Finite fundamental groups, free over $\mathbb{Z}/c\mathbb{Z}$, for Galois covers of $\mathbb{C}P^2(a)$

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Introduction, construction of the surfaces, formulation of the main results

In this article we give an explicit description of finite fundamental groups of certain surfaces with positive index. These surfaces were constructed in [MoTe1].

We recall the construction. Take $X = \mathbb{C}P^1 \times \mathbb{C}P^1$. Let $\ell_i \subset X$, $\ell_i = \mathbb{C}P^1$, $i = 1, 2$; $\ell_1, \ell_2$ meeting transversally. Let $E = a_1 \ell_1 + b_2 \ell_2$, $a, b \in \mathbb{N}$. Let $X_{ab}$ be the embedding of $X$ into $\mathbb{C}P^2$ with respect to the linear system $|E|$. Take a canonical projection $f$ of $X_{ab}$ to $\mathbb{C}P^2$, $n = \deg(f) = 2ab$. Let $Y_{ab} = \text{Gal}(X_{ab})$ be its Galois cover that corresponds to the full symmetric group $S_n$. $Y_{ab}$ are the surfaces we studied in [MoTe1].

In [MoTe1, (0.2)] we computed the index of these surfaces and we got that

$$\tau = \tau(Y_{ab}) = (1/3)(2ab)(ab - 3a - 3b + 5).$$

Thus $\tau > 0$ for $a \geq 6$, $b \geq 5$.

Let $C$ be the "generic" affine part in $\mathbb{C}P^2$. Let $Y_{ab}^{\text{aff}}$ be the part of $Y_{ab}$ that lies over $C$. In [MoTe1] we proved that $x^a = x^b = 1 \forall x \in \pi_1(Y_{ab}^{\text{aff}})$. For $a, b$ relatively prime, we thus got $\pi_1(Y_{ab}^{\text{aff}}) = 0$ and thus $\pi_1(Y_{ab}) = 0$.

In this work we prove two theorems:

Theorem 10.1. The fundamental group $\pi_1(Y_{ab}^{\text{aff}})$ is a finite abelian group on $n - 1$ generators each of order $\text{g.c.d.}(a, b)$ and there are no further relations.

Theorem 10.2. The fundamental group $\pi_1(Y_{ab})$ is a finite abelian group on $n - 2$ generators each of order $\text{g.c.d.}(a, b)$ and there are no further relations.

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The basic idea and strategy of the proof of these results can be outlined as follows:

Denote by $S$ the branch curve of $f : X_{ab} \to \mathbb{CP}^2$. Using the braid monodromy technique [Mo2, Mo3] and results of [MoTe1], we write a finite presentation of $\pi_1(\mathbb{CP}^2 - S, *)$, with so-called good (or 'geometric') generators $\{f_I\}$.

Let $S_n$ be the symmetric group on $n$ letters. There is a canonical surjection

$$\psi : \pi_1(\mathbb{CP}^2 - S, *) \to S_n.$$  

Indeed, $\text{Ker}\psi$ is isomorphic to $\pi_1(Y_{ab} - f^{-1}(S))$ where $f : Y_{ab} \to \mathbb{CP}^2$ is the map canonically corresponding to $f : X_{ab} \to \mathbb{CP}^2$. Denote by $\langle F^2 \rangle$ the normal subgroup of $\pi_1(\mathbb{CP}^2 - S, *)$ (normally) generated by squares $\{F^2_i\}$ of geometric generators $\{f_I\}$. It is not difficult to see that actually $\langle F^2 \rangle$ is contained in $\text{Ker}\psi$ and that $\text{Ker}\psi/\langle F^2 \rangle$ is isomorphic to $\pi_1(Y_{ab})$. So to get our result we prove that $\text{Ker}\psi/\langle F^2 \rangle$ [which is an important geometrical invariant of $\pi_1(\mathbb{CP}^2 - S)$] is a finite abelian group. There are some indications that the last fact reflects much more general phenomena in the theory of algebraic surfaces (see [MoTe1, Introduction]). This work is related to the problem of the classification of surfaces of general type, where we expect to find new invariants distinguishing different components of moduli spaces.

The groups $\pi_1(Y_{ab})$ and $\pi_1(Y_{ab}^*)$ are different because the finite presentation of $\pi_1(\mathbb{CP}^2 - S)$ in terms of a finite good set of geometric generators $\{f^*_I\}$ has one more relation than $\pi_1(\mathbb{CP}^2 - S)$, namely $\|F^2 = 1$.

All the results from Chaps. 1–5 in [MoTe1] are true for arbitrary $a, b$. Following the proof of Theorem 6 [MoTe1] we see that for arbitrary $a, b$ we have for every $x \in \pi_1(Y_{ab}^*)^*$, $x^c = 1$, $c = \text{g.c.d.}(a, b)$ [For $a, b$ relatively prime we thus got $\pi_1(Y_{ab}^*)^* = 1$.] In this paper we shall compute $\pi_1(Y^*)$ explicitly for arbitrary $a, b$ (Theorem 10.1), and we shall also compute $\pi_1(Y_{ab})$ (Theorem 10.2).

We denote $\pi_1(Y_{ab}^*)$ by $\mathcal{A}^*$.

Proofs of the theorems

1. Reidemeister-Schreier method (RMS method)

We recall the Reidemeister-Schreier method (RMS method) from Magnus, Combinatorial group theory, p. 39. Consider the following situation:

(a) A short exact sequence $1 \to \mathcal{A} \to G \overset{\beta}{\longrightarrow} S \to 1$ of groups.

(b) $\{g_i\}_{i=1}^k$, a finite set of generators for $G$.

(c) $\{R\}$, a finite complete set of relations for $G$.

(d) $\psi$, a splitting of $\beta$.

The RMS method gives us a finite presentation of $\mathcal{A}$ as follows: Denote $\bar{x} = \psi(x)$. Then a set of generators for $\mathcal{A}$ is:

$$\{g(\bar{x})g_i^{-1}g(\bar{x})^{-1} | g \in G, i \in I\}.$$  

A complete set of relations for $\mathcal{A}$ is induced from $\{R\}$ as follows: Each $R = g_1 \cdots g_k$ induces the following relation on $\mathcal{A}$:

$$A(R) = \prod A_i(R) \text{ where } A_i(R) = \bar{g}_1 \cdots \bar{g}_{i-1}g_i\bar{g}_{i}^{-1} \cdots \bar{g}_k^{-1}.$$  

We are going to compute $\mathcal{A} = \pi_1(Y_{aff})$ by using the RMS method. In the next few sections we shall show that $\mathcal{A}$ can be a part of the situation described above.

2. Introducing a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$, $S$ the branch curve of $f$.

Proposition 2 [proof in [MoTe1]]. The projection $f: X \to \mathbb{C}^2$ can be degenerated into $f_0: X_0 \to \mathbb{C}^2$, where $X_0$ is a union of $n$ planes ($n = 2ab$) as in the following configuration:

![Diagram](image)

Fig. 1. Global numeration of triangles (here $a = 4$, $b = 3$)

Each triangle is a plane.

We shall numerate the intersection lines between the planes as follows:

![Diagram](image)

Fig. 2. Global numeration of intersection lines (here $a = 4$, $b = 3$)

The union of these lines is $S_0$, the "branch curve" of $f_0$. $S_0 = \bigcup \{p \text{ lines} \}$, $p = 3ab - a - b$.

