IDENTIFICATION OF BOUNDARY CONDITIONS USING NATURAL FREQUENCIES IN CASE OF A RING MEMBRANE

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Abstract

The problem of finding boundary conditions for fastening of a ring membrane, which are inaccessible for direct observation from the natural frequencies of its flexural oscillations, is considered.

Two theorems on the uniqueness of this problem are proved, and a method for establishing the unknown conditions for fastening of the membrane to the walls is indicated.

An approximate formula for determining the unknown conditions is obtained, using first three natural frequencies. The method of approximate calculation of unknown boundary conditions, is explained with the help of an example.

Keywords: Boundary conditions, inverse spectral problem, membrane, natural frequencies, Plücker coordinates, Plücker relation.

1. Formulation of the Inverse Problem.

Let us give the mathematical formulation of the problem, before describing the method of its solution. The problem of the radial oscillations of membrane, is reduced after making a separation of variables, to the following eigenvalue problem (see \cite{5})

\[
\ddot{y}(r) + \frac{\dot{y}(r)}{r} + \lambda^2 y(r) = 0, \tag{1}
\]

\[
U_1(y) = k_1 \dot{y}(a) - k_2 y(a) = 0, \quad U_2(y) = k_3 \dot{y}(b) + k_4 y(b) = 0, \tag{2}
\]

where \( \lambda \) is the spectral parameter, \( a \) and \( b \) are the short and long radius of the ring membrane respectively.

Now we formulate the inverse of the eigenvalue problem:

\textbf{Problem 1}(inverse problem). \textit{It is required to find the unknown linear forms \( U_1(y) \) and \( U_2(y) \) from the eigenvalues of the problem (1)–(2).}

Let \( L(y, \lambda) = \ddot{y}(r) + \frac{\dot{y}(r)}{r} + \lambda^2 y(r), \ (U_1(y), U_2(y))^T = AX^T, \)

where \( A = \{a_{ij} | i = 1, 2; j = 1, 2, 3, 4\} \) is a matrix of order 2 \( \times \) 4 such that \( a_{11} = k_1, \ a_{12} = -k_2, \ a_{23} = k_3, \ a_{24} = k_4, \ a_{13} = a_{14} = a_{21} = a_{22} = 0, \) and

\( X(y) = (\dot{y}(a), y(a), \dot{y}(b), y(b)) \) is a row-vector. Then the eigenvalue problem (1)–(2) is equivalent to the following eigenvalue problem

\[
L(y, \lambda) = 0, \quad AX^T(y) = 0. \tag{3}
\]

We consider also another eigenvalue problem

\[
L(y, \lambda) = 0, \quad \widetilde{A} X^T(y) = 0. \tag{4}
\]
In these problems only the matrices of boundary conditions are different.

**Proposition 1.** If matrix $\tilde{A}$ is equivalent to matrix $A$, i.e. there exist non-singular matrix $S$ such that $A = S\tilde{A}$, then problem (4) is equivalent to problem (3).

The proof is trivial.

**Proposition 2.** Matrix $\tilde{A}$ is equivalent to matrix $A$, if and only if, their correspond minors of order 2 are equal, accurate to coefficient.

The proof is in [3].

In the projective geometry we define the Plücker coordinates of the set of equivalent matrices of order $m \times n$ to be any one of the equivalent $C_{(m+1)(n+1)}$-tuples of its minors of order $m$.

Now we shell give the following

**Definition 1.** Plücker coordinates of boundary conditions of the eigenvalue problem (3) are called any one of the equivalent 8-tuples of minors (of order 2) of its matrix $A$: $(\ldots, A_{ij}, \ldots)$.

From Proposition 1 and Proposition 2 we get

**Proposition 3.** If Plücker coordinates of boundary conditions of the eigenvalue problems (3) and (4) are identical, then problems (3) and (4) are also identical.

Then Problem 1 is equivalent to the following problem:

**Problem 2.** It is required to find Plücker coordinates of boundary conditions of the problem (3) from its eigenvalues, if its differential equation is known.

2. The Uniqueness of the Solution of the Inverse Problem.

Together with problem (3), let us consider problem (4).

**Theorem 1** (on the uniqueness of the solution of the inverse problem). Suppose the following conditions are satisfied:

$$\text{rank } A = \text{rank } \tilde{A} = 2. \quad (5)$$

If non-zero eigenvalues of problems (3) and (4) are identical, with account taken for their multiplicities, then problems (3) and (4) are also identical.

**Proof.** A general solution of the differential equation of the problem (3) is the function $y(r) = y(r, \lambda) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$, where standard notations for cylinder functions are used (see [2]).

The boundary conditions are used to determine the constants $C_1$ and $C_2$.

