$ for $i = 1$ or 2 or 3, $q, q' \in Q$, $Q = MU = UM$ so $q = u p$, $q' = p' u'$ for $u, u' \in U$ and $p, p' \in M$. Now $p = m_{ij}, p' = m_{i'}^{m_j}$, for $m \in M \setminus M$ and $m' \in M/M_1, m' \in M_1$, and we have $q = u m_{ij}, m' m' ; u'$ where $u m_{ij}, m' u' \in D$. So $\{ m_{ij}, m' \}$, $i, j \in \{ 1, 2, 3 \}, m \in M \setminus M$ and $m' \in M/M_1$, is a set of representatives for $D \setminus G/D$.

Each $m \in M$ is of the form

$$m = \begin{pmatrix} x & n \\ n & x^{-1} \end{pmatrix}$$

for $n \in O(L)$.

$L = \text{Span}(e_1, \ldots, e_{n-1})$ and $x \in k^*$. We write

$$\tilde{m} = \begin{pmatrix} 1 \\ n \\ 1 \end{pmatrix}$$

and

$$\tilde{x} = \begin{pmatrix} 1 \\ \cdots \\ 1 \\ x^{-1} \end{pmatrix}.$$

So $m = \tilde{m} \tilde{x}$. We shall use this form in the forthcoming calculation. The following proposition states that $K = \gamma K$ for “almost all” the representatives.

**Proposition 2.** $K$ equals $\gamma K$ on representatives of type I, type III and representatives of type I1a for which

$$m = \begin{pmatrix} x & n \\ n & x^{-1} \end{pmatrix}, \quad m' = \begin{pmatrix} x' & n' \\ n' & x'^{-1} \end{pmatrix}$$

and $\langle (x n + x'^{-1} n')^{-1} - e_1, t \rangle \neq 0$ or $\langle (x n + x'^{-1} n')^{-1} - e_{n-1}, t \rangle \neq 0$.

*Proof.* Type I, $g = m$.

Type I1a, $g = m$, $m \neq \pm I$.

For each $u \in U m u^{-1} u m \in U$. Since $\langle , , \rangle$ is nondegenerate and $X$ is nontrivial, there exist $u \in U$ such that $\gamma(u) \neq \gamma(m^{-1} u m)$. Let $K e x$. Then

$$\gamma(u) \langle K(m) = K(u m) = K(m^{-1} u m) = K(m) \gamma(m^{-1} u m) \rangle.$$ 

Thus $K(m)$ must vanish. This is true for every $K$ and thus $\gamma K(m) = K(m)$.

Type I1b, $g = \pm I$.

$$\gamma = \delta g \delta^{-1} = g \Rightarrow \gamma K(g) = K(g).$$
Type IIA, $g = mm't$,

$$m = \begin{pmatrix} x \\ n \\ x^{-1} \end{pmatrix}, \quad m' = \begin{pmatrix} x' \\ n' \\ x'^{-1} \end{pmatrix}$$

and $\langle xn + x'^{-1}n'^{-1} e_2, t \rangle \neq 0$ or $\langle (xn + x'^{-1}n'^{-1}) e_2, t \rangle \neq 0$.

Without loss of generality we assume that $\langle (xn + x'^{-1}n'^{-1}) e_2, t \rangle \neq 0$.

$\psi$ is nontrivial so there is $y$ in $\kappa$ such that $\psi(\langle xn + x'^{-1}n'^{-1} e_2, t \rangle) \neq 1$.

So $\psi(\langle xy e_2, t \rangle) \neq \psi(\langle x'^{-1}n'^{-1} (-y) e_2, t \rangle)$.

Under the correspondence $L \to U$, we have

$$xe_2 \leftrightarrow \begin{pmatrix} 1 & 0 & \cdots & y & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & y & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = U(y),$$

where $x(ye_2) \mapsto mu(y) m^{-1}$ and $x'^{-1}n'^{-1} (-ye_2) \mapsto m'^{-1} u(y) m'$. So we have $\gamma(mu(y) m^{-1}) \neq \gamma(m'^{-1}u(y) m')$. We also have $\eta_2 u(-y) = u(y) \eta_2$.