Let $\pi: \mathbb{C}^2 \to \mathbb{C}$ be the projection on the first coordinate. Let $M \subseteq S \subseteq \mathbb{C}^2$ be the points of $S$ where $\pi_{|S}$ is not etale. Let $M = \pi(M')$ ($M$ is finite). Let $u \in \mathbb{C} - M$. $(\pi'|u)^{-1}(u)$ is a "good" fibre. Let us take $u$ real, "far enough" from the "bad" points. Let $u_0$ be a point in $\mathbb{C}^1 = \pi^{-1}(u)$, $\nu_0 \neq S$. $S_0 \cap \mathbb{C}^2$ has $p$ points. $S_0 \cap \mathbb{C}^2 = \{L_1, L_2, \ldots, L_p\}$, When re-degenerating lines are "doubled" and thus $S \cap \mathbb{C}^1$ has $2p$ points. Let $q_i, q_j$ be $2$ points in $S \cap \mathbb{C}^1$ that arise from the line $L_i$ in $S_0$, i.e. $(q_i, q_j)$ is the intersection of $\mathbb{C}^1$ with the re-degenerated object of $L_i$. We then have $S \cap \mathbb{C}^2 = \{q_i, q_j, \ldots, q_p, q_j\}$.

Let $I_i = y_i\eta_i^{-1}$ where $y_i$ is a small circle around $q_i$, and $\eta_i$ a path from $u_0$ to $q_i$. Let $I_j = y_j\eta_j^{-1}$ be a path around $q_j$, accordingly. Let us take $\{I_i, I_j\}_{i=1}^{p}$, that meet each other in $u_0$ only. $\{I_i, I_j\}_{i=1}^{p}$ is called a good system of generators for $\pi_1(\mathbb{C}^1 - S, u_0)$.

We have a surjection:

$$\pi_1(\mathbb{C}^1 - S, u_0) \to \pi_1(\mathbb{C}^2 - S, u_0) \to 1$$

$$(\nu(I_i), \nu(I_j))_{i=1}^{p}$$ generates $\pi_1(\mathbb{C}^2 - S, u_0)$. By abuse of notation we shall denote $\nu(I_i)$ by $I_i$ and $\nu(I_j)$ by $I_j$. Thus $\{I_i, I_j\}_{i=1}^{p}$ generates $\pi_1(\mathbb{C}^2 - S, u_0)$.

The branch curve $S_0$ has 3 types of singular points, namely: "6-point", "3-point", and "2-point". A "6-point" is an intersection point of 6 planes. A
"3-point" is an intersection point of 3 planes. We number locally the lines around a "6-point" by 1...6 as in Fig. 3. Around a "3-point" the lines are numbered with respect to the numeration around a "6-point" (see Fig. 4).

3 Introducing a set of generators for $\pi_2(\mathcal{C}^1 - S, u_0)/\langle \Gamma^2_1, \Gamma^2_2 \rangle$

When regarding $\Gamma^2_1$ and $\Gamma^2_2$ as elements of $\pi_2(\mathcal{C}^2 - S, u_0)/\langle \Gamma^2_1, \Gamma^2_2 \rangle$ they form a set of generators for $\pi_2(\mathcal{C}^2 - S, u_0)/\langle \Gamma^2_1, \Gamma^2_2 \rangle$ each of order 2.

4 Van Kampen's method for obtaining a set of relations for $\pi_1(\mathcal{C}^2 - S, u_0)$ and for $\pi_1(\mathcal{C}^2 - S, u_0)/\langle \Gamma^2_2 \rangle$

There is a classical theorem of Van Kampen from the 30's that all the relations in $\pi_1(\mathcal{C}^2 - S, u_0)$ come from the Braid group via the braid monodromy (see [V, Z or Mo1, p. 127]).

Let $B_{2p} = B_p(\mathcal{C}^1, \mathcal{C}^1 \cap S)$ be the Braid group of $\mathcal{C}^1$ and $\mathcal{C}^1 \cap S$. There is a natural defined homomorphism $\Theta : \pi_1(\mathcal{C}^1 - M, u) \to B_{2p}$ which is called the braid monodromy with respect to $S \subset \mathcal{C}^1$.

The map $f$ is generic, and so for $c_i \in M, \pi^{-1}(c_i) \cap M$ is a single point, say $c_i'$ where $c_i'$ is either a nonsingular tangent point of $S$, a node or a cusp. Let $\{s_i\}_{1 \leq i \leq p}$ be a good system of generators for $\pi_1(\mathcal{C}^1 - M, u)$ ordered naturally going in a positive direction around $u$.

By Van Kampen's theory each of the braids $\Theta(s_i)$ induce a relation on $\pi_1(\mathcal{C}^2 - S, u_0)$. The relation is of one of the following types: (1) $A = B$; (2) $AB = BA$ (i.e. $[A, B] = 1$), (3) $AB = BAB$ (i.e. $[A, B] = -1$). All these relations form a complete set of relations for $\pi_1(\mathcal{C}^2 - S, u_0)$. When adding $\Gamma^2_1 = \Gamma^2_2 - 1$, we get a complete set of relations for $\pi_1(\mathcal{C}^2 - S, u_0)/\langle \Gamma^2_1, \Gamma^2_2 \rangle$.

We shall not quote here the formulas for the braid monodromy $\Theta$, which we computed in [Mo1], nor shall we quote the exact list of relations obtained from $\Theta$, which appeared in [MoTe1]. Where necessary we shall refer the reader to
Galois covers of \( \mathbb{C}P^2 \)

[MoTel1]. The relations are arranged there in a few tables, named as follows:

- \( \Delta_2 \)-relations, \( \alpha \) a "6-point",
- \( \Delta_3 \)-relations, \( \alpha \) a "3-point",
- \( \Delta_4 \)-relations, \( \alpha \) a corner point,
- \( C \)-table.

In the next section we shall give a complete list of relations for

\[
\pi_1(\mathbb{C}^2 - S, u_0) / \langle T_1, T_2 \rangle
\]

which is equivalent to the original set of the relations in the above tables.

5. A complete set of relations for \( \pi_1(\mathbb{C}^2 - S, u_0) / \langle T_1, T_2 \rangle \)

**Proposition 5.** The following list is complete list of relations for

\[
\pi_1(\mathbb{C}^2 - S, u_0) / \langle T_1, T_2 \rangle :
\]

(R1) \( T_1^2 = T_2^2 = 1 \).

(R2) \( [T_i, T_j] = 1 \) for \( i \) and \( j \) such that \( L_i \) and \( L_j \) are not edges of the same triangle

\( T_{ij} = T_i \) or \( T_j \).

(R3) \( \langle T_0, T_{ij} \rangle = 1 \) for \( i \) and \( j \) such that \( L_i \) and \( L_j \) are 2 edges of some triangle.

\[
T_0 = \left[ \left[ T_{i_1} T_{i_2} T_{i_3} \ldots T_{i_k} \right] \right]^{\frac{1}{k}} \cdot \left[ T_{j_1} T_{j_2} T_{j_3} \ldots T_{j_k} \right]^{\frac{1}{k}}.
\]

When numerated locally around a "6-point" \( \alpha \) (see Fig. 2)

(R4) \( T_1 = T_3 T_2 T_1 T_2 T_3 T_2 T_3 T_2 T_3 T_2 \).

(R5) \( T_0 = T_3 T_2 T_1 T_2 T_3 T_4 T_2 T_1 T_2 T_3 T_2 \).

(R6) \( T_3 T_1 T_2 T_3 T_4 = T_3 T_2 T_0 T_2 T_3 T_2 \).

(R7) \( T_3 T_1 T_2 T_3 T_4 = T_3 T_2 T_0 T_2 T_3 T_2 \).

(R8) \( T_3 T_1 T_2 T_3 T_4 = T_3 T_2 T_0 T_2 T_3 T_2 \).

(R9) \( T_3 T_1 T_2 T_3 T_4 = T_3 T_2 T_0 T_2 T_3 T_2 \).

When numerated locally around a "3-point" \( \alpha \) (see Fig. 3)

(R10) \( T_1 = T_3 T_2 T_1 T_2 T_3 T_2 \) (\( \alpha \) of type \( (1, 3) \)).

(R11) \( T_1 = T_3 T_2 T_1 T_2 T_3 T_2 \) (\( \alpha \) of type \( (1, 2) \)).

