Fundamental groups of some quadric-line arrangements

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Received 15 June 2002; received in revised form 29 July 2002

Abstract

In this paper we obtain presentations of fundamental groups of the complements of three quadric-line arrangements in $\mathbb{P}^2$. The first arrangement is a smooth quadric $Q$ with $n$ tangent lines to $Q$, and the second one is a quadric $Q$ with $n$ lines passing through a point $p \notin Q$. The last arrangement consists of a quadric $Q$ with $n$ lines passing through a point $p \in Q$.
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Keywords: Fundamental groups; Complements of curves; Conic-line arrangements

1. Introduction

This is the first of a series of articles in which we shall study the fundamental groups of complements of some quadric-line arrangements. In contrast with the extensive literature on line arrangements and the fundamental groups of their complements, (see, e.g., [14, 7, 15]), only a little known about the quadric-line arrangements (see [12, 1, 2]). The present article is dedicated to the computation of the fundamental groups of the complements of three infinite families of such arrangements. A similar analysis for the quadric-line arrangements up to degree six will be done in our next paper.

Let $C \subset \mathbb{P}^2$ be a plane curve and $\ast \in \mathbb{P}^2 \setminus C$ a base point. By abuse of language we will call the group $\pi_1(\mathbb{P}^2 \setminus C, \ast)$ the fundamental group of $C$, and we shall frequently omit base

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doi:10.1016/S0166-8641(02)0218-3
points and write $\pi_1(\mathbb{P}^2 \setminus C)$. One is interested in the group $\pi_1(\mathbb{P}^2 \setminus C)$ mainly for the study of the Galois coverings $X \to \mathbb{P}^2$ branched along $C$. Many interested surfaces have been constructed as branched Galois coverings of the plane, for example for the arrangement $A_3$ in Fig. 1, there are Galois coverings $X \to \mathbb{P}^2$ branched along $A_3$ such that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, or $X$ is an abelian surface, a K3 surface, or a quotient of the two-ball $\mathbb{B}_2$ (see [9,8,17]). Moreover, some line arrangements defined by unitary reflection groups studied in [13] are related to $A_3$ via orbifold coverings. For example, if $L$ is the line arrangement given by the equation

$$xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z) = 0$$

then the image of $L$ under the branched covering map $[x : y : z] \in \mathbb{P}^2 \to [x^2 : y^2 : z^2] \in \mathbb{P}^2$ is the arrangement $A_3$, see [17] for details.

The standard tool for fundamental group computations is the Zariski–van Kampen algorithm [19,18], see [3] for a modern approach. We use a variation of this algorithm developed in [16] for computing the fundamental groups of real line arrangements and avoids lengthy monodromy computations. The arrangements $E_n$ and $C_n$ discussed below are of fiber type, so presentations of their fundamental groups could be easily found as an extension of a free group by a free group. However, our approach has the advantage that it permits to capture the local fundamental groups around the singular points of these arrangements. The local fundamental groups are needed for the study of the singularities of branched of $\mathbb{P}^2$ branched along these arrangements.

In Section 2 below, we give fundamental group presentations and prove some immediate corollaries. In Section 3 we deal with the computations of fundamental group presentations given in Section 2.

2. Results

Let $C \subset \mathbb{P}^2$ be a plane curve and $B$ an irreducible component of $C$. Recall that a meridian $\mu$ of $B$ in $\mathbb{P}^2 \setminus C$ with the base point $* \in \mathbb{P}^2$ is a loop in $\mathbb{P}^2 \setminus C$ obtained by following a path $\omega$ with $\omega(0) = *$ and $\omega(1)$ belonging to a small neighborhood of a smooth point $p \in B \setminus C$, turning around $C$ in the positive sense along the boundary of a small disc $\Delta$ intersecting $B$ transversally at $p$, and then turning back to $*$ along $\omega$. The meridian $\mu$ represents a homotopy class in $\pi_1(\mathbb{P}^2 \setminus C, *)$, which we also call a meridian of $B$. Any
two meridians of $B$ in $\mathbb{P}^2 \setminus C$ are conjugate elements of $\pi_1(\mathbb{P}^2 \setminus C)$ (see, e.g., [10, 7.5]), hence the meridians of irreducible components of $C$ are supplementary invariants of the pair $(\mathbb{P}^2, C)$. These meridians are specified in presentations of the fundamental group given below, they will be used in orbifold-fundamental group computations in [17].

2.1. The arrangement $\mathcal{A}_n$

Theorem 1. Let $\mathcal{A}_n := Q \cup T_1 \cup \ldots \cup T_n$ be an arrangement consisting of a smooth quadric $Q$ with $n$ distinct tangent lines $T_1, \ldots, T_n$. Then

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \cong \left\{ \begin{array}{l}
\tau_1, \ldots, \tau_n, \\
\kappa_1, \ldots, \kappa_n
\end{array} \right\}$$

$$\begin{array}{l}
\kappa_i = \tau_i \tau_{i-1} \tau_i^{-1}, \ 2 \leq i \leq n \\
(\kappa_i \tau_i)^2 = (\tau_i \kappa_i)^2, \ 1 \leq i \leq n \\
[\kappa_i, \tau_i \kappa_i, \tau_i] = 1, \ 1 \leq i < j \leq n \\
\tau_1 \cdots \tau_n \kappa_i^2 = 1
\end{array}$$

where $\kappa_i$ are meridians of $Q$ and $\tau_i$ is a meridian of $T_i$ for $1 \leq i \leq n$. Local fundamental groups around the singular points of $\mathcal{A}_n$ are generated by $(\kappa_i^{-1} \tau_i \kappa_i, \tau_i)$ for the nodes $T_i \cap T_j$ and by $(\kappa_i, \tau_i)$ for the tangent points $T_i \cap Q$.