So

$$\gamma(mu(y) m^{-1}) K(g) = \gamma(mu(y) m^{-1}) K(mm_2' mm')$$

$$= K(mu(y) \eta_2 m')$$

$$= K(mn_2 u(-y) m')$$

$$= K(mn_2 m'm'^{-1}u(-y) m')$$

$$= K(mn_2 m') \chi(m'^{-1}u(-y) m')$$

Since $\gamma(mu(y) m^{-1}) \neq \chi(m'^{-1}u(-y) m')$ we get $K(g) = 0$. Thus $K(g) = 0$ for every $K \in A$.

Type III, $g = mm_3 m'$.

We shall first prove the following lemma.

**Lemma.** $(mt, t) = (m't, t) m, m' \in M \Rightarrow m' = m_1 m_2 m_3$ for $m_1, m_2 \in M_1$.

**Proof.** $mt$ and $m't \in L$ since $t \in L$ and $m', m \in M (= O(L))$. So we can put: $mt = x_1 t + x_2$, $m't = x_2 t + x_3$, $(x_1, t) = (x_3, t) = 0$, $x_1, x_2 \in L$.

$(mt, t) = x_1$, $(m't, t) = x_2 \Rightarrow x_1 = x_2$, $(mt, mt) = (x_1^2 (t, t) + (x_1, x_1)$,
\((m't, m') = 2\mathfrak{t}(t, t) \pm (x_2, x_2) \Rightarrow (x_1, x_1) = (x_2, x_2)\). \(M_t\) is transitive on \(\{y \in L, (y, y) = c, (y, t) = 0\}\) there exist \(m_1 \in M_t\) such that \(m_1(x_1) = x_2\).

Look at \(m_1(m't) = m_1(z_1,t + x_1) = x_1, t + x_2 = m't\). This implies that \(m^{-1}m_1m = t\) so \(m^{-1}m_1m \in M_t\Rightarrow m^{-1}m_1m = (m_1')^{-1}, m' = m_1m\).

**Continuation of Proof of Proposition 2.** Let us take \(g = mn_1m'\). Each \(m \in M\) is of the form

\[
\begin{pmatrix}
X \\
N \\
X^{-1}
\end{pmatrix}
\]

for \(x \in k\) and \(n \in O(L)\), and thus \(\eta_1M = M\eta_3\). So \(g\) is in fact of the form \(mn_1\) for some \(m \in M\). Recall that \(\tilde{a}(t) = -t, \tilde{a}_1|_{(5)} = \text{Id}\). \(K(g) = K(\tilde{a}^{-1}g\tilde{a})\). Now

\[
\tilde{a} = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} I_{n-2} \\ & 1 \end{pmatrix},
\]

\[
\tilde{x} = \begin{pmatrix} x \\ I_{n-2} \\ x^{-1} \end{pmatrix}, \quad \tilde{n} = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}
\]

for \(n \in O(L)\).

We have \(\eta_3 - \eta_3^{-1}, \eta_3 \tilde{a} = 2\eta_3, \eta_3 \tilde{x} = \tilde{x}^{-1} \eta_3, \tilde{a} \eta_3^{-1} = \eta_3^{-1} \tilde{a}, \tilde{a} \tilde{x}^{-1} = \tilde{x}^{-1} \tilde{a}, \tilde{x} \eta_3^{-1} = \eta_3^{-1} \tilde{x}, \tilde{x} \tilde{n} = \tilde{n} \tilde{x}^{-1} \eta_3, \tilde{n} \tilde{x}^{-1} = \tilde{x}^{-1} \tilde{n}\). As denoted before \(m = \tilde{a} \tilde{n} \tilde{x}\), so \(3^{-1} \tilde{a} \tilde{x}^{-1} = \tilde{x}^{-1} \tilde{n} \tilde{x}^{-1} \eta_3 = \tilde{x}^{-1} \tilde{n} \tilde{x}^{-1} \eta_3\). Now \((\tilde{a}^{-1} \tilde{n} \tilde{x}^{-1} t), t) = (\tilde{n} \tilde{x}^{-1} t, t) = (\tilde{n} t, t)\). By the above lemma \(\tilde{a}^{-1} \tilde{n}^{-1} \tilde{x}^{-1} = m_1 \tilde{n} \tilde{x}^{-1} \eta_3\) for some \(m_1, \tilde{m}' \in M_t\). So \(\tilde{a}^{-1} \tilde{n} \tilde{x}^{-1} = m_1 \tilde{n} \tilde{x}^{-1} \eta_3\) because \(m_1\) is of the form \(\tilde{n}\) and thus commutes with \(\tilde{a}^{-1}\) and \(\eta_3\). Since \(\eta_3 \tilde{x} = \tilde{x}^{-1} \eta_3\) we actually have \(\tilde{x}^{-1} \eta_3 = m_1 \tilde{n} \tilde{x}^{-1} \eta_3\). Now \(\det(m_1)^{-1} = \det(\tilde{n} \tilde{x}^{-1})\det(\tilde{a}^{-1}) = 1\). Let \(K\) be in \(A\), \(K(\tilde{x}^{-1} \eta_3) = K(m_1 \eta_3 \tilde{n} \tilde{x}^{-1} \eta_3) = X(m_1, m')K(\eta_3 \tilde{x}) = K(g)\). So \(K(\tilde{x}^{-1} \eta_3) = K(g)\) for every \(K\) in \(A\).