(R12) \( T_1 = T_3 T_2 T_1 T_2 T_3 T_2 \) (\( \alpha \) of type \( (4, 6) \)).

(R13) \( T_1 = T_3 T_2 T_1 T_2 T_3 T_2 \) (\( \alpha \) of type \( (5, 6) \)).

(R14) \( T_1 = T_1 \) (globally numerated).

(R15) \( T_1 = T_1 \) (globally numerated).
Proof: In the table $\Delta^2$-relations, $\alpha$ a “6-point”, which appears in [MoTe1], there are 18 relations. In [MoTe1, Chap. 1] we proved that relations 1–6 and 15–18 imply $B^3$ formula. In fact 1–6 and $B^3$ formula also imply 15–18 so we replace 15–18 by $B^3$ formula. In [MoTe, Lemma 1.1 and Proposition 2] we proved that relations 1–6, 7–12, and 15–18 imply $[l_{xy}, l_{yj}] = 1$ for $i, j$ s.t. $L_i$ and $L_j$ meet in $x$ and for $[\psi(l_i), \psi(l_j)] = 1$. In fact reversing the arguments one can show that 1–6, 15–18 and $[l_{xy}, l_{yj}] = 1$ imply 7–12. So 7–12 is replaceable by $[l_{xy}, l_{yj}] = 1$ for $i$ and $j$, s.t. $\psi(l_i), \psi(l_j) = 1$. So instead of considering the table $\Delta^2$-relations, $\alpha$ a “6-point”, we shall consider the following table:

1. relations 1–6 from $\Delta^2$-relations.
2. $[l_{xy}, l_{yj}] = 1$ for $i, j$ s.t. $[\psi(l_i), \psi(l_j)] = 1$.
3. $B^3$ formula.
4. Relations 13 and 14 from $\Delta^2$-relations.

Consider the remark on p. 343 in [MoTe]. Let us state it:

**Remark.** All explicit curves which we discuss were defined over $\mathbf{R}$. Thus in all our arguments we can change roles of upper and lower-half planes in $\mathbf{C}^1$ and $x$-axis. Such a change requires also to reverse the order of the factors in the expressions for braid monodromy. Thus we see that our final expression

$$A^2 = \prod_{j=1}^{5} C_j H_{j,0}$$

is equivalent to

$$A^2 = \prod_{j=1}^{5} B_{j,0} C_j$$

where $H_{j,0}$ (resp. $C_j$) is obtained from $H_{1,0}$ (resp. $C_0$) as follows: each factor in $H_{1,0}$ (resp. $C_0$) is replaced by the complex conjugate braid (in $\mathbf{C}^1$) and the order of factors is reversed.

In our case we need more than the remark, because we need that each $H_{j,0}$ differ from $H_{j,0}$ by a finite number of elementary transformations. ([] $E_i$ differ from $[\psi(l_i)]$ by an elementary transformation if $3k$ s.t. $F_k = E_i E_{k+1} E_{k-1}^{-1}$ and $F_{k+1} = E_i$.) It is true because all the curves that we used were defined over real numbers.

Using this remark we took for the $C$-table the relations that are induced by complex conjugate braids and get equivalent sets of relations (local sets of braids are replaced by equivalent sets). When adding the $C$-table and $\Delta^2$-relation, $\alpha$ a 3-point, to the relations (1)–(4) that appear before the remark, we get the proposition.

6. **Constructing a short exact sequence** $1 \rightarrow \mathcal{A} \rightarrow \pi_1 \rightarrow S_n \rightarrow 1$

Consider the natural epimorphism

$$\pi_1(\mathbf{C}^2 - S_n, u_0) \rightarrow S_n \rightarrow 1.$$
standard isomorphism theorems we get

\[ 1 \to \ker \psi \ni \langle \langle I_1^2, I_1^2 \rangle \to \pi_1(\mathbb{C}^2 - S, u_o) \ni \langle I_1^2, I_1^2 \rangle \to S_\pi \to 1. \]

It is possible to prove that \( \mathcal{A} \cong \ker \psi \ni \langle I_1^2, I_1^2 \rangle \). We denote

\[ \pi_1(\mathbb{C}^2 - S, u_o) \ni \langle I_1^2, I_1^2 \rangle \]

by \( \tilde{x} \) and \( \tilde{S}_\pi \) by \( \psi \) (abuse of notation). Thus we have

\[ 1 \to \mathcal{A} \ni \tilde{x} \to S_\pi \to 1. \]

It turns out that \( \psi(I) = \psi(I') = (k, l') \) where \( L_j \) is the intersection line of the planes \( P_h \) and \( P_L \).

7 \( \psi \) splits.

Let us denote \( \psi(I) = \psi(I') \). Since \( \alpha_1, \ldots, \alpha_n \) generates \( S_\pi \), it is enough to define a splitting by defining it on \( \alpha_i \). We define \( \tilde{q} : S_\pi \to \tilde{S}_\pi \) by induction: \( \tilde{q}(\alpha_i) = \tilde{q}(I) \) where \( \tilde{q}(I) = \tilde{I} \), \( \tilde{q}^{-1}(I) = I \). \( \tilde{q} = q \), \( \tilde{I} = I \), \( \tilde{I} = I \). \( \tilde{q}^{-1}(I) = I \). \( \ell_1 = 1 \), \( \ell_2 = 0 \), \( \ell_3 = 0 \), and:

\[ \ell_j = \begin{cases} 
\ell_j - 1 & \text{if } L_j \text{ is diagonal } \exists j < k; L_j \text{ diagonal and } L_j \text{ meets } L_k \\
\ell_j + 1 & \text{if } L_j \text{ is vertical } \exists j < k; L_j \text{ vertical and } L_j \text{ meets } L_k \\
0 & \text{otherwise}.
\end{cases} \]

Denote \( \tilde{x} = \tilde{q}(x) \).

**Proposition 7** [MoTcI, Proposition 3]. \( \tilde{q} \) is a splitting. Moreover, in local numeration any relation that \( I_1, I_2, I_3, I_4, I_5, I_6 \) satisfy, \( \tilde{I}_1, \ldots, \tilde{I}_6 \) satisfy and vice versa.

8 Applying the RMS method on \( 1 \to \mathcal{A} \ni \tilde{x} \to S_\pi \to 1 \)

By Sect. 6 we have a short exact sequence \( 1 \to \mathcal{A} \ni \tilde{x} \to S_\pi \to 1 \). In Sect. 3 we introduced a finite set of generators for \( \tilde{S}_\pi \). In Proposition 5 we introduced a complete finite set of relations for \( \tilde{x} \). In Proposition 7 we introduced a splitting of \( \psi \). Thus we can apply the RMS method.

By the RMS method and by some easy observations, we get the following complete set of generators for \( \mathcal{A} \):

\[ \{ \tilde{q}(\alpha I_j I I^{-1} | \alpha \in S_\pi, j = 1 \ldots p) \}. \]

We shall denote:

\[ A_{K_j} = K I_j I I^{-1}, \quad K \in \tilde{S}_\pi, \]

\[ A_{\alpha_j} = A_{\tilde{q}(\alpha I_j I I^{-1}), \quad \alpha \in S_\pi}, \]

\[ A_{\sigma K_j} = A_{\tilde{q}(\alpha K_j), \quad \sigma \in S_\pi, K \in \tilde{S}_\pi}. \]

Thus \( \{ A_{j,1} \} \) is a set of generators for \( \mathcal{A} \).
We shall use the RMS method to obtain a complete set of relations but first we shall quote from [MoTe1] a list of relations which are valid in $\mathcal{A}$.

**Proposition 8** (Proof in [MoTe1]).

(P1) $A_{K_{r,i}j} = A_{K_{r,j}}^{-1} \forall K_{r,i}$ (trivial).