Part (i) of the corollary below is almost trivial. Part (ii) appears in [6], and part (iii) was given in [4].

Corollary 2.

(i) One has: $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \cong \mathbb{Z}$.

(ii) The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ admits the presentation

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2) \cong \{ \tau, \kappa \mid (\tau \kappa)^2 = (\kappa \tau)^2 \},$$

where $\kappa$ is a meridian of $Q$ and $\tau$ is a meridian of $T_1$. A meridian of $T_2$ is given by $\kappa^{-2} \tau^{-1}$.

(iii) The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3)$ admits the presentation

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \cong \{ \tau, \sigma, \kappa \mid (\tau \kappa)^2 = (\kappa \tau)^2, (\sigma \kappa)^2 = (\kappa \sigma)^2, [\sigma, \tau] = 1 \},$$

where $\sigma$, $\tau$ are meridians of $T_1$ and $T_3$, respectively, and $\kappa$ is a meridian of $Q$. A meridian of $T_2$ is given by $(\kappa \tau \kappa \sigma)^{-1}$.

A group $G$ is said to be big if it contains a non-abelian free subgroup, and small if $G$ is almost solvable. In [6], it was proved by V. Lin that the group (2) is big. Below we give an alternative proof:

Proposition 3. For $n \geq 1$, the group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ is big.

Proof. A group with a big quotient is big. Since $\tau_{n+1}$ is a meridian of $T_{n+1}$ in $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n+1})$, one has

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n) \cong \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{n+1}) / \langle \tau_{n+1} \rangle.$$
and it suffices to show that the group $\pi_1(\mathbb{P}^2 \setminus A_2)$ is big. In the presentation (2), applying the change of generators $\alpha := \tau \kappa$, $\beta := \tau$ gives

$$\pi_1(\mathbb{P}^2 \setminus A_2) \cong \langle \alpha, \beta \mid [\alpha^2, \beta] = 1 \rangle.$$  

Adding the relations $\alpha^2 = \beta^3 = 1$ to the latter presentation gives a surjection $\pi_1(\mathbb{P}^2 \setminus A_2) \twoheadrightarrow \mathbb{Z}/2 \ast \mathbb{Z}/3$. Since the commutator subgroup of $\mathbb{Z}/2 \ast \mathbb{Z}/3$ is the free group on two generators (see [5]), we get the desired result. □

2.2. The arrangement $B_n$

**Theorem 4.** Let $B_n := Q \cup T_1 \cup T_2 \cup L_1 \cup \ldots \cup L_n$ be an arrangement consisting of a smooth quadric $Q$ with $n + 2$ distinct lines $T_1, T_2, L_1, \ldots, L_n$ all passing through a point $p \notin Q$ such that $T_1, T_2$ are tangent to $Q$. Then one has

$$\pi_1(\mathbb{P}^2 \setminus B_n) \cong \left\{ \sigma : \frac{\kappa \tau}{(\kappa \tau)^2} = (\kappa \tau)^2 \quad \frac{(\kappa \tau \lambda_i)}{1} = 1, \; 1 \leq i \leq n \right\}$$

where $\tau$ is a meridian of $T_1$, $\lambda_i$ is a meridian of $L_i$ for $1 \leq i \leq n$, and $\kappa$ is a meridian of $Q$.

A meridian $\lambda$ of $T_2$ is given by $\sigma := (\lambda_n \ldots \lambda_1 \lambda_2 \tau)^{-1}$. Local fundamental groups around the singular points of $B_n$ are generated by $(\kappa, \lambda_i)$ and $(\tau^{-1} \kappa \tau, \lambda_i)$ for the nodes $L_i \cap Q$, by $\langle \kappa, \tau \rangle$ for the tangent point $T_1 \cap Q$, and by $\langle \kappa, \sigma \rangle$ for the tangent point $T_2 \cap Q$.

**Corollary 5.** (i) Put $B'_n := B_n \setminus T_1$ and $B''_n := B'_n \setminus T_2$. Then

$$\pi_1(\mathbb{P}^2 \setminus B'_n) \cong \pi_1(\mathbb{P}^2 \setminus B''_n) \cong \pi_1(\mathbb{P}^2 \setminus B'_n) \cong \pi_1(\mathbb{P}^2 \setminus B''_n).$$

**Proof.** One has $\pi_1(\mathbb{P}^2 \setminus B'_n) \cong \pi_1(\mathbb{P}^2 \setminus B''_n) \cong \langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\kappa, \lambda_i] = 1, \; 1 \leq i \leq n \rangle$.

Setting $\sigma = 1$ in the presentation (4) gives

$$\pi_1(\mathbb{P}^2 \setminus B'_n) \cong \langle \kappa, \lambda_1, \ldots, \lambda_n \mid [\kappa, \lambda_i] = 1, \; 1 \leq i \leq n \rangle.$$  

Setting $\sigma = 1$ in the expression for a meridian $\lambda$ of $T_2$ given in Theorem 4 shows that $(\lambda_n \ldots \lambda_1 \lambda_2 \tau)^{-1}$ is a meridian of $T_2$ in $\pi_1(\mathbb{P}^2 \setminus B'_n)$. In order to find $\pi_1(\mathbb{P}^2 \setminus B''_n)$, it suffices to set $\lambda_n \ldots \lambda_1 \lambda_2 = 1$ in the presentation of $\pi_1(\mathbb{P}^2 \setminus B'_n)$. Eliminating $\lambda_n$ by this relation yields the presentation

$$\pi_1(\mathbb{P}^2 \setminus B''_n) \cong \langle \kappa, \lambda_1, \ldots, \lambda_{n-1} \mid [\lambda_i, \kappa] = [\lambda_{n-1} \ldots \lambda_1 \lambda_2, \kappa] = 1 \rangle.$$  

