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We need a model.

Landau & Vishkin 1994
developed a model for
discretizing the digitization process.

Fredriksson & Ukkonen 1998
developed a similar
Geometric Model
for discrete rotations.
Fig. 1. The text grid and pixel centers of a 7 x 7 text.
**PROPOSED SOLUTION:**

Construct all possible rotated patterns.

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5a  5b  5c

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5d  5e  5f

**Fig. 5.** An example of some possible 2-dimensional arrays that represent one pattern. Fig 5a - the original pattern. Figures 5b-d are computed in the "pattern over the text" model. Fig 5b - a representation of the pattern rotated by 19°. Fig 5c - Pattern rotated by 21°. Fig 5d - Pattern rotated by 20°. Figures 5e-f are computed in the "pattern under the text" model. Fig 5e - Pattern rotated by 17°. Fig 5f - Pattern rotated by 26°.
Every rotated pattern can be found in the text using FFT in time:

\[ O(n^2 \log m) \]

If there are \( N \) rotated patterns the total time is:

\[ O(N n^2 \log m) \]

WHAT IS \( N \)?
UPPER BOUND

$m^2$ pixels

Each pixel center crosses at most $4m$ grid lines.

$\Rightarrow \bigO(m^3)$ different rotated patterns.
Could many points cross a gridline together?

We will show:

**Lower Bound:** $\Omega(m^3)$

**Restriction:**

Set $P =$

1) Points in Quadrant I

2) Points $(x, y)$ where $x$ and $y$ are co-prime.
   \[
   (\gcd(x, y) = 1)
   \]
WE WILL SHOW: \( \forall X_1, X_2 \in \mathbb{P} \)

It is impossible that \( X_1 \) and \( X_2 \) cross grid line at same angle.

How does it help?

\[
\| P \| = \frac{6m^2}{\pi^2} + o(m \log m)
\]

(\textit{Geometry Thm}).

Consider:

\[
P_\frac{m}{4} = \left\{ (x,y) \mid (x,y) \in \mathbb{P}, \quad 0 \leq x, y \leq \frac{m}{4} \right\}
\]
Schematically: $P \cap$ shaded area

In shaded area: $\frac{m^2}{16}$ points.
So in $P - P \leq \frac{m}{4}$, at least

$$\frac{6m^2}{\pi^2} - \frac{m^2}{16}$$
points, i.e.

$$\frac{96 - \pi^2}{16 \pi^2} m^2 = \Theta(m^2)$$
points.
Each of the $\Theta(m^2)$ points in $P_1P_2\leq\frac{m^3}{r}$ crosses the grid $\Omega(m)$ times, and no two of them cross together.

Conclude: There are $\Omega(m^3)$ different rotated patterns.

Left to show:
Lemma: $\forall x_1, x_2 \in \mathcal{P}$

It is impossible for them to cross a grid line at the same angle.

Proof:
We discuss the case where $x_1$ crosses horizontal grid line to $x_2$, and $x_2$ crosses horizontal grid line to $x_1$.

(Other cases, both crossing vertical or one vertical & one horizontal, are similar.)

Let $X_i = (c, s)$, $Y_i = (c', s')$.
$c, s$ are odd.

$$c = 2k_1 + 1$$
$$s = 2k_2 + 1$$
$S'$ is even \[ S' = 2l_1 \]

\[ c^2 + s^2 = r^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 \]

\[ c'^2 + s'^2 = r^2 \]

\[ c'^2 + 4l_1^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 \]

\[ c'^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 - 4l_1^2 = 4(k_1^2 + k_1 + k_2^2 + k_2l_1^2) + 2 \]

So \[ c'^2 \] is even. \[ c'^2 = 2l_2 \]

\[ 2l_2 = 4(k_1^2 + k_1 + k_2^2 + k_2l_1^2) + 2 \]

\[ l_2 = 2(k_1^2 + k_1 + k_2^2 + k_2l_1^2) + 1 \]
Conclude:

\[ c' = \sqrt{2} l_2 \quad \text{where } l_2 \text{ is odd} \]

\[ s' = 2 l_1 \]

We can say even more:

\[ c' = m \sqrt{2} m \]

where \( n \in \mathbb{Z}^+ \)

and \( m \) is a square-free odd number

(does not have a square factor)

i.e. \( c' \) is an irrational number.
Fig. 8. Points $X_1$ and $X_2$ each have coprime integer coordinates. Orbits of them under rotation around point $O$ cross an horizontal line at points $Y_1$ and $Y_2$ respectively. Then, $\angle X_1OX_2 \neq \angle Y_1OX_2$ by Claim 3.
CLAIM: $\angle x_1 o y_1 \neq \angle x_2 o y_2$.

Proof: We show $\angle x_1 o x_2 \neq \angle y_1 o y_2$.

$x_1 = (c_1, s_1)$  \hspace{1cm} $y_1 = (c_1', s_1')$

$x_2 = (c_2, s_2)$  \hspace{1cm} $y_2 = (c_2', s_2')$

$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

\[
\sin \angle x_1 o x_2 = \frac{c_2 s_1 - s_2 c_1}{r_2 r_1}
\]

Similarly:

$\sin \angle y_1 o y_2 = \frac{s_1' c_2' - s_2' c_1'}{r_1 r_2}$
If $\angle x, ox_2 = \angle y, oy_2$ then
\[
\sin \angle x, ox_2 = \sin \angle y, oy_2 \quad \text{i.e.}
\]
\[
s_1c_2 - s_2c_1 = s'_1c'_2 - s'_2c'_1
\]
\[
m \in \mathbb{Z}
\]

What do we know?
\[
s'_1, s'_2 \in 2\mathbb{Z}
\]
\[
c'_1 = m_1\sqrt{2m_1}
\]
\[
c'_2 = n_2\sqrt{2m_2}
\]

where $c'_1, c'_2$ irrational.

We have:
\[
a\sqrt{2m_1} + b\sqrt{2m_2} \in \mathbb{Z}, \text{ where } a, b \in \mathbb{Z}
\]

When can this happen?
Options:

1) \( m_1 \neq m_2 \)

Can not happen since \( a, b \in \mathbb{Z} \) and \( \sqrt{2m_1}, \sqrt{2m_2} \) are linearly independent in \( \mathbb{Z} [\sqrt{2m_1}, \sqrt{2m_2}] \).

2) \( m_1 = m_2 \)

\[ a \sqrt{2m_1} + b \sqrt{2m_1} \in \mathbb{Z} \]
\[ \sqrt{2m_1} (a + b) \in \mathbb{Z} \quad \text{iff} \quad a = -b \]

\[ a \sqrt{2m_1} + b \sqrt{2m_1} = 0 \]
\[ \Rightarrow \]
\[ S_1 C_2 - S_2 C_1 \]
When can \[ S_1 C_2 = S_2 C_1 \]?

\[
\frac{S_1}{C_1} = \frac{S_2}{C_2}
\]

Since \((c, s)\) are relatively prime, this can only happen if

\[ S_1 = S_2 \quad \text{and} \quad C_1 = C_2 \]

i.e. \[ X_1 = X_2. \]
WHAT IS LEFT?

Faster rotation.
Approximate rotation.
Approximate scaling.
Integration.

We have come a long way
but have a lot longer way
to go...