(P2) $A_{K_{r,i}j} = A_{K_{r,i}}(A_{K_{r,j}})^{-1} \forall K_{r,i}$ s.t. $\langle T_f, T_f \rangle = 1$ (Lemma 1.3).

(P3) $\mathcal{A}$ is commutative (Proposition 3).

(P4) $A_{r,i} = A_{r,i} \forall r, i \in \text{global numeration} (Table A_r^2)$.

(P5) $A_{\sigma \tau \rho} = A_{\sigma \tau \rho} \forall \sigma \in \text{Stab(supp}(\psi(f)) \text{ Result 3.1)}$.

(P6) In local numeration around a “6-point” $\alpha$:

$A_{r,3} = A_{r,1}^{-1} A_{r,2} \forall r, i, j, i' \neq j, i' = 1, A_{r,1} A_{r,2}, 3$

$A_{r,5} = A_{r,3}^{-1} A_{r,4} A_{r,2}^{-1}$

$A_{r,7} = A_{r,5}^{-1} A_{r,6}$

$A_{r,9} = A_{r,7}^{-1} A_{r,8}$

In local numeration around a “3-point” $\alpha$ of type (4, 6):

$A_{r,4} = A_{r,2} A_{r,6}^{-1}$

In local numeration around a “3-point” $\alpha$ of type (5, 6):

$A_{r,5} = A_{r,3} A_{r,7}^{-1}$

In local numeration around a “3-point” $\alpha$ of type (1, 2):

$A_{r,1} = A_{r,3} A_{r,5}^{-1}$

In local numeration around a “3-point” $\alpha$ of type (1, 3):

$A_{r,3} = A_{r,1} A_{r,5}^{-1} 1$ (proof of Proposition 4).

(P7) $(A_{r,1})^c = 1$ for $c = \text{g.c.d.}(a, b)$ (Proposition 6).

The references from [MoTe1] are given in brackets.

Property PS asserts that $A_{r, i} = A_{r, j}$ for $\sigma \in \text{Stab}(|f_0|)$. Thus $A_{r, i}$ depends on the values of $\sigma^{-1}$ on $\text{Supp}(\psi(f))$. Let $A_{r, i}$ be $A_{r, j}$ for $\sigma$ s.t. $\sigma^{-1}(a) = k, \sigma^{-1}(b) = \ell$ where $\psi(f) = (a \beta b)$. Then $\mathcal{A}$ is generated by $\{A_{r, i}\}_{k, \ell, a, b}$. $\mathcal{A}$ is generated by $\{A_{r, i}\}$ but a priori we have in $\mathcal{A}$ also elements of the form $A_{K_{r,i}}$ for $K$ a general element of $\hat{f}_1$ and we even use such elements in the various proofs. In fact each such element $A_{K_{r,i}}$ equals $A_{r, i}$ for $\sigma = \psi(K)$. We prove it in the following lemma.

**Lemma 8.1.** $A_{K_{r,i}} = A_{r,i}$.

**Proof.** By definition $A_{K_{r,i}} = K_{r,i}^{-1} K_{r,i} K_{r,i}^{-1} A_{r,i} K_{r,i} K_{r,i}^{-1} / (\psi(K))^{-1}$. Thus

$A_{K_{r,i}} = K_{r,i}^{-1} K_{r,i} K_{r,i}^{-1} / (\psi(K))^{-1} = K_{r,i} K_{r,i}^{-1} / (\psi(K))^{-1}$.

Since $\psi = 1$ and $\psi^{-1}(K) = 1$, $K_{r,i} K_{r,i}^{-1} / (\psi(K))^{-1} = 1$. Since $\mathcal{A}$ is commutative (see Proposition 8, P3), $K_{r,i} K_{r,i}^{-1} / (\psi(K))$ commutes with $T_f, T_f^{-1}$ and thus $A_{K_{r,i}} = A_{r,i}$.

So given $A_{K_{r,i}}$ it equals $A_{r,i}$ for $K$ and $\ell$ the inverse images of $\psi(K)$ on $\text{Supp}(f)$.

The next lemmas will be used to reduce the number of generators that we got by these methods.

**Lemma 8.2.** $A_{r,i} = (A_{r,i})^{-1}$.

**Proof.** Write $\psi(f) = (a \beta b), a < b$. Let $\sigma$ be such that $\sigma^{-1}(a) = \ell, \sigma^{-1}(b) = \ell$. Then $A_{r,i} = A_{r, i} / (a \beta (a \beta b))^{-1} = \ell, (a \beta (a \beta b))^{-1} = k$, thus $A_{r,i} = A_{r, i} / (a \beta)$. By Lemma 1,
\[ A_{\alpha}(e_{1}; e_{1}, e_{2}) = A_{\alpha}(e_{1}, e_{2}) \]. By property P1 of Proposition 8, \( A_{\alpha}(e_{1}; e_{1}, e_{2}) = A_{\alpha}(e_{1})^{-1} \) so \( A_{\alpha}(e_{1}; e_{1}) = A_{\alpha}(e_{1})^{-1} \) so \( A_{\alpha}(e_{1}) = (A_{\alpha}(e_{1}))^{-1} \).

**Lemma 8.3.** In local numeration around a “6-point” \( \alpha \)

(a) \( A_{\alpha}(e) = A_{\alpha}(e) A_{\alpha}(e) \).
(b) \( A_{\alpha}(e) = A_{\alpha}(e) \).
(c) \( A_{\alpha}(e) = A_{\alpha}(e) \).
(d) \( A_{\alpha}(e) = A_{\alpha}(e) \).

**Proof.** We use property P6 of Proposition 8. Denote locally the planes around the “6-point” \( \alpha \) as in Fig. 5. As we proved in [MoTe1, Chap. 0.8]:

\[ \psi(T_1) = (12), \quad \psi(T_2) = (13), \quad \psi(T_3) = (24), \]
\[ \psi(T_4) = (35), \quad \psi(T_5) = (46), \quad \psi(T_6) = (56). \]

Fig. 5. Local numeration of triangles meeting in a 6-point \( \alpha \)

(a) By property P6 of Proposition 8, commutativity, and property P2 of Proposition 8,

\[ A_{\alpha}(e, e; e, e, e) = (A_{\alpha}(e, e; e, e, e))^{-1}. \]

Let \( \sigma \) be such that \( \sigma^{-1}(2) = \psi^{-1}(4) = \ell \). Then \( A_{\alpha}(e, e; e, e, e) \). By Lemma 8.1

\[ A_{\alpha}(e, e; e, e, e) = A_{\psi}(e, e; e, e, e). \]

Let \( i = \sigma(T_1) = \sigma(T_2) = \sigma(T_3) \). We want to translate it into \( A_{\alpha}(e) \) language. \( \psi^{-1}(T_1) = \sigma^{-1}(T_1) = \ell \). Then \( \psi(T_1) = A_{\alpha}(e) \).

By Lemma 8.1, \( A_{\alpha}(e, e; e, e, e) = A_{\psi}(e, e; e, e, e) \) Let

\[ i = \sigma(T_1) = \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = \sigma(T_5) = \sigma(T_6). \]

We have \( A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e) \). We want to translate it into \( A_{\alpha}(e) \) language. \( \psi^{-1}(T_1) = \sigma^{-1}(T_1) = \ell \). Then \( \psi(T_1) = A_{\alpha}(e) \).

By Lemma 8.1

\[ A_{\alpha}(e, e; e, e, e) = A_{\psi}(e, e; e, e, e). \]

We substitute the last results in P6 to get \( A_{\alpha}(e) = A_{\alpha}(e) \).

(b) By property P6 of Proposition 8, \( A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e, e; e, e, e) \). Denote \( \sigma^{-1}(3) = k \), \( \sigma^{-1}(5) = \ell \). Then \( A_{\alpha}(e) = A_{\alpha}(e) \). By Lemma 8.1

\[ A_{\alpha}(e, e; e, e, e) = A_{\psi}(e, e; e, e, e). \]

Let

\[ i = \sigma(T_1) = \sigma(T_2) = \sigma(T_3). \]

Then \( \psi^{-1}(2) = \sigma^{-1}(3) = k \) and \( \psi^{-1}(4) = \sigma^{-1}(5) = \ell \). Thus \( A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e) \) and so \( A_{\alpha}(e) = A_{\alpha}(e) \).