Thus for almost every representative of \(D|G|D\), \(K(g) = K(\tilde{x}\eta_3)\). We can rephrase it and get the following: \(\text{Result } \text{Ind}^G_A X\) is almost multiplicity-free.

For the other representatives we also have results concerning equality between \(K(g)\) and \(K(\tilde{g})\). The other \(g\)'s are \(g\)'s of type \(\text{IIb}\), i.e., \(g = mn_1m'\),

\[
m = \begin{pmatrix} X \\
N \\
X^{-1} \end{pmatrix}, \quad m' = \begin{pmatrix} X' \\
N' \\
X'^{-1} \end{pmatrix}
\]
and \((x^n + x^{n-1} + \cdots + x^{n-k+1}) \cdot e_2, \tau) = \langle (x^n + x^{n-1} + \cdots + x^{n-k+1}) \cdot e_{n-1}, \tau \rangle = 0 \). We shall consider now such \(g\)'s. For such \(g\)'s we will neither prove nor disprove \(K(g) = K_G(g)\) for every \(K\), but we found conditions on \(m\) and \(m'\) to ensure that \(g\) and \(g'\) are in the same double coset of \(D/G; D\), i.e., \(g = g_d \cdot g_d^{-1}\) with \(g_d \cdot g_d^{-1} = 1\). This is clearly enough to have \(K(g) = K_G(g)\) for every \(K\) in \(A\).

In fact if there exist \(K \in A\) such that \(K(g) \neq 0\) then this condition turns out to be a necessary condition to have \(K(g) = K_G(g)\). Since for \(g\)'s of type IIb we do not have \(K(g) = 0\) for every \(K\), it is natural to look for conditions on such \(g\)'s to have \(g\) and \(g'\) in the same double coset.

Next we shall state the theorem concerning elements of type IIb. For that purpose we remark that every element of \(O(e_2, \ldots, e_{n-1}, \tau)\) that stabilizes \(\text{Span}\{e_2\}\) is of the form

\[
\begin{pmatrix}
  r & -r & 0 & C & 0 \\
  0 & C & -r & C_T & C_1 \cdot C_T / 2 \\
  0 & 0 & 0 & r^{-1} & 0
\end{pmatrix}
\]

where \(C \in \text{Span}\{e_2, \ldots, e_{n-2}\}\). We denote such an element by \(S(r, C, C_1)\). Such matrices satisfy \(S(r, C, C_1) S(r', C', C_1') = S(r r', C C', C_1 + r^{-1} C_1 C')\). So \(S(r, C, C_1)^{-1} = S(r^{-1}, -r^{-1} C_1 C_T)\).

We shall refer to \(C_1\) as an \(n - 4\) vector or as an \(n - 2\) vector with zero 2nd and \((n - 1)\text{st}\) coordinate, as necessary.

**Proposition 3.** Let \(g\) be of type IIb, i.e., \((x^n + x^{n-1} + \cdots + x^{n-k+1}) \cdot e_2, \tau) = 0\), \(k = 2, n - 1\). Then there exist \(d, d' \in D\) such that \(g = g d d'\), \((d, d') = 1\) if and only if there exist \(C_1 \in \text{Span}\{e_2, \ldots, e_{n-2}\}\), \(n_1, n_1' \in O(e_2, \ldots, e_{n-2})\), \(d_1 \in \text{Span}\{e_2, \ldots, e_{n-2}\}\) such that \((1) n_1 = n_1' S((x^n)^{-1}, A, A_1) n^{-1}\) stabilizes \(\tau\). (2) \(P = S((x^n)^{-1}, A, A_1)\) stabilizes \(n^{-1}\).