The following function is a characteristic determinant of the problem (3):

$$\Delta(\lambda) \equiv \begin{vmatrix} U_1(J_0(\lambda a)) & U_1(Y_0(\lambda a)) \\ U_2(J_0(\lambda b)) & U_2(Y_0(\lambda b)) \end{vmatrix}$$
\[ \Delta(\lambda) \equiv \det(AB^T), \] where \( B \) is a matrix of order \( 2 \times 4 \) such that \( b_{11} = J_0(\lambda a), b_{12} = J_0(\lambda b), b_{13} = J_0(\lambda a), b_{14} = J_0(\lambda b), b_{21} = Y_0(\lambda a), b_{22} = Y_0(\lambda a), b_{23} = Y_0(\lambda b), b_{24} = Y_0(\lambda b). \) The equation for frequencies, is obtained from the condition of the existence of a non-zero solution for \( C_i \). The latter solution exists if and only if, the determinant \( \Delta(\lambda) \) is equal to zero (see [5]).

Using the Binet-Cauchy formula, we obtain

\[ \Delta(\lambda) = \sum_{1 \leq i < j \leq 4} A_{ij} \cdot B_{ij}(\lambda) \equiv 0. \] (6)

It follows from asymptotic estimations for cylindrical functions

\[ J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \nu \pi/2 - \pi/4), \quad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin(z - \nu \pi/2 - \pi/4), \]

that \( \Delta(\lambda) \) is an entire function (see [4]). Hence, it follows from Hadamard’s factorization theorem (see [4]), that characteristic determinant \( \Delta(\lambda) \) of problem (3) and characteristic determinant \( \tilde{\Delta}(\lambda) \) of problem (4), are connected by the relation

\[ \Delta(\lambda) \equiv C \lambda^k \tilde{\Delta}(\lambda), \] (7)

where \( k \) is a certain non-negative integer and \( C \) is a certain non-zero constant. From this, we obtain the identity

\[ \sum_{1 \leq i < j \leq 4} A_{ij} \cdot B_{ij}(\lambda) \equiv C \lambda^k \sum_{1 \leq i < j \leq 4} \tilde{A}_{ij} \cdot B_{ij}(\lambda). \]

It follows from this, that

\[ \sum_{1 \leq i < j \leq 4} (A_{ij} \cdot B_{ij}(\lambda) - \tilde{A}_{ij} \cdot C \lambda^k B_{ij}(\lambda)) \equiv 0. \] (8)

Note, that number \( k \) in this identity is equal to zero. Actually, let us assume the opposite: \( k \neq 0 \). Using Maple, we get that the functions \( B_{ij}(\lambda) (1 \leq i < j \leq 4) \) and also the same functions multiplied by \( \lambda^k \), are linearly independent.

From this and identity (8), we obtain \( A_{ij} = \tilde{A}_{ij} = 0 (1 \leq i < j \leq 4) \), which contradict the condition (5) of the theorem.

Hence, \( k = 0 \) and

\[ \sum_{1 \leq i < j \leq 4} (A_{ij} - C \tilde{A}_{ij}) \cdot B_{ij}(\lambda) \equiv 0. \] (9)

From this, by virtue of the linearly independence of the corresponding functions, we obtain \( A_{ij} = C \tilde{A}_{ij} \), i.e. Plücker coordinates of boundary conditions of the eigenvalue problems (3) and (4) are identical.
Then, it follows from Proposition 3, that problems (3) and (4) are also identical. The theorem is proved.

3. **Exact Solution of The Inverse Problem and Its Stability.**

This section is concerned with solving this problem and constructing an exact solution of the inverse problem.

Suppose $\lambda_1$, $\lambda_2$, $\lambda_3$, are the eigenvalues of problem (3). We substitute the values $\lambda_i$, $(i = 1, 2, 3)$ into (6). Since $A_{12} = A_{34} = 0$, we obtain a system of three homogeneous algebraic equations in the four unknowns $A_{13}$, $A_{14}$, $A_{23}$, $A_{24}$:

$$F \cdot Z^T = 0,$$

where $F = \{f_{ij}| i = 1, 2; j = 1, 2, 3, 4\}$ is a matrix of order $3 \times 4$ such that $f_{i1} = B_{13}(\lambda_i)$, $f_{i2} = B_{14}(\lambda_i)$, $f_{i3} = B_{23}(\lambda_i)$, $a_{i4} = B_{24}(\lambda_i)$, $(i = 1, 2, 3)$, and $Z = (A_{13}, A_{14}, A_{23}, A_{24})$ is a row-vector.

The resulting system has an infinite set of solutions. It follows from the uniqueness theorem, which has been proved that the unknown minors $A_{13}$, $A_{14}$, $A_{23}$, $A_{24}$ can be found apart from a constant. Hence, the resulting system must have a rank of 3 and a solution, determined apart from a constant multiplier.

If the minors (Plücker coordinates) $A_{13}$, $A_{14}$, $A_{23}$, $A_{24}$ are found apart from a constant, then then Problem 2 is solved. We can reconstruct any one of the equivalent matrix of boundary conditions by its Plücker coordinates (see [3]).

For example, if $A_{13} = 1$, $A_{12} = A_{34} = 0$ then

$$A = \begin{bmatrix} 1 & A_{23} & 0 & 0 \\ 0 & 1 & 0 & A_{14} \end{bmatrix}. \quad (11)$$

This reasoning proves

**Theorem 2.** *If the matrix $F$ of system (10) has a rank of 3, then the solution of the inverse problem of the reconstruction boundary conditions (3), is unique.*

Note, that theorem 2 is stronger than theorem 1. Theorem 2 used only three natural frequencies for the reconstruction of the boundary conditions and not all natural frequencies as theorem 1, are used.