(c) By property P6 of Proposition 8, \( A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e, e; e, e, e) \). We let \( k = \sigma^{-1}(4) \), \( \ell = \sigma^{-1}(6) \). Then \( A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e) \). Let \( \tau = \sigma(T_1) = \sigma(T_2) = \sigma(T_3). \) Then

\[ A_{\alpha}(e, e; e, e, e) = A_{\alpha}(e). \]
Now \( \tau^{-1}(1)=k \) and \( \tau^{-1}(3)=\ell \) and thus \( A_{k,\ell}=A_{k,\ell}^3 \) and so \( A_{k,\ell}^3=A_{k,\ell}^2 \).

In the same way we get \( A_{k,\ell}^5=A_{k,\ell}^4 \).

**Lemma 8.4.** (a) If \( \alpha \) is a "3-point" of type \((5,6)\) then \( A_{k,\ell}^3=(A_{k,\ell}^3)^{-1} \).
(b) If \( \alpha \) is a "3-point" of type \((4,6)\) then \( A_{k,\ell}^4=A_{k,\ell}^3 \).
(c) If \( \alpha \) is a "3-point" of type \((1,2)\) then \( A_{k,\ell}^3=(A_{k,\ell}^3)^{-1} \).
(d) If \( \alpha \) is a "3-point" of type \((1,3)\) then \( A_{k,\ell}^3=A_{k,\ell}^2 \).

**Proof.** The proof is based on property P6 of Proposition 8 and Lemma 8.1. The proof of the 4 claims are similar so we shall give here only the proof of (a).

By property P6 of Proposition 8 we have \( A_{k,\ell}^3=A_{k,\ell}^4 \). By Lemma 1 we have \( \psi(\Gamma_3)=(46) \). Let \( k=\sigma^{-1}(5) \), \( \ell=\sigma^{-1}(6) \). Then \( A_{k,\ell}^3=A_{k,\ell}^4 \). Denote \( \tau=\sigma(46) \). Then \( \tau^{-1}(4)=\ell, \tau^{-1}(6)=k \). Then \( A_{k,\ell}^3=A_{k,\ell}^4 \).

By Lemma 8.2 \( A_{k,\ell}^3=(A_{k,\ell}^3)^{-1} \). Thus \( A_{k,\ell}^3=(A_{k,\ell}^3)^{-1} \).

In the next lemma we introduce our algorithm for producing relations in \( \mathcal{A} \) which is somehow different from the RMS method. From each relation of \( \Gamma_1, \Gamma_2, \ldots, \Gamma_6, \) numerated locally we induce a relation \( B(R) \) in \( \mathcal{A} \). In Lemma 5 we prove that \( B(R) \) is equivalent to \( A(R) \), which is the relation induced by \( R \) on \( \mathcal{A} \) via the RMS method. We prefer the \( B(R) \)'s because they are simpler.

**Definition.** \( B(R), B_\alpha \). Take \( \Gamma_1, \Gamma_2, \ldots, \Gamma_6, \Gamma_6 \in \pi_4 \) in local numeration around a "6-point" \( \alpha \). Define:

\[
\bar{R}_i = \frac{\bar{\Gamma}_i}{\bar{\Gamma}_i}, \quad i=2,3,4,6
\]

Let \( R = g_1 \cdot \ldots \cdot g_k \) be a relation in \( \Gamma_1, \Gamma_2, \ldots, \Gamma_6 \). Define:

\[
B(R) = g_1 \cdot \ldots \cdot g_k (g_1 \cdot \ldots \cdot g_k)^{-1}
\]

\[
B(R) = \prod_{i=1}^k B_i(R).
\]

**Lemma 8.5.** Let \( R \) be a relation in \( \Gamma_1, \Gamma_2, \ldots, \Gamma_6 \) numerated locally around a "6-point" \( \alpha \). Then \( B(R) = 1 \) iff \( A(R) = 1 \).

**Proof.** Let \( g_i = \Gamma_i \) or \( \Gamma_i \), \( i=1 \ldots 6 \) numerated locally around a "6-point" \( \alpha \). By Proposition 7 any relation that \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6 \) satisfy, \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6 \) satisfy also and vice versa. Since \( \psi(\Gamma_1) = \psi(\Gamma_1) = \Gamma_1 \). So \( g_i = \Gamma_i \) or \( \Gamma_i \), \( i=1 \ldots 6 \). Thus we can replace \( \Gamma_i \) by \( g_i \). Since

\[
\bar{g}_i = \begin{cases} \Gamma_i & \text{for } i=2,3,4,6 \\ \Gamma_i & \text{for } i=1,5 \end{cases}
\]

we can replace \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6 \) by \( g_1, \ldots, g_6 \) respectively. Thus any relation that \( g_1, \ldots, g_6 \) satisfy \( g_1, \ldots, g_6 \) satisfy too and vice versa. Assume \( R = g_1 \cdot \ldots \cdot g_6 \) is a relation on \( \bar{\pi}_4 \). In fact

\[
\prod A(R) = (g_1 \cdot \ldots \cdot g_6)(g_1 \cdot \ldots \cdot g_6)^{-1}
\]

and

\[
\prod B(R) = (g_1 \cdot \ldots \cdot g_6)(g_1 \cdot \ldots \cdot g_6)^{-1}.
\]

Since \( g_1, \ldots, g_6 = 1 \),

\[
A(R) = (g_1 \cdot \ldots \cdot g_6)^{-1} \quad \text{and} \quad B(R) = (g_1 \cdot \ldots \cdot g_6)^{-1}.
\]
By the above, $g_i \ldots g_n = 1 \iff \bar{g}_i \ldots \bar{g}_n = 1$. Thus $A(R) = 1 \iff B(R) = 1$. \hfill \Box

In Lemmas 6, 7, 8 we compute $\prod B_i(R)$ in some cases.

**Lemma 8.6.** Let $R$ be a relation on $\bar{z}_i$, expressed in $I_\alpha, I_\beta, i = 1, \ldots, 6$ (in local numeration around a “6-point” $z$). Say $R = g_1 \ldots g_m$. Then

$$B_i(R) = \begin{cases} 1 & \text{if } g_i = \bar{g}_i; \\ A_{g_i \cdot h \cdot i, h}^{-1} & \text{if } g_i = \bar{g}_i \cdot g_i = I_{h}; \\ A_{g_i \cdot h \cdot i, -h}^{-1} & \text{if } g_i = \bar{g}_i \cdot g_i = I_{-h}. \end{cases}$$

**Proof:** $B_i(R) = g_1 \ldots g_i \ldots g_n$ so if $g_i = \bar{g}_i$ certainly $B_i(R) = 1$. If $g_i \neq \bar{g}_i$ and $g_i = I_{h1, j}$, one of the indices 2, 3, 4, or 6 and then

$$g_i = I_{h_2} \quad \text{and} \quad B_i(R) = g_1 \ldots g_i \ldots g_{i-1} I_{h_2} I_{j} I_{h_2-1} \ldots g_1 = (A_{g_i \cdot h_1 \cdot j, j})_{h_2}.$$  

If $g_i \neq \bar{g}_i$ and $g_i = I_{h_2, j}$ must be 1 or 5 and then $g_i = I_{h_2}$ and thus

$$B_i(R) = g_1 \ldots g_i \ldots g_{i-1} I_{h_2} I_{j} I_{h_2-1} \ldots g_1 = A_{g_i \cdot h_2 \cdot j, j}.$$ \hfill \Box

**Lemma 8.7.** Let $R$ be a triple relation, i.e. $R \in \langle I_{\alpha}, I_{\beta} \rangle$. Then

$$B(R) = (A_{I_{\alpha}, I_{\beta}})_{I_{\alpha}} (A_{I_{\beta}, I_{\alpha}})_{I_{\beta}}$$

for some $k, f, m, r, s$.