![Fig. 2. Arrangements $B_2$ and $B'_2$.](image-url)
Since the last relation above is redundant, we get the desired isomorphism \( \pi_1(P^2 \setminus B'_n) \cong \pi_1(P^2 \setminus B_{n+1}) \). \( \square \)

Note that the groups \( \pi_1(P^2 \setminus B'_n) \) are abelian for \( i = 0, 1, 2 \), hence, the groups \( \pi_1(P^2 \setminus B'_i) \) are abelian for \( i = 0, 1 \). Otherwise, setting \( \kappa = 1 \) in presentation (5) gives the free group on \( n - 1 \) generators, which shows that these groups are big. The groups \( \pi_1(P^2 \setminus B_n) \) are always big, since the arrangement \( \mathcal{B}_0 \) is same as \( \mathcal{A}_2 \), and \( \pi_1(P^2 \setminus \mathcal{A}_2) \) is big by Proposition 3.

2.3. The arrangement \( C_n \)

**Theorem 6.** Let \( C_n := Q \cup T \cup L_1 \cup \ldots \cup L_n \) be an arrangement consisting of a smooth quadric \( Q \) with \( n + 1 \) distinct lines \( T, L_1, \ldots, L_n \), all passing through a point \( p \in Q \) such that \( T \) is tangent to \( Q \). Then one has

\[
\pi_1(P^2 \setminus C_n) \cong \langle \kappa, \lambda_1, \ldots, \lambda_n \mid \{\kappa, \lambda_i\} = 1, \ 1 \leq i \leq n \rangle,
\]

where \( \kappa \) is a meridian of \( Q \) and \( \lambda_i \) is a meridian of \( L_i \) for \( 1 \leq i \leq n \). A meridian \( \tau \) of \( T \) is given by \( \tau := (\lambda_1 \ldots \lambda_n)^{-1} \). Local fundamental groups around the singular points of \( C_n \) are generated by \( \{\kappa, \lambda_i\} \) for the nodes \( L_i \cap Q \) and by \( \{\tau, \lambda_1, \ldots, \lambda_n, \kappa\} \) for the point \( p \).

Note that the arrangement \( C_n \) is a degeneration (in the sense of Zariski) of the arrangement \( B'_n \) as the point \( p \) approaches to \( Q \). By Zariski's "semicontinuity" theorem of the fundamental group [19] (see also [5]), there is a surjection \( \pi_1(P^2 \setminus C_n) \twoheadrightarrow \pi_1(P^2 \setminus B'_n) \).

In our case, this is also an injection:

**Corollary 7.**

(i) \( \pi_1(P^2 \setminus B'_n) \cong \pi_1(P^2 \setminus C_n) \).

(ii) \( \pi_1(P^2 \setminus C'_n) \cong \pi_1(P^2 \setminus C_{n+1}) \).

**Proof.** Part (i) is obvious. The proof of part (ii) is same as the proof of Corollary 5, (ii). \( \square \)
3. The arrangement $\mathcal{A}_n$

It is easily seen that any two arrangements $\mathcal{A}_n$ with fixed $n$ are isotopic. In particular, the groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_n)$ are isomorphic. Hence one can take as a model of the arrangements $\mathcal{A}_n$ the quadric $Q$ defined by $x^2 + y^2 = z^2$, where $[x : y : z] \in \mathbb{P}^2$ is a fixed coordinate system in $\mathbb{P}^2$. Pass to the affine coordinates in $\mathbb{C}^2 \cong \mathbb{P}^2 \setminus \{z = 0\}$. Choose real numbers $x_1, \ldots, x_n$ such that $-1 < x_1 < x_2 < \cdots < x_n < 0$, and define $y_i$ to be the positive solution of $x_i^2 + y_i^2 = 1$ for $1 \leq i \leq n$. Put $t_i := (x_i, y_i) \in Q$, and take $T_i$ to be the tangent line to $Q$ at the point $t_i$ (see Fig. 4).

Let $\text{pr}_1 : \mathbb{C}^2 \setminus \mathcal{A}_n \to \mathbb{C}$ be the first projection. The base of this projection will be denoted by $B$. Put $F_x := \text{pr}_1^{-1}(x)$, and denote by $S$ the set of singular fibers of $\text{pr}_1$. It is clear that if $F_x \in S$, then $x \in [-1, 1]$. There are three types of singular fibers:

(i) The fibers $F_1$ and $F_{-1}$, corresponding to the 'branch points' $(-1, 0)$ and $(1, 0)$.

(ii) The fibers $F_{s_i}$ ($1 \leq i \leq n$) corresponding to the 'tangent points' $t_i = (x_i, y_i) = T_i \cap Q$.

(iii) The fibers $F_{s_i, j}$ ($1 \leq i \neq j \leq n$) corresponding to the nodes $n_{i,j} = (a_{i,j}, b_{i,j}) := T_i \cap T_j$. One can arrange the lines $T_i$ such that

$$-1 < x_1 < a_{1,2} < a_{1,3} < \cdots < a_{1,n} < x_2 < a_{2,3} < \cdots < x_n < 1.$$

Identify the base $B$ of the projection $\text{pr}_1$ with the line $y = -2 \in \mathbb{C}^2$. Let $N$ be the number of singular fibers and let $-1 = s_1 < s_2 < \cdots < s_{N-1} < s_N = 1$ be the elements of $S \cap B$ (so that $s_2 = x_1, s_3 = a_{1,2}, s_4 = a_{1,3}$, and so on). In $B$, take small discs $\Delta_i$ around the points $s_i$, and denote by $c_i, d_i$ ($c_i < d_i$) the points $\partial \Delta_i \cap \mathbb{R}$ for $1 \leq i \leq N$ (see Fig. 5).