**Proof.** We shall divide the proof into a few lemmas.

**Lemma 1.** The existence of \(d_1, d_2 \in D\) s.t. \(d_1 d_2 = 1\) is equivalent to the existence of \(y, y' \in L\) and \(n_1, n_1' \in \text{Stab}_{\tau}(\tau)\) such that the following eight equations are satisfied:

\[
\begin{align*}
(a) & \quad 0 = x^{-y} \cdot y T n_1 n e_2, \\
(b) & \quad 0 = -x^{-y} \cdot y T n_1 n e_{n-1} + \frac{1}{2} (y', y') x' \cdot y T n_1 n e_2 + x' e_2 n' n' y' - y' T n_1 n K n' n' y', \\
(c) & \quad 0 = -x^{-y} \cdot e_{n-1} n' b_{y'}, \\
(d) & \quad x^{-y} \cdot e_{n-1} n' = x^{-y} y T n_1 n e_2 y' T + x' e_2 n' n' - y' T n_1 n K n' n' y', \\
(e) & \quad x n_1' \cdot e_{n-1} = x^{-y} n_1 n e_{n-1} - \frac{1}{2} (\mu, \mu') x' n_1 n e_2 + n_1 n K n' n' y' + x^{-y} \cdot e_{n-1} n' n' y'.
\end{align*}
\]
(f) \( x^T e_{n-1} n^{-1} \gamma_2 = x^{-1} n^{-1} e_{n-1} n'b' \).

(g) \( x^{-1} n^{-1} e_2 = x'n n'e_2 \).

(h) \( x^n n^{-1} Kn^{-1} \gamma = -x'n n'e_2 \gamma T + n_n nK_n n_i + x^{-1} n^{-1} e_{n-1} n'b' \).

Proof of Lemma 1.

\[
\eta_{e_2}(e_{n-1} K e_{n-1}) = 0
\]

where \( e_2, \ldots, e_{n-1} \) are written as the \( n-2 \) coordinate vectors in \( L \) and

\[
K = \begin{pmatrix} 0 & I_{n-1} \\ E_{n-1} & 0 \end{pmatrix}.
\]

We have

\[
g = \begin{pmatrix} x^n e_2 & x^n n' e_{n-1} \\ nKn^{-1} e_2 & n^{-1} n'e_{n-1} \end{pmatrix}
\]

and thus

\[
g^{-1} = \begin{pmatrix} 0 & x^{-1} n^{-1} e_2 & x^{-1} n^{-1} \gamma \end{pmatrix}
\]

and thus

\[
\gamma = 2g^{-1} \gamma = \begin{pmatrix} 0 & x^{-1} n^{-1} e_2 & x^{-1} n^{-1} K n^{-1} \gamma \end{pmatrix}
\]

Every \( u \in U \) is determined by \( y \in L \) as follows:

\[
u = \begin{pmatrix} 1 - \gamma \mu & -\mu \gamma/2 \\ \mu & \mu \end{pmatrix}
\]

Every \( m_i \in M_i \) is of the form \( \hat{m}_i \) for \( n_i \in \text{Stab}_{n \in L}(i) \). An element \( d \in D \) is of the form \( d = m \mu = d' m' \) for some \( m_1, m_2 \in M_i \) and \( u, u' \in U \). Thus a typical
element of the form \( g dg' \) is of the form \( u_{1}g_{1}u'_{1} \) for some \( u, u' \in L \) and \( n_{1}, n_{1}' \in \text{Stab}(L, T) \). Multiplications of matrices give us

\[
dgd' = \begin{pmatrix}
-x'yt_{n_{1}}ne_{2} & x'y_{T_{n_{1}}}ne_{2} & y'T \\
+x'v_{e_{2}n_{1}'} & -y'T_{n_{1}}nK'n_{1}' & -\frac{1}{2}x^{-1}(u'u)\epsilon_{v_{e_{2}n_{1}'}}n_{1}' \\
-x'n_{1}ne_{2} & -x'n_{1}ne_{2} & -\frac{1}{2}(y'u)\epsilon_{v_{e_{2}n_{1}'}}n_{1}' \\
+x'v_{e_{2}n_{1}'} & -yT_{n_{1}}nK'n_{1}' & -\frac{1}{2}(y'u)\epsilon_{v_{e_{2}n_{1}'}}n_{1}' \\
0 & x^{-1}v_{e_{2}n_{1}'n_{1}'} & -\frac{1}{2}x^{-1}(u'u)\epsilon_{v_{e_{2}n_{1}'n_{1}'}n_{1}'} \\
\end{pmatrix}
\]