Furthermore, the same is true for $R = \langle I_{\alpha}, I_{\beta}, I_{\gamma} \rangle$ for $I_{\alpha}$, a certain conjugation of $I_{\alpha}$ by $I_{\beta}$ or $I_{\gamma}$ which appears in $A_{I_{\alpha}}$ [MoTeC1, Chap. 1].

**Proof:** As we quoted in the list of notations

$$R = \langle I_{\alpha}^{c} I_{\beta} I_{\beta} I_{\gamma} I_{\gamma} I_{\gamma} \rangle$$

where $c, \mu = 0$ or 1 and $\langle f, g \rangle = f g^{-1} f^{-1} g^{-1}$. We shall consider only the case $c = \mu = 0$, $I_{\alpha} = I_{\alpha}$ and $I_{\beta} = I_{\beta}$. The other 15 cases are similar.

If $I_{\alpha}$ and $I_{\beta}$ are of order 2 so $R$ is actually

$$R = (I_{\alpha} I_{\beta})^{c} (I_{\alpha} I_{\beta} I_{\alpha} I_{\beta} I_{\alpha} I_{\beta} I_{\alpha} I_{\beta} I_{\alpha} I_{\beta} I_{\alpha} I_{\beta} I_{\alpha} I_{\beta})^{c}.$$  

To compute $B_i(R)$ we use the previous lemma.

$$I_{\alpha} = I_{\alpha} \quad \text{so} \quad B_i(R) = 1;$$

$$I_{\beta} = I_{\beta} + I_{\alpha} \quad \text{so} \quad B_i(R) = A_{I_{\alpha}, I_{\beta}};$$

$$I_{\beta} = I_{\alpha} \quad \text{so} \quad B_i(R) = 1;$$

$$I_{\gamma} = I_{\alpha} + I_{\beta} \quad \text{so} \quad B_i(R) = A_{I_{\alpha}, I_{\gamma}};$$

and

$$B_{2 \alpha, x+1}(R) = 1, \quad i = 0, \ldots, x;$$

$$B_{2 \beta, x+1}(R) = (A_{I_{\alpha}, I_{\beta}})^{-1}, \quad i = 1, \ldots, x;$$

$$I_{\alpha} + I_{\beta} \quad \text{so} \quad B_{2 \alpha, x+2}(R) = A_{I_{\alpha}, I_{\beta}} I_{\alpha} I_{\beta} = A_{I_{\alpha}, I_{\beta}};$$

$$I_{\alpha} = I_{\alpha} \quad \text{so} \quad B_{2 \beta, x+1}(R) = 1;$$

and so on.

$$B_{2 \alpha, x+2}(R) = A_{I_{\alpha}, I_{\beta}}; \quad i = x + 1, \ldots, x + \beta + 1,$$

$$B_{2 \beta, x+1}(R) = 1, \quad i = x + 1, \ldots, x + \beta.$$
We keep computing $B(R)$ using the previous lemma and get

\[
B_{2t+1}(R) = \text{Id}, \quad t = \alpha + \beta + 1, \ldots, 2\alpha + \beta + 1.
\]

\[
B_{2t+1}(R) = A_{t+\beta+1}, \quad t = \alpha + \beta + 2, \ldots, 2\alpha + \beta + 1.
\]

\[
B_{2t+1}(R) = A_{n+1}, \quad t = 2\alpha + \beta + 2, \ldots, 2\alpha + 2\beta + 2.
\]

\[
B_{2t+1}(R) = \text{Id}, \quad t = 2\alpha + \beta + 2, \ldots, 2\alpha + 2\beta + 1.
\]

\[
B_{2t+1}(R) = \text{Id}, \quad t = 2\alpha + 2\beta + 2, \ldots, 3\alpha + 2\beta + 2.
\]

\[
B_{2t+1}(R) = A_{t+\beta+1}, \quad t = 2\alpha + 2\beta + 3, \ldots, 3\alpha + 2\beta + 3.
\]

\[
B_{2t+1}(R) = \text{Id}, \quad t = 2\alpha + 2\beta + 3, \ldots, 3\alpha + 2\beta + 3.
\]

Then

\[
B(R) = \prod \sum_{t=1}^{n} B(R) = (A_{t+\beta+1})(A_{n+1})(A_{t+\beta+1})(A_{t+\beta+1})(A_{n+1})^{\beta+1}
\]

\[
\times (A_{t+\beta+1})(A_{n+1})(A_{t+\beta+1})(A_{n+1})^{\beta+1}.
\]

Since $\prod 2^t \prod 2^t = 1$ we thus have

\[
B(R) = \prod \sum_{t=1}^{n} B(R) = (A_{t+\beta+1})(A_{n+1})(A_{t+\beta+1})(A_{n+1})^{\beta+1}
\]

\[
\times (A_{t+\beta+1})(A_{n+1})^{\beta+1}.
\]

By property P4 of Proposition 8, $A_{k+1} = A_{k+1}$, so

\[
B(R) = \prod \sum_{t=1}^{n} B(R) = (A_{t+\beta+1})(A_{n+1})(A_{t+\beta+1})(A_{n+1})^{\beta+1}
\]

\[
\times (A_{t+\beta+1})(A_{n+1})^{\beta+1}.
\]

Since $\mathcal{A}$ is commutative we have

\[
B(R) = (A_{t+\beta+1})(A_{n+1})(A_{t+\beta+1})(A_{n+1})^{\beta+1}
\]

\[
\prod 2^t \prod 2^t = 1 \text{ so } \prod 2^t \prod 2^t = 1 \Rightarrow [T_1, T_2] = 1.
\]

By Proposition 5 $[\psi(T_1), \psi(T_2)] = 1$. Thus $\psi(T_1)$ and $\psi(T_2)$ have a common index say $\psi(T_1) = (k')$, $\psi(T_2) = (m')$. If we translate $B(R)$ into $A_{k'}$ language we get

\[
B(R) = (A_{k'})(A_{k'})(A_{k'})^{-1} \cdot (A_{k'})^{-1}(A_{k'})^{-1}(A_{k'})^{\beta+1}.
\]

We use Lemma 8.2 and commutativity of $\mathcal{A}$ to get

\[
B(R) = (A_{k'})(A_{k'})(A_{k'})^{-1} \cdot (A_{k'})^{-1}(A_{k'})^{-1}(A_{k'})^{\beta+1}.
\]

To prove the second assertion we take $R^-$ to be the relation obtained from $R$ by replacing $I_1$ by $I_2$, and $I_2$ by $I_1$, and $I_3$ by $I_4$, $I_4$ differs from $I_3$ by multiplication by $I_4$. Thus by the previous lemma, the factors $B(R)$ which come from the multipliers in $R$ that do not exist in $R^-$ are the identity. So $B(R) = B(R^-)$ and the lemma is proved. \hfill \Box

**Lemma 8.8.** If $R$ admits $[T_1, T_2] = 1$ then $B(R)$ for every $R$ in $[T_1, T_2]$ is the identity relation or $A_{k'} = (A_{k'})^{-1}$ or a commutativity relation.
Proof. $R$ is actually one of the following relations:

$$R_1: \Gamma^T \Gamma^T \Gamma^T,$$
$$R_2: \Gamma^T \Gamma^T \Gamma^T,$$
$$R_3: \Gamma^T \Gamma^T \Gamma^T,$$
$$R_4: \Gamma^T \Gamma^T \Gamma^T.$$