Put $B_1 := [c_1, c_2] \cup \Delta_1$ and for $2 \leq i \leq N$ let $B_i := [c_i, c_{i+1}] \cup \Delta_i \cup \cdots \cup \Delta_i$. Let $X_i := \text{pr}_1^{-1}(B_i)$ be the restriction of the fibration $pr$ to $B_i$. Let

$$A_i := \Delta_i \cup \partial \left( \{\Re(x) \leq 0, c_2 \leq \Re(x) \leq c_1 \} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1}) \right)$$

and let $Y_i := \text{pr}_1^{-1}(A_i)$ be the restriction of the fibration $pr$ to $A_i$ (see Fig. 6).

Clearly, $X_i = X_{i-1} \cup Y_i$ for $2 \leq i \leq N$. We will use this fact to compute the groups $\pi_1(X_i, \ast)$ recursively, where $\ast := (c_2, -2)$ is the base point. For details of the algorithm we apply below, see [16].

![Fig. 4](image-url)
Identify the fibers of \( \text{pr}_1 \) with \( F_0 \) via the second projection \( \text{pr}_2 : (x, y) \in \mathbb{C}^2 \to y \in \mathbb{C} \). In each one of the fibers \( F_{c_i} \) (respectively \( F_{d_i} \)) take a basis for \( \pi_1(F_{c_i}, -2) \) (respectively for \( \pi_1(F_{d_i}, -2) \)) as in Fig. 7 (for \( F_{d_i} \), just replace \( \gamma \)'s by \( \theta \)'s in Fig. 7). We shall denote these basis by the vectors \( \Gamma_i := \{ \gamma_1^{(i)}, \ldots, \gamma_{n+2}^{(i)} \} \) (respectively \( \Theta_i := \{ \theta_1^{(i)}, \ldots, \theta_{n+2}^{(i)} \} \)).

Let \( v_i \subset B_i \subset B \) be a path starting at \( v_i(0) = c_2 \), ending at \( v_i(1) = c_1 \) and such that

\[
v_i([0, 1]) = \partial \left( \{ \Im(z) \leq 0, \ c_2 \leq \Re(z) \leq c_1 \} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_{i-1}) \right).
\]

Similarly, let \( \eta_i \subset B_i \subset B \) be a path starting at \( \eta(0) = c_2 \), ending at \( \eta(0) = d_i \) and such that

\[
\eta_i([0, 1]) = \partial \left( \{ \Im(z) \leq 0, \ c_2 \leq \Re(z) \leq d_i \} \setminus (\Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_i) \right).
\]

For \( 2 \leq i \leq N \) and \( 1 \leq j \leq n + 2 \) each loop \( \tilde{\gamma}_j^{(i)} := v_i \cdot \gamma_j^{(i)} \cdot v_i^{-1} \) represents a homotopy class in \( \pi_1(X_i, \ast) \), where \( \ast := (c_2, -2) \) is the base point. Similarly, each loop \( \tilde{\theta}_j^{(i)} := \eta_i \cdot \theta_j \cdot \eta_i^{-1} \) represents a homotopy class in \( \pi_1(X_i, \ast) \). Denote \( \tilde{\gamma}_i := [\tilde{\gamma}_1^{(i)}, \ldots, \tilde{\gamma}_{n+2}^{(i)}] \), and \( \tilde{\Theta}_i := [\tilde{\theta}_1^{(i)}, \ldots, \tilde{\theta}_{n+2}^{(i)}] \).
It is well known that the group \( \pi_1(Y_i, *) \) has the presentation

\[
(y_j^{\circ} \mid 1 \leq j \leq n+2)
\]

where \( M_i : \pi_1(F_i, *) \to \pi_1(F_i, *) \) is the monodromy operator around the singular fiber above \( s_i \). It is also well known that if it is the branches of \( A_n \) corresponding to the loops \( y_k^{\circ} \) and \( y_{k+1}^{\circ} \) that meet above \( s_i \), then the only non-trivial relation in (7) is \( y_k^{\circ} y_{k+1}^{\circ} y_{k+1}^{\circ} y_k^{\circ} \) in case of a branch point, \( [y_k^{\circ}, y_{k+1}^{\circ}] = 1 \) in case of a node, and \( (y_k^{\circ} y_{k+1}^{\circ})^2 = (y_k^{\circ})^2 \) in case of a tangent point.

Now suppose that the group \( \pi_1(X_{i-1}, *) \) is known, with generators \( \tilde{\gamma}_j \). Recall that \( X_i = X_{i-1} \cup Y_i \). In order to find the group \( \pi_1(X_i, *) \), one has to express the base \( \tilde{\gamma}_j \) in terms of the base \( \tilde{\gamma}_i \). Adding to the presentation of \( \pi_1(X_{i-1}) \) the relation obtained by writing the relation of \( \pi_1(Y_i) \) in the new base then yields a presentation of \( \pi_1(X_i) \). Note that, since the space \( Y_i \) is eventually glued to \( X_{i-1} \), it suffices to find an expression of \( \tilde{\gamma}_j \) in terms of the base \( \tilde{\gamma}_2 \) in the group \( \pi_1(X_{i-1}, *) \).

Since all the points of \( A_n \) above the interval \([d_{i-1}, c_i]\) are smooth and real, one has

**Fact.** The loops \( \tilde{\gamma}_j^{i-1} \) and \( \tilde{\gamma}_j^i \) are homotopic in \( X_i \) (or in \( Y_i \)) for \( 2 \leq i \leq N \) and \( 1 \leq j \leq n+2 \). In other words, the bases \( \tilde{\gamma}_{i-1} \) and \( \tilde{\gamma}_i \) are homotopic.