It is significant that we have the following theorem on stability of the solution:

**Theorem 3.** *Suppose that one of the third-order minors of matrix $F$ is substantially non-zero. If $|\tilde{\lambda}_i - \lambda_i| < \delta << 1$, then the boundary conditions of problem (3), are close to the boundary conditions of problem (4).*

The proof is analogous to the proof of the theorem on stability in [1].

4. **Approximate Solution.**

Since small errors are possible when measuring natural frequencies, the problem arises to find an algorithm for the approximate determination of the type of fastening from the natural frequencies, which are found with a certain error.
Let $A_{13}^o, A_{14}^o, A_{23}^o, A_{24}^o$ be solution of the system of homogeneous algebraic equations (10). It is unnecessary for these liquids to be totally immiscible.

Aside from measurement and calculation errors, it is unnecessary for the values $A_{13}^o, A_{14}^o, A_{23}^o, A_{24}^o$ to be minors of a matrix. So this problem is not trivial. We must find minors $A_{13}, A_{14}, A_{23}, A_{24}$, closer to the values $A_{13}^o, A_{14}^o, A_{23}^o, A_{24}^o$.

It is known from algebraic geometry [3] that the numbers $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}$ are minors of some $2 \times 4$ matrix, if and only if, the following condition is satisfied: $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0$. This condition is called the Plücker relation.

If the Plücker relation for these numbers is realized, then $A_{13}^o, A_{14}^o, A_{23}^o, A_{24}^o$ are minors of some matrix and corresponding boundary conditions are found.

If the Plücker relation for numbers $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}$ is not realized, the required minors $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}$ are found with the help of orthogonal projection. By definition, put $x_1 = A_{12}, x_2 = A_{34}, x_3 = A_{13}, x_4 = -A_{24}, x_5 = A_{14}, x_6 = A_{23}$. Using this definition, we get the Plücker relation

$$x_1x_2 + x_3x_4 + x_5x_6 = 0, \quad (12)$$

which characterizes a surface $S$ in the 6-dimensional space. By definition, put $y_1 = A_{12}, y_2 = A_{34}, y_3 = A_{13}, y_4 = -A_{24}, y_5 = A_{14}, y_6 = A_{23}$.

By definition, put

$$X = (x_1, x_2, x_3, x_4, x_5, x_6), \quad Y = (y_1, y_2, y_3, y_4, y_5, y_6),$$

$$X^* = (x_2, x_1, x_4, x_3, x_6, x_5), \quad Y^* = (y_2, y_1, y_4, y_3, y_6, y_5),$$

$$(\vec{X}, \vec{Y}) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6,$$

where $\vec{X} = \overrightarrow{OX}$, $\vec{Y} = \overrightarrow{OY}$, $O$ is the origin.

Let $X$ be an orthogonal projection of $Y$ on surface (12). The vector $X^*$ is normal for surface (12) at the point $X$. It is identical to the following equations

$$\vec{Y} = \vec{X} + p \vec{X}^*, \quad (13)$$

where $p$ is a real number. Having solved a set of linear equations (13), with the unknowns $x_1, x_2, x_3, x_4, x_5, x_6$, we obtain

$$\vec{X} = \frac{1}{1 - p^2}(\vec{Y} - p\vec{Y}^*). \quad (14)$$

From (15), it is easy to obtain (see [1])

$$p = \frac{(\vec{Y}, \vec{Y}) - \sqrt{(\vec{Y}, \vec{Y})^2 - (\vec{Y}, \vec{Y}^*)^2}}{(\vec{Y}, \vec{Y}^*)}. \quad (15)$$

5
5. Example.

If $\lambda_1 = 2.93$, $\lambda_2 = 6.16$, $\lambda_3 = 9.34$ corresponding to the first three natural frequencies $\omega_i$ determined, using instruments for measuring the natural frequencies, then the solution of system (10), apart from a constant, has the form $A_{i3}^0 = 1.00C$, $A_{14}^0 = 2.00C$, $A_{24}^2 = -2.00C$, $A_{23}^2 = -C$. Using (14) and (15), we get $A_{13} = 0.72C$, $A_{14} = 1.17C$, $A_{23} = -1.17C$, $A_{24} = -1.89C$. Suppose $C = 1/A_{13}$; then from (11), we obtain

$$A = \begin{bmatrix} 1 & -1.62 & 0 & 0 \\ 0 & 0 & 1 & 1.62 \end{bmatrix}.$$

Note that the numbers $\lambda_1 = 2.93$, $\lambda_2 = 6.16$, $\lambda_3 = 9.34$ presented above are almost the same as the first three exact values, that correspond to a case with

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This means that the unknown fixing of the membrane, which is inaccessible for direct observation, has been satisfactorily determined. The errors of determination of boundary conditions, are stipulated with round-off errors of calculations.

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