Without loss of generality assume $f^T = f$, $f^T = f$. By Lemma 6, $B(R) = \text{Id}$ and thus $B(R) = \text{Id}$. By Lemma 6, $B(R) = B(R) = \text{Id}$. By Lemma 6, $B(R) = B(R) = \text{Id}$, and $B(R) = B(R) = \text{Id}$. By $R$, $\Gamma^T$ commutes with $\Gamma^T$ and thus $\psi(\Gamma^T)$ and $\psi(\Gamma^T)$ have no common index. $\psi(\Gamma^T)$ stabilizes $\psi(\Gamma^T)$, and $B(R)$ stabilizes $\psi(\Gamma^T)$. Now $\psi(\Gamma^T) = \psi(\Gamma^T)$ (by $R$) so $B(R) = B(R) = \text{Id}$. We translate $B(R)$ and $B(R)$ to $\mathcal{A}'$, then language and get $B(R) = (\mathcal{A}')^{-1}$, $B(R) = (\mathcal{A}')^{-1}$ and thus $B(R) = (\mathcal{A}')^{-1}$, $B(R)$ is of the same type.

$$B(R) = (\mathcal{A}')^{-1}(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}.$$ 

Using the fact that $\psi(\Gamma^T)$ and $\psi(\Gamma^T)$ have no common index we translate $B(R)$ to $\mathcal{A}'$, language and get $(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}$. Using $B(R)$ we get $(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}(\mathcal{A}')^{-1}=1$.

a commutativity relation.

Theorems 8.9, 9, 10 are finite presentations of $\mathcal{A}$. 

Theorem 8.9. $\mathcal{A}$ is generated by $\{A_{\mathcal{A}_i}\}_{i \in \mathcal{I}}$ and the only relations are

(T1) $\mathcal{A}$ is commutative.

(T2) $(A_{\mathcal{A}_i})'=1, c = \text{g.c.d.}(a, b) \forall k_i, \ell \forall h_i,

(T3) $A_{\mathcal{A}_i} = (A_{\mathcal{A}_i})^{-1} \forall k_i, \ell,$

(T4) $A_{\mathcal{A}_i}A_{\mathcal{A}_j} = 1 \forall k_i, \ell, m \forall j_i.$

(T5) When numerated locally around a "6-point":

$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i},$$
$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i},$$
$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i},$$
$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i}.$$

(T6) When numerated locally around a "2-point" $x$ of type $(i, j)$

$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i}^2$$ if $(i, j) = (1, 3)$ or $(4, 6),$ 
$$A_{\mathcal{A}_i}^2 = A_{\mathcal{A}_i} A_{\mathcal{A}_i}^{-1}$$ if $(i, j) = (1, 2)$ or $(5, 6).$

(T7) For 1 global numeration $A_{\mathcal{A}_i}^2 = 1 \forall k_i, \ell.$

For $p$ global numeration $A_{\mathcal{A}_i}^2 = 1 \forall k_i, \ell.$

Proof. $\mathcal{A}$ is generated by $\{A_{\mathcal{A}_i}\}_{i \in \mathcal{I}}$. Each $A_{\mathcal{A}_i}$ is of the form $A_{\mathcal{A}_i}$ for some $1 \leq k_i, \ell \leq n$. Relations T1, T2, T3, T7 are translations of properties P3, P7, P1,
P4 of Proposition 8 into $A_3^I$ language. For each $j$ there is an $i$ such that $\psi(I_j)$ and $\psi(I_i)$ have one common index and $\langle I_{iP}, I_{jI} \rangle = 1$ and so property P2 is valid for every $j$. P2 translated into the $A_3^I$ language is T4. T5 and T6 are Lemmas 8.3 and 8.4. Thus T1, T2, T3 are valid.

To prove that these are the only relations we shall consider the complete set of relations for $\mathfrak{A}_1$ that appear in Proposition 5. For each $R$ that appears there we shall show that $A(R)$ or $B(R)$ is a consequence of T1 to T7. Since $\{A(R)\}_R$ is a complete set of relations for $\mathfrak{A}_1$ and $B(R) \sim A(R)$ that will be enough.

(1) $B(R_2)$ translated to $A_3^I$ language is T3.
(2) $B(R_3)$ is a consequence of T1 and T3 or it is the trivial relation by Lemma 8.8.
(3) $B(R_4)$ is a consequence of T4 by Lemma 8.7.
(4) $B(R_4) = A_{r_2}A_{r_3}A_{r_4}A_{r_5}A_{r_6}A_{r_7}A_{r_8}$.

Recall that numerated locally we have: $\psi(I_1) = (12), \psi(I_2) = (13), \psi(I_3) = (14), \psi(I_4) = (35), \psi(I_5) = (46), \psi(I_6) = (56)$. We use this and Lemma 8.1 to translate $B(R_4)$, a relation in $A_{r,j}$ to a relation in $A_3^I$. When translated into $A_3^I$ language we get

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$

This is a consequence of T4 and T5 as follows: By T5 we have the relation $(A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1})$. By T4: $(A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1})$. We multiply these 3 relations and use commutativity to get: $B(R_4)$.

(5) $B(R_4)$ is a consequence of T4 and T5 in a similar way.

(6) $B(R_4)$ is the identity because all generators of $\mathfrak{A}_1$ that appear there satisfy $j = j'$.

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$

Translating into $A_3^I$ language we get

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$

This is a consequence of T1, T3, T4, and T5 as follows:

$$(A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}) = (T5),$$

$$(A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}) = (T4),$$

$$(A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}) = (T4, T3).$$

We multiply all these to get what we need.

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$

In $A_3^I$ language it is $A_{r_1}^{-1}A_{r_2}^{-1}$ which is an immediate consequence of T3 and T5.

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$

In $A_3^I$ language it is

$$B(R_4) = A_{r_1}^{-1}A_{r_2}^{-1}A_{r_3}^{-1}A_{r_4}^{-1}A_{r_5}^{-1}A_{r_6}^{-1}A_{r_7}^{-1}A_{r_8}^{-1}$$
This is a consequence of T1, T3, T4, and T5 as follows:

\[ A_2^3 (A_4^2)^{-1} = 1 \quad \text{(T5, T3)}, \quad A_2^3 (A_4^2)^{-1} = 1 \quad \text{(T5)}, \]

\[ (A_2^2)^{-1} (A_4^2)^{-1} (A_2^2)^{-1} = 1 \quad \text{(T4)}, \quad A_2^3 (A_4^2)^{-1} (A_4^2)^{-1} = 1 \quad \text{(T4, T3)}. \]

We multiply the last 4 relations to get what we want.

\[ B(R_{10}), ..., B(R_{13}) \text{ are similar and we shall compute here only } B(R_{12}). \]

\[ B(R_{12}) = A_2^3 A_4^2 A_2^3 A_4^2 A_4^2 A_4^2 A_4^2. \]

Locally we have \( \psi(I_4) = (35), \psi(I_0) = (56) \) (see Fig. 4). Translating \( B(R_{12}) \) into \( A_4^e \) language we get:

\[ B(R_{12}) = (A_4^6)^{-1} (A_4^6)^{-1} (A_4^6)^{-1}. \]

By T6: \( (A_4^6)^{-1} (A_4^6)^{-1} = 1 \). By T4: \( (A_4^6)^{-1} (A_4^6)^{-1} (A_4^6)^{-1} \). Multiplying these two relations we get \( B(R_{12}) \).

\[ B(R_{12}) \text{ is T7.} \]

\[ B(R_{15}) \text{ is T7.} \]

9. Obtaining smaller sets of generators on \( \mathcal{A} \)

**Theorem 9.** \( \mathcal{A} \) is generated by \( \{A_i^e\}_{i \in \mathcal{I}} \) (2 global numeration) and the only relations are

\( \text{(T1) } \mathcal{A} \text{ is commutative.} \)

\( \text{(T2) } A_1^e A_2^e = 1, \text{ g.c.d. } (a, b). \)

\( \text{(T3) } A_{i}^e = (A_{i+1}^e)^{-1}. \)

\( \text{(T4) } A_1^e A_2^e A_3^e A_4^e = 1. \)

**Proof:** By Theorem 9, \( \mathcal{A} \) is generated by \( \{A_i^e\}_{i \in \mathcal{I}} \). We shall prove that it is enough to take \( \{A_i^e\}_{i \in \mathcal{I}} \). By induction on global numeration of lines. By T5 and T6 one can see immediately that each \( A_i^e \) for \( i \leq j \) can be expressed as a product of 1 or 2 elements of the form \( A_i^e \) for \( i < j \). Moreover \( A_i^e = 1 \) \( \forall i \not\in \mathcal{I} \). To prove the other assertions we need to have a precise expression of \( A_i^e \) in terms of \( A_i^e \).