In order to express the base \( \tilde{\gamma}_i \) in terms of the base \( \tilde{\gamma}_1 \) the following lemma will be helpful.

**Lemma 8.** Let \( C_k : x^2 - y^{k+1} = 0 \) be an \( A_k \) singularity, where \( k = 1 \) or \( k = 3 \). Put \( D := \{(x, y) : |x| \leq 1, |y| \leq 1\} \) and let \( \text{pr}_1 := (x, y) \in D \setminus C_k \to (x, -1) \) be the first projection. Denote by \( F_k \) the fiber of \( \text{pr}_1 \) above \((x, -1)\). Identify the fibers of \( \text{pr}_1 \) via the second projection. Let \(-1 < c < 0 \) be a real number, and put \( d := -c \). In \( F_c \) (respectively in \( F_{d} \)) take a basis \( \Gamma := \{y_1, y_2\} \) for \( \pi_1(F_c, *) \) (respectively a basis \( \Theta := \{\theta_1, \theta_2\} \) for \( \pi_1(F_d, -1) \)) as in Fig. 8. Let \( \gamma_1 \) be the path \( \gamma(t) := e^{\pi ti}, \) and put \( \delta_i := \gamma_1 \cdot \theta_i \cdot \gamma_1^{-1} \) for \( i = 1, 2 \). Then \( \gamma_1, \delta_i \) are loops in \( D \setminus C_k \) based at \( * := (c, -1) \), and one has

(i) If \( k = 1 \), then \( \delta_1 \) is homotopic to \( y_2 \), and \( \delta_2 \) is homotopic to \( y_1 \), in other words,

\[
\Theta = \{y_2, y_1\}.
\]

(ii) If \( k = 3 \), then \( \Theta = \{y_2 y_1^{-1}, y_1^{-1} y_2 y_1\} \).

![Fig. 8.](image-url)
Proof. Since \( \pi_1(D \setminus C_2) \) is abelian, part (i) is obvious. For part (ii), note that the points of intersection \( F_{\eta(t)} \cap C_2 \) are \( \gamma_1 := e^{2\pi i} \) and \( \gamma_2 := -e^{2\pi i} \). Hence, when we move the fiber \( F_c \) over \( F_d \) along the path \( \eta \), \( \gamma_1 \) and \( \gamma_2 \) make one complete turn around the origin in the positive sense. The loops \( \gamma_1, \gamma_2 \) are transformed to loops \( \bar{\gamma}_1, \bar{\gamma}_2 \subset F_d \) as in Fig. 9.

It follows that the loop \( \eta \cdot \bar{\gamma}_1 \cdot \eta^{-1} \) is homotopic to \( \gamma_i \) for \( i = 1, 2 \). This homotopy can be constructed explicitly as follows: Let \( \Phi_{\eta(t)} : F_c \to F_{\eta(t)} \) be the corresponding Leftschez homeomorphism (see [11]). Then

\[
H(s, t) := \begin{cases} 
\Phi_{\eta(t)}(\gamma_1(3(s-t)/3)/(3-2t)), & 0 \leq s \leq t/3, \\
\eta(3(1-s)), & 1-t/3 \leq s \leq 1 
\end{cases}
\]

gives a homotopy between \( \gamma_1 \) and \( \bar{\gamma}_1 \). Expressing \( \bar{\delta}_i \) in terms of \( \bar{\gamma}_i \), we get

\[
\bar{\delta}_1 = \bar{\gamma}_1^{-1} \bar{\gamma}_2^{-1} \gamma_1 \gamma_2 \gamma_1, \\
\bar{\delta}_2 = \bar{\gamma}_1^{-1} \bar{\gamma}_2 \gamma_1 = \gamma_1^{-1} \gamma_2 \gamma_1.
\]

Since from the monodromy one has the relation \( (\gamma_1 \gamma_2)^2 = (\gamma_2 \gamma_1)^2 \), the expression for \( \bar{\delta}_1 \) can be simplified to get \( \bar{\delta}_1 = \gamma_2 \gamma_1 \gamma_2^{-1} \). \( \Box \)

Now we proceed with the computation of the groups \( \pi_1(X_i) \). Clearly, the group \( \pi_1(X_2) \) is generated by the base

\[
\bar{\gamma}_2 = [\gamma_1^{(2)}, \gamma_2^{(2)}, \ldots, \gamma_{n+2}^{(2)}]
\]

with the only relations

\[
\gamma_1^{(2)} = \gamma_2^{(2)} \tag{8}
\]

and

\[
(\gamma_2^{(2)}, \gamma_3^{(2)})^2 = (\gamma_3^{(2)}, \gamma_2^{(2)})^2. \tag{9}
\]

Put

\[
[k_1, k_1, \tau_1, \ldots, \tau_n] := \Omega_2.
\]

Then relation (9) becomes

\[
(k_1 \tau_1)^2 = (\tau_1 k_1)^2. \tag{10}
\]
By Lemma 8 and the above Fact, one has
\[ \tilde{\mathcal{F}}_3 = \tilde{\mathcal{S}}_2 = [\kappa_1, \tau_1, \kappa_1, \tau_1^{-1}, \kappa_1^{-1}, \tau_1, \kappa_1, \tau_2, \ldots, \tau_n]. \]
Since \( s_3 \) corresponds to the node \( T_1 \cap T_2 \), the next relation is
\[ [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_2] = 1. \] (11)