\( g = g dg' \) iff all the blocks are equal so the lemma follows. \( \square \)

**Lemma 2.** \( n_{1} \in \text{Stab}(t) \) satisfies (g) if and only if there exist \( A \in O(e_{3}, ..., e_{n-2}) \) and \( A_{1} \in \text{Span}(e_{3}, ..., e_{n-2}) \) such that \( n'n_{1}n = S((xx')^{-1}, A, A_{1}) \).

\( n_{1}' \in \text{Stab}(t) \) satisfies (f) if and only if there exist \( B \in O(e_{3}, ..., e_{n-2}) \) and \( B_{1} \in \text{Span}(e_{3}, ..., e_{n-2}) \) such that \( n'n_{1}'n = S((xx')^{-1}, B, B_{1}) \).

**Proof.** Taking the transpose on two sides of (f) gives the following equivalent form of (f):

\[ x't'\epsilon_{e_{n-1}} = x^{-1}(n')^{-1}\epsilon_{e_{n-1}}. \]

We use \( T_{e} = v_{e_{1}} \), \( n^{-1}T = Tn \) and \( n'T = Tn \) and get

\[ x'Tne_{2} = x^{-1}T_{n}n^{-1}e_{2}. \]

Since \( T \) is nonsingular we get the following equivalent form of (f):

\[ x'nne_{2} = x^{-1}n^{-1}e_{2}. \]

We change sides and use \( x = \text{id} \) to get

\[ xne_{2} = an^{-1}x^{-1}n^{-1}e_{2}. \]
Since $n'_1$ stabilizes $i$ it stabilizes $\langle i \rangle^+$ and thus it commutes with $x$ and we get the following equivalent form of (f)

$$(n'n'_1) e_2 = (x^x)^{-1} e_2.$$  

So, $n'_1$ satisfies (f) or equivalently the last equation if and only if there exist $B \in O(e_{-1}, e_{-2})$ and $B_1 \in \text{Span}(e_{-1}, e_{-2})$ such that $n'n'_1 n = S((x^x)^{-1}, B, B_1).$

It is immediate that (g) is equivalent to

$$(n'n_1 n) e_2 = (x^x)^{-1} e_2.$$  

It is in $O(e_{-1}, e_{-2}, T)$ so it is of the form $S((x^x)^{-1}, A, A_1)$ for $A \in O(e_{-1}, e_{-2})$ and $A_1 \in \text{Span}(e_{-1}, e_{-2}).$

**Lemma 3.** Given $n_1$ and $n'_1$ in $\text{Stab}(i)$ such that $n'n_1 n = S((x^x)^{-1}, A, A_1)$ and $n'n'_1 n = S((x^x)^{-1}, B, B_1).$ Then $n_1, n'_1, y$ and $y'$ satisfy (a)-(c) if and only if

$$y = (-x^{-1} b_{-1} T + b' e_{-1}) n^{-1} n_1^{-1} T$$
$$y' = n_1^{-1} n'^{-1} (-x^{-1} b_{-1} A^{-1} A_{-1} + a' e_2),$$

where $xa' - x^{-1} b = x^{-1} b_{-1} A_{-1}.$

**Proof.** Any $y, y' \in \text{Span}(e_{-1}, e_{-2})$ can be written in the form

$$(b' e_2 + b e_{-1}) n^{-1} n_1^{-1} T,$$

where $b \in \text{Span}(e_{-1}, e_{-2})$ and $b, b' \in k$

$$(a' e_2 + a e_{-1}),$$

where $a \in \text{Span}(e_{-1}, e_{-2})$ and $a, a' \in k.$

(a) is satisfied if and only if $b' = 0,$

(c) is satisfied if and only if $a = 0,$

(b) is satisfied if and only if $-x^{-1} b + xa' - b g = 0,$

(d) + (a) is equivalent to

$$e_2 (n'n_1 n)^{-1} = xx' e_2 - x' y n_1 n K - \frac{1}{2} x x'^{-1} \langle y, y' \rangle e_{-1}.$$  