Using \( A_i^e = A_i^e \) around \( \alpha \)'s on the same horizontal line we get that \( A_i^e \) is independent of \( i \)'s on the same horizontal line (see Fig. 5). The same is true for \( j \)'s on the same vertical line, and for \( i \)'s on the same diagonal line (see Fig. 5). We want to determine \( A_i^e \) \( \forall j \) in terms of \( A_i^e \).

We shall use the following 3 relations:

1. \( A_i^e = A_i^e \) around \( \alpha \) a "6-point".
2. \( A_i^e = A_i^e \) around \( \alpha \) a "3-point" of type (4, 6).
3. \( A_i^e = (A_{i+1}^e)^{-1} \) around \( \alpha \) a "3-point" of type (5, 6).

We use global numeration on lines as in Fig. 2. Fix \( k, \ell \). Denote \( \alpha = A_i^e \). Then \( A_i^e = x \forall j \) on the same vertical line as \( L_{i} \) (see Fig. 6). \( A_i^e = I \) so by relation 1 for \( \alpha = a + 2 \) we get \( A_i^e = x \). And so \( A_i^e = x \forall j \) on the same horizontal line as \( L_{i} \). Consider \( \alpha = 2 \) and relation 3. We get \( A_i^e = (A_{i+1}^e)^{-1} = x^{-1}. \) And thus \( A_i^e = x^{-1} \forall j \) where \( L_{i} \) is on the same diagonal line as \( L_{i} \). Consider \( \alpha = a + 3 \) and relation 1 we get:

\[ A_i^e = A_i^e \cdot (A_{i+1}^e)^{-1} = x \cdot (x^{-1})^{-1} = x^2; \]
Thus $A_{L_i} = x^{2^i}$ for $L_i$ is on the same vertical line as $L_i$. We use now relation 3 for $x = 2$ to get $A_{L_i} = x^{2^i}$ and thus $A_{L_i} = x^{2^{i+1}}$ for $L_i$ on the same diagonal line as $L_i$, and so on. We get that $A_{L_i}$ for $L_i$ on a vertical line equals a power of $x$. The power is determined by the distance of the line from $x = 1$. We also have that $A_{L_j}$ for $L_j$ on a diagonal line to the right of the main diagonal equals a power of $x^{-1}$. The power determined by the distance from $x = 1$.

By the same method, using the above relation 2 and 3, we get that on horizontal lines and on diagonal lines to the left of the main diagonal $A_{L_j}$ equals a certain power of $x$. The power is determined by the distance from $x = 1$. We put all that information in Fig. 6.

By Theorem 8.9, $T_1', ... , T_4'$ are valid. To show that these are the only relations we have to take any of the relations mentioned in Theorem 8.9 and substitute there instead of any $A_{L_i}$ its expression as the product of $A_{L_j}$'s and show that we get one of the relations $T_1', ... , T_4'$ or a product of them.

Each $A_{L_i}$ is a power of $A_{L_j}$. The power is determined by $j$ and not by $k, l$. So when substituting $A_{L_i} = (A_{L_j})^j$ in relations $T_2, T_3,$ and $T_4$ we get the $j$-th power of $T_2', ... , T_4'$ respectively.

Fix $k, l$. One can see from Fig. 6 that around a “6-point”, $A_{L_i}$ is a power of $x$ (3 local numeration). The power is the sum of the powers of $x$ that gives $A_{L_i}$ and $A_{L_j}$ (1, 2 local numeration). Thus when substituting in $A_{L_i} = (A_{L_j})^j$ instead of each of the factors, the correct power of $x$, we get the identity. The same is true for every $k$ and $l$. The same is true for every other relation in $T_5$. The same is true for the relations around a “3-point” of type $(4, 6)$ and $(5, 6)$. We have to be more careful with relations around a “3-point” of type $(1, 2)$ or $(1, 3)$.

Take $c$ a point of type $(1, 3)$. From Fig. 6 we see that around $x$, $A_{L_i} = x^c$ and $A_{L_j} = x^{c'c}$ so when substituting it into $A_{L_i} = (A_{L_j})^j$, we get $x^{c'c} = x^j$ which is equivalent to $x^{c'c} = 1$ which is $c'c$ times $T_2$. The same technique applies for the relations in $T_6$ that come from a “3-point” of type $(1, 2)$.
In $\mathbb{T}_n$ we have two relations concerning the triviality of $A_{jk}$ for $j = 1$ or $p$. In our expression of $A_{jk}$ in terms of $A_{jk}$, we already have $A_{jk} = 1$. To get $A_{jk} = 1$ from $\mathbb{T}_1, \ldots, \mathbb{T}_4$, we have to note that $L_{jk}$ is on the diagonal whose distance from the main diagonal is $|a - b|$. Thus $A_{jk}^{|a-b|} = x^{e+b}$ or $x^{e-b}$. Now, $c = \text{gcd}(a, b)$ divides $a - b$ and $b - a$ and thus $x^{e+b}$ and $x^{e-b}$ are powers of $x$. But $x^{e+b} = 1$ (T2'), so $x^{e-b} = x^{e+b} = 1$ and thus $A_{jk}^p = 1 \forall k, l$.

10 Main theorems

**Theorem 10.1.** The fundamental group $\pi_1(Y_{ab}^{m})$ (denoted $s''$) is a finite commutative group on $n - 1$ generators each of order $\text{gcd}(a, b)$ (denoted $c$) and there are no further relations.

**Proof.** Consider $\{A_{21}, A_{22}\}$ for $a 
 T_3'$ and $T_4'$,

$$A_{21} = (A_{21}^e)^{-1} = (A_{21}^c)^{-1}.$$ 

So $\{A_{21}, A_{22}\}$ is a set of generators each of order $c$. There are no other relations. In fact, when substituting $A_{21} = A_{21}'(A_{21}^{-1})$ in $T_3'$ and $T_4'$ we get the trivial relation.

**Theorem 10.2.** The fundamental group $\pi_1(Y_{ab})$ is a finite abelian group with $n - 2$ generators of order $\text{gcd}(a, b)$ and there are no further relations.

**Proof.** By Van Kampen $\pi_1(CP^2 - S)$ is generated by $\{T_i\}$ with the relations listed above for $\pi_1(CP^2 - S)$ plus one relation: $\prod_i T_i = 1$.

We have as before

$$\pi_1(CP^2 - S) \xrightarrow{\psi} S_n \rightarrow 1,$$

$1 \rightarrow \ker \psi \rightarrow <T_i^2> \rightarrow \pi_1(CP^2 - S)/<T_i^2> \rightarrow S_n \rightarrow 1,$

and $\pi_1(Y_{ab}) = \ker \psi <T_i^2>$. Thus we can use the Reidemeister-Schreier method again to get a finite presentation of $\pi_1(Y_{ab})$. It will consist of the same generator as $\pi_1(Y_{ab}^{m})$ with the extra relation induced from $T_1 T_1 \cdots T_p T_p = 1$. The induced relation when transcribed to $A_{12}$ language equals $\prod_{j=2}^{p} A_{12} = 1$. Thus $A_{12}$ is expressed in terms of the other generators and we are left with only $n - 2$ generators.

References