Hence,
\[ \pi_1(X_3, *) \simeq (\kappa_1, \tau_1, \ldots, \tau_n | (\kappa_1 \tau_1)^2 = (\tau_1 \kappa_1)^2, [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_2] = 1). \]
By Lemma 8, one has
\[ \tilde{\mathcal{F}}_4 = \tilde{\mathcal{S}}_3 = [\kappa_1, \tau_1, \kappa_1, \tau_1^{-1}, \tau_2, \kappa_1^{-1}, \tau_1, \kappa_1, \tau_2, \tau_3, \ldots, \tau_n]. \]
Since \( s_4 \) corresponds to the node \( T_1 \cap T_3 \), one has the relation
\[ [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_1, \tau_2] = 1. \]
Hence,
\[ \pi_1(X_4, *) \simeq (\kappa_1, \tau_1, \ldots, \tau_n | (\kappa_1 \tau_1)^2 = (\tau_1 \kappa_1)^2, [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_1, \tau_2] = [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_1, \tau_2] = 1). \]
By Lemma 8, one has
\[ \tilde{\mathcal{F}}_5 = \tilde{\mathcal{S}}_4 = [\kappa_1, \tau_1, \kappa_1, \tau_1^{-1}, \tau_2, \tau_3, \kappa_1^{-1}, \tau_1, \kappa_1, \tau_1, \tau_2, \tau_3, \ldots, \tau_n]. \]
Since \( s_5 \) corresponds to the node \( T_1 \cap T_{k-1} \) for \( 2 \leq k \leq n + 1 \), repeating the above procedure gives the presentation
\[ \pi_1(X_{n+1}, *) \simeq (\kappa_1, \tau_1, \ldots, \tau_n | (\kappa_1 \tau_1)^2 = (\tau_1 \kappa_1)^2, [\kappa_1^{-1}, \tau_1, \kappa_1, \tau_2] = 1, 2 \leq k \leq n) \]
and
\[ \tilde{\mathcal{F}}_{n+2} = \tilde{\mathcal{S}}_{n+1} = [\kappa_1, \kappa_2, \tau_2, \ldots, \tau_n, \kappa_1^{-1}, \tau_1 \kappa_1], \]
where we put \( \kappa_{n+1} := \tau_1 \kappa_1 \tau_1^{-1} \) for \( 1 \leq i \leq n - 1 \).

The next point \( s_{n+2} \) corresponds to the tangent point \( T_2 \cap Q \). This gives the relation
\[ (\kappa_2 \tau_2)^2 = (\tau_2 \kappa_2)^2 \] (12)
and
\[ \tilde{\mathcal{F}}_{n+2} = \tilde{\mathcal{S}}_{n+1} = [\kappa_1, \tau_2 \kappa_2 \tau_2^{-1}, \kappa_2^{-1}, \tau_2 \kappa_2, \tau_3, \ldots, \tau_n, \kappa_1^{-1}, \tau_1 \kappa_1]. \]
Now comes the \( n - 2 \) points \( s_k \) corresponding to the nodes \( T_2 \cap T_{k-n} \) for \( n + 3 \leq 2n + 1 \). These give the relations
\[ [\kappa_2^{-1}, \tau_2, \kappa_2, \tau_2] = 1, \quad 3 \leq k \leq n. \]
Hence, one has
\[ \pi_1(X_{n+1}, *) \simeq (\kappa_1, \kappa_2, \tau_1, \ldots, \tau_n | \kappa_2 = \tau_1 \kappa_1 \tau_1^{-1}, (\kappa_1 \tau_1)^2 = (\tau_1 \kappa_1)^2, [\kappa_1^{-1}, \tau_1 \kappa_1, \tau_1] = 1, \quad i \leq k \leq n, \quad i = 1, 2). \]
We proceed in this manner until the last singular fiber $s_N$. Since this is a branch point, the final relation is

$$\kappa_n = \kappa_1.$$  \hspace{1cm} (13)

This gives the presentation

$$\pi_1(X_N, *) \simeq \left\{ \tau_i, \kappa_i, 1 \leq i \leq n \left| \begin{array}{l}
\kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1}, 2 \leq i \leq n \\
(k_1 \tau_i)^2 = (\tau_i \kappa_i)^2, 1 \leq i \leq n \\
[k_{i}^{-1} \tau_i \kappa_i, \tau_j] = 1, 1 \leq i < j \leq n \\
\kappa_1 = \kappa_n
\end{array} \right. \right\}. \hspace{1cm} (14)$$

Adding to this presentation of $\pi_1(X_N, *)$ the projective relation $\tau_n \cdots \tau_1 \kappa_1^2 = 1$ gives the presentation

$$\pi_1(\mathbb{P}^2 \setminus A_n) \simeq \left\{ \tau_i, \kappa_i, 1 \leq i \leq n \left| \begin{array}{l}
\kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1}, 2 \leq i \leq n \\
(k_1 \tau_i)^2 = (\tau_i \kappa_i)^2, 1 \leq i \leq n \\
[k_{i}^{-1} \tau_i \kappa_i, \tau_j] = 1, 1 \leq i < j \leq n \\
\tau_n \cdots \tau_1 \kappa_1^2 = 1, \kappa_1 = \kappa_n
\end{array} \right. \right\}. \hspace{1cm} (15)$$

Note that the relation $\kappa_1 = \kappa_n$ is redundant. Indeed, since $\kappa_i = \tau_i \kappa_{i-1} \tau_i^{-1}$, one has

$$\kappa_n = (\tau_n \cdots \tau_1) \kappa_1 (\tau_n \cdots \tau_1)^{-1}. \hspace{1cm} (16)$$

But $\tau_n \cdots \tau_1 = \kappa_1^{-1}$ by the projective relation. Substituting this in (16) yields the relation $\kappa_1 = \kappa_n$. This finally gives the presentation (1) and proves Theorem 1. Claims regarding the local fundamental groups around the singular points of $A_n$ are direct consequences of the above algorithm.

3.1. Proof of Corollary 2

(i) The arrangement $A_1$. Writing down the presentation (1) explicitly for $n = 1$ gives

$$\pi_1(\mathbb{P}^2 \setminus A_1) \simeq \left\{ \kappa_1, \tau_1 \left| (k_1 \tau_1)^2 = (\kappa_1 \tau_1)^2 \right. \right\}. \hspace{1cm}$$

Eliminating $\tau_1$ from the last relation shows that $\pi_1(\mathbb{P}^2 \setminus A_1) \simeq \mathbb{Z}$.

(ii) The arrangement $A_2$. Writing down the presentation (1) explicitly for $n = 2$ gives

$$\pi_1(\mathbb{P}^2 \setminus A_2) \simeq \left\{ \kappa_1, \kappa_2, \tau_1, \tau_2 \left| \left. \begin{array}{l}
(1) \ k_2 = \tau_1 \kappa_1 \kappa_1^{-1} \\
(2) \ (k_1 \tau_1)^2 = (\tau_1 \kappa_1)^2 \\
(3) \ (k_2 \tau_2)^2 = (\tau_2 \kappa_2)^2 \\
(4) \ [k_1^{-1} \tau_1 \kappa_1, \tau_2] = 1 \\
(5) \ \tau_2 \tau_1 \kappa_1^2 = 1
\end{array} \right. \right\}. \hspace{1cm}$$

Eliminating $\kappa_2$ by (1) and $\tau_2$ by (5) one easily shows that the relations (3) and (4) are redundant. This leaves (2) and gives the desired presentation.
(iii) The arrangement $\mathcal{A}_3$. Writing down the presentation (1) explicitly for $n = 3$ gives

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \cong \left\{ \begin{array}{ll}
(1) \ k_2 = r_1 k_1 r_1^{-1} & (2) \ k_3 = (r_2 r_1) k_1 (r_2 r_1)^{-1} \\
(3) (k_1 r_1)^2 = (r_1 k_1)^2 & (4) \ (k_2 r_2)^2 = (r_2 k_2)^2 \\
(5) (k_3 r_3)^2 = (r_3 k_3)^2 & (6) \ [k_1^{-1} r_1 k_1, r_3] = 1 \\
(7) \ [k_1^{-1} r_1 k_1, r_3] = 1 & (8) \ [k_2^{-1} r_2 k_2, r_3] = 1 \\
(9) \ r_3 r_2 r_1 k_1^2 = 1
\end{array} \right.$$

Eliminate $k_2$ by (1), $k_3$ by (2), and $r_2$ by (9). It can be shown that the relations (4), (6) and (8) are consequences of the remaining relations. The relation (5) becomes $(k_1 r_3)^2 = (r_3 k_1)^2$. This gives the presentation

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_3) \cong \langle k_1, r_1, r_3 | (k_1 r_1)^2 = (r_1 k_1)^2, \ (k_1 r_3)^2 = (r_3 k_1)^2, \ [k_1^{-1} r_1 k_1, r_3] = 1 \rangle.$$

Finally, put $\kappa := k_1$, $\tau := k_1^{-1} r_1 k_1$ and $\sigma := r_3$. Then $r_1 = \kappa \tau \kappa^{-1}$, and the first relation in the above presentation becomes $(\kappa^2 \tau \kappa^{-1})^2 = (\kappa \tau)^2 \Rightarrow (\kappa \tau)^2 = (\tau \kappa)^2$. This gives the desired presentation.

4. The arrangement $\mathcal{B}_n$

As in the case of the arrangements $\mathcal{A}_n$, it is readily seen that arrangements $\mathcal{B}_n$ are all isotopic to each other for fixed $n$, so one can compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_n)$ from the following model for $\mathcal{B}_n$'s (see Fig. 10): The quadric $Q$ is given by the equation $x^2 + y^2 = 1$, and $p$ is the point $(2, 0)$. The lines $L_1$ intersect $Q$ above the $x$-axis.

The projection to the $x$-axis has four types of singular fibers:

![Fig. 10](image-url)
(i) The fibers $F_1$ and $F_{-1}$, corresponding to 'branch points'.

(ii) The fiber $F_x$ corresponding to the 'tangent points' $(x, y) = t_1 := T_1 \cap Q$ and $(x, -y) = t_2 := T_2 \cap Q$.

(iii) The fibers $F_{a_1}, \ldots, F_{a_n}$ ($-1 < a_n < \cdots < a_1 < x$) corresponding to the nodes $L_i \cap Q$ lying on the right of the tangent points and the fibers $F_{b_1}, \ldots, F_{b_n}$ ($x < b_1 < \cdots < b_n < 1$) corresponding to the nodes $L_i \cap Q$ lying on the left of the tangent points.

(iv) The fiber $F_2$, corresponding to the point $p$.

In order to find the group $\pi_1(\mathbb{P}^2 \setminus B_n)$, we shall apply the same procedure as in the computation of $\pi_1(\mathbb{P}^2 \setminus A_n)$. Let $y \in \mathbb{R}$ be such that $-1 < y < a_n$, and take $F_y$ to be the base fiber. Let $s_1 := -1$, $s_{i+1} := b_i$ for $1 \leq i \leq n$, $s_{n+2} := x$, $s_{n+2+i} := a_{n+1-i}$ for $1 \leq i \leq n$, and $s_{2n+4} := 1$, and $s_{2n+4} = 2$. Take a basis

$$\tilde{\mathcal{F}} := \{\tau, \kappa, \lambda_1, \ldots, \lambda_n, \sigma\}$$

for $F_y$ as in Fig. 7.

Since $s_1$ corresponds to a branch point, one has the relation $\kappa_1 = \kappa_2$. Put $\kappa := \kappa_1 = \kappa_2$.

The point $s_2$ is a node, and yields the relation $[\kappa, \lambda_1] = 1$, and one has

$$\tilde{\mathcal{F}} := \{\tau, \kappa, \lambda_1, \ldots, \lambda_n, \sigma\}.$$  

Repeating this for the nodes $s_3, \ldots, s_{n+1}$ gives the relations $[\kappa, \lambda_i] = 1$ for $1 \leq i \leq n$, and one has

$$\tilde{\mathcal{F}} := \{\tau, \kappa, \lambda_1, \ldots, \lambda_n, \kappa, \sigma\}.$$  

The monodromy around the fiber $F_x$ gives the relations $(\tau \kappa)^2 = (\kappa \tau)^2$ and $(\sigma \kappa)^2 = (\kappa \sigma)^2$.

One has

$$\tilde{\mathcal{F}} := \{\kappa \tau \kappa^{-1}, \tau^{-1} \kappa \tau, \lambda_1, \ldots, \lambda_n, \sigma \kappa \sigma^{-1}, \kappa^{-1} \sigma \kappa\}.$$  

Since the points $s_{n+3}, \ldots, s_{2n+2}$ corresponds to nodes, one has the relations $[\lambda_i, \sigma \kappa \sigma^{-1}] = 1$, and

$$\tilde{\mathcal{F}} := \{\kappa \tau \kappa^{-1}, \tau^{-1} \kappa \tau, \sigma \kappa \sigma^{-1}, \lambda_1, \ldots, \lambda_n, \kappa^{-1} \sigma \kappa\}.$$  

The branch point corresponding to $s_{2n+3}$ yields the relation

$$\tau^{-1} \kappa \tau = \sigma \kappa \sigma^{-1}.$$  

Together with the projective relation $\sigma \lambda_n \cdots \lambda_1 \kappa^2 \tau = 1$ these relations already gives a presentation of $\pi_1(\mathbb{P}^2 \setminus B_n)$, since one can always ignore one of the singular fibers when computing the monodromy (see [16]).

We obtained the presentation

$$\pi_1(\mathbb{P}^2 \setminus B_n) \simeq \left\{\Lambda, \tau, \kappa, \lambda_1, \ldots, \lambda_n, \sigma \right\},$$

where

1. $$(\tau \kappa)^2 = (\kappa \tau)^2$$
2. $$(\kappa \sigma)^2 = (\sigma \kappa)^2$$
3. $$\tau^{-1} \kappa \tau = \sigma \kappa \sigma^{-1}$$
4. $$[\kappa, \lambda_i] = 1 \quad 1 \leq i \leq n$$
5. $$[\sigma \kappa \sigma^{-1}, \lambda_i] = 1 \quad 1 \leq i \leq n$$
6. $$\sigma \lambda_n \cdots \lambda_1 \kappa^2 \tau = 1$$
Put $A := \lambda_n \ldots \lambda_1$. Eliminating $\sigma$ by (7), it is easily seen that (3) is redundant. Relation (2) becomes

\[(\Lambda \kappa^2 \tau \kappa^{-1})^2 = (\kappa^{-1} \Lambda \kappa^2 \tau)^2 \implies [\tau^{-1} \kappa \tau, A] = 1.\]

But this relation is a consequence of (4), so that (2) is also redundant. Since $\tau^{-1} \kappa \tau = \sigma \kappa \sigma^{-1}$ by (3), the relation (5) can be written as $[\tau^{-1} \kappa \tau, \lambda_i] = 1$. This gives the presentation (4) and proves Theorem 4.

5. The arrangement $C_n$

In order to compute the group, consider the model of $C_n$ shown in Fig. 11, where $Q$ is given by $x^2 + y^2 = 1$. Suppose that the second points of intersection of the lines $L_i$ with $Q$ lie above the $x$-axis. As in the previous cases, take an initial base

\[F_2 := [\kappa_1, \kappa_2, \lambda_1, \ldots, \lambda_n, \tau].\]

The relation induced by the branch point is $\kappa_1 = \kappa_2 = : \kappa$. The nodes of $C_n$ will give the relations $[\kappa, \lambda_i] = 1$ for $1 \leq i \leq n$, and one has

\[F_{n+2} = [\kappa, \lambda_1, \ldots, \lambda_n, \kappa, \tau].\]

One can simplify the computation of the monodromy around the complicated singular fiber as follows: Put $A := \lambda_n \ldots \lambda_1$. By the projective relation one has $\tau A \kappa^2 = 1 \implies \tau = \kappa^{-2} A^{-1}$. Hence, $[\kappa, \tau] = 1$. Since we also have $[\kappa, \lambda_i] = 1$ for $1 \leq i \leq n$, this means that when computing the monodromy around this fiber, one can ignore the branch $Q$. This leaves $n + 1$ branches intersecting transversally, and the induced relation is (see [16])

\[[\tau A, \lambda_i] = [\tau A, \tau] = 1,\]

and one has

\[F_{n+3} = [\kappa, \kappa, \ldots].\]

The last relation induced by the branch point yields the trivial relation $\kappa = \kappa$, as expected.

![Fig. 11.](image-url)
Eliminating $\tau$ shows that the relations (17) are redundant and gives the presentation
\[ \pi_1(\mathbb{P}^2 \setminus C_n) \cong \langle \kappa, \lambda_1, \ldots, \lambda_n | [\lambda_i, \kappa] = 1 \rangle. \]

Acknowledgements

This work was partially supported by the Emmy Noether Research Institute for Mathematics (center of the Minerva Foundation of Germany), the Excellency Center “Group Theoretic Methods in the Study of Algebraic Varieties” of the Israel Science Foundation, and EAGER (EU network, HPRN-CT-2009-00099).

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