We assume that $n'n_1 n = S((x^x)^{-1}, B, B_1),$ so

$$e_2 (n'n_1 n)^{-1} = xx' e_2 + b_{-1} T - \frac{1}{2} \langle B_1, B_1 \rangle e_{-1}.$$
So (d) + (a) is satisfied if and only if

\[-x'Tn_1 nK = -B_1 T \quad \text{and} \quad x'(u, u) = x'(-1)\langle B_1, B_1 \rangle.\]

Substituting the above form of \(y\) we get that (d) + (a) is satisfied if and only if

\[-x'b = -x'(-1)\langle B_1, T \rangle \quad \text{and} \quad x'(b, b) = x'(-1)\langle B_1, B_1 \rangle.\]

Thus (d) + (a) is satisfied if and only if \(y = -x'(-1)\langle B_1, T \rangle\). Similarly (e) + (c) is satisfied if and only if \(y = -x'(-1)\langle A_1, A_1 \rangle\).

When we sum up the above results we get the lemma.

**Lemma 4.** \(n_1, n'_1, u, u' \) satisfy (a)-(b) if and only if there exist \(A \in O(e_{1}, \ldots, e_{n-2})\) and \(A_1, B_1 \in \text{Span}(e_{1}, \ldots, e_{n-2})\) such that \(n_1n_2 = S(x'A)\), \((A_1, A_1)\), \(n'_1n_2 = S(x'A)\), \((A_1, A_1)\), \(y = -x'(-1)\langle B_1T + b'e_{n-1} \rangle n_1^{-1}T\), \(y' = n'_1n_2^{-1}\langle a'e_{2} - x'(-1)\langle A_1, A_1 \rangle \rangle \) where \(x'a' = x'(-1)b = x'(-1)\langle B_1T, A_1, A_1 \rangle \).

**Proof.** By Lemma 2, \(n_1, n'_1, u, u' \) satisfy (f) and (g) if and only if there exist \(A, B \in O(e_{1}, \ldots, e_{n-2})\) and \(A_1, B_1 \in \text{Span}(e_{1}, \ldots, e_{n-2})\) such that \(n_1n_2 = S((xx')^{-1}, A_1, A_1)\) and \(n'_1n_2 = S((xx')^{-1}, B_1, B_1)\). By Lemma 3 these \(n_1, n'_1, u, u' \) satisfy (a)-(c) if and only if \(y = -x'(-1)\langle B_1T + b'e_{n-1} \rangle\), \(y' = n'_1n_2^{-1}\langle a'e_{2} - x'(-1)\langle A_1, A_1 \rangle \rangle \) where \(x'a' = x'(-1)b = x'(-1)\langle B_1T, A_1, A_1 \rangle \).

Assume \(n_1, n'_1, u, u' \) satisfy (a)-(g).

We shall look for extra conditions on \(n_1, n'_1, u, u'\) to satisfy (h), too. Using (f) and (g), (h) is equivalent to

\[K = (n_1n_2)K(n_1n_2, n) + x'n'_2uy'\langle e_{n-1} - x^{-1}e_{2}, y'Tn_2 \rangle.\]

Since \(n_1, n'_1, u, u' \) satisfy (a)-(c) we get by Lemma 3 that (h) is equivalent to

\[K = (n_1n_2)K(n_1n_2, n) + x'n'_2uy'(x'e_{n-1} - x^{-1}A_1, A_1) \langle n_1n_2, n \rangle.\]

Since \(n_1, n'_1 \) satisfy (f), (g) we get by Lemma 2 that (h) is equivalent to

\[K = S((xx')^{-1}, A_1, A_1)K((xx')^{-1}, B, B_1) + x'S((xx')^{-1}, A_1, A_1)(b'e_{n-1} - x'(-1)\langle B_1, B_1 \rangle)\langle n_1n_2 \rangle - x'e_{2}(a'e_{n-1} - x'(-1)\langle A_1, A_1 \rangle)S((xx')^{-1}, B, B_1).\]
We multiply matrices and get that (h) is equivalent to

\[
K = \begin{pmatrix}
0 & -(xx')^{-1}A_1TAB & -(xx')^{-1}A_1TAB_1 \\
0 & AB & AB_1 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & -A\hat{B}_1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 & -(xx')^{-1}A_1TAB & -(xx')^{-1}A_1TAB_1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

or

\[
K = \begin{pmatrix}
0 & 0 & +x^{-1}b - x'a + (xx')^{-1}A_1TAB_1 \\
0 & AB & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

So (h) is satisfied if and only if \(AB = I\) and \(x^{-1}b - x'a = -(xx')^{-1}A_1TAB_1\). The last equation is a scalar multiplication of the equation in the lemma. If we sum up all the results we get the lemma.

Proof of the Proposition 3. Assume that the first condition of the proposition is satisfied. By Lemma 1 there exist \(n, n_1 \in \text{Stab}_2(I)\) and \(a, b, c \in L\) such that (a)-(h) are satisfied. By Lemma 4 there exist \(\hat{A} \in O(e_1, \ldots, e_{n-2}), \hat{B}_1 \in \text{Span}(e_1, \ldots, e_{n-2})\) such that \(n' \equiv n \equiv S((x')^{-1}, A, \hat{A}_1)\) and \(n' \equiv n \equiv S((x')^{-1}, A, \hat{B}_1)\). Thus \(n = 2n' - S((x')^{-1}, A, \hat{A}_1)n - t\) stabilizes \(I\) and claim 1 is satisfied.

Denote \(S_1 = S((x')^{-1}, A, \hat{A}_1), S_2 = S((x')^{-1}, A, \hat{B}_1)\). Then \(n' \equiv n \equiv S_1, n' \equiv n \equiv S_2\) and \(n(t) = n_1(t) = n_1(t) = t\) and \(n'(t) = t\). So \(S_1(n'(t)) = -n'(t)\) and \(S_2(n'(t)) = -n'(t)\). So \(S_1, S_2, S_1^{-1} = S_1, S_2^{-1} = S_1^{-1} = -n'(t)\). Let \(\hat{C}_1 = (x')^{-1}(-A^{-1}A_1 + \hat{B}_1)\). Then \(S_1, S_2, S_1^{-1} = -n'(t)\) and claim 2 is satisfied.

Assume the second condition of the theorem is satisfied. Let \(n = 2n' - S((x')^{-1}, A, A_1)n - t\). Let \(n' \equiv n \equiv Pn_1 \equiv n_1 \equiv S((x')^{-1}, A, A_1)\) and \(n' \equiv n \equiv Pn_1 \equiv n_1 \equiv S((x')^{-1}, A, A_1)\). Denote \(A = A^{-1}A_1 + xx'\hat{C}_1\). Let \(y = t((x')^{-1}B_1T + b \equiv e_{n-1}n^{-1}a^{-1}T_h, y = n_1^{-1}n^{-1}a^{-1}(-x')^{-1}A_1 + a'\epsilon_{n-1})\) for \(a'\) and \(b\) that satisfy \(x'a' = xx'\equiv b' \equiv (x')^{-1}B_1T, A_1\). For every \(b\) there is an \(a'\) that satisfies this equation.

By Lemma 4, \(n_1, n_1', y, y'\) satisfy (a)-(h).
By Lemma 1 there exist $d, d \in D$ such that $q = d_1 g d_2$. Furthermore $d = u_1 \delta$, $u = \delta'; u$. Thus $q(d_1 d_2) = \det \delta'; \psi'([u T]) \det \delta'; \psi'([u' T]) = \det(n_1, n_2)$

$\psi'([u + u']) T = \det(zn^{-1} P \lambda zn') \psi'([u + u'] T) = \det P(d_1) = 2$

$\psi'([u + u'] T) = \det A^{-2} \det A' \psi'([u + u'] T) = \psi'([u + u'] T) = (-x^C T - x^{-1} (\lambda A', T A (\lambda A^{-1} T + n'(T))) + \lambda n^{-1} (bn^{-1} T + a (n'(T)))$.

Since there is some freedom of choice of $a'$ and $b$ there are $a'$ and $b$ that would give $\psi(d_1 d_2) = 1$.

ACKNOWLEDGMENTS

I would like to express my gratitude to Professor Piatetski Shapiro for suggesting these theorems and for his helpful remarks. I would also like to express my gratitude to the Institute for Advanced Study for its hospitality during the first part of this work.

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Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium