Proof:

⇒ Obvious.

⇐ Assume all generated 4-leaf subtrees are homeomorphic.
Prove: 2 trees are homeomorphic.

Def. Let \( u \) be a node of degree \( \geq 2 \). We call \( u \) a super-2 node. Two nodes \( a, b \) are twins if in the path from \( a \) to \( b \) are at most one internal node which is a super-2 node, and \( a, b \) are leaves.
Claim: A tree with at least 2 leaves and a super 2 node has a pair of twins.

Proof: Let $T$ be a tree with at least 2 nodes. Consider $a, b$ such that the path from $a$ to $b$ has the largest number of internal super 2 nodes. Let $v$ be the closest super 2 node to $a$.

$v$ is super-2 so $3$ leaf $c$ such that there is a path from $v$ to $c$. 
It is clear that there are no super-2 nodes on the path from $v$ to $c$ otherwise the path from $c$ to $b$ has more super-2 nodes than the path from $a$ to $b$, contradicting its maximality.

So $a$ and $c$ are twins.

Now we prove the theorem by induction on the number of leaves $n$.

**Base Case:** $n \leq 4$. Theorem true immediately.
Ind. hyp. Thm true for trees with $< n$ leaves. Prove for $n \ (n > y)$.

Let $x, y$ be twins in tree $T_1$ and let $v$ be super-2 node in path.

Claim: $x$ & $y$ are twins in $T_2$.

Proof: If $x, y$ not twins in $T_2$ the situation in $T_2$ is:

In $T_1$ the situation is:
Consider the tree induced by $x, y, p, q$.

In $T_2$:

```
    y
   /\
  p  q
```

Homeomorphic image: $T'_2$

In $T_1$: Either

```
    y
   /\
  p  q
```

or:

```
    y
   /\
  p  q
```

$T'_1 \neq T'_2$. Contradiction.
Return to proof of thm.
Let \( x, y \) be twins in \( T_1 \) and \( T_2 \).
For reasons similar to previous claim, either the super-2, node \( w \) on the path from \( x \) to \( y \) has degree 3 in both \( T_1, T_2 \) or degree \( \geq 3 \) in both \( T_1, T_2 \).

Cases: 1)

\[ \begin{array}{c}
  \begin{array}{c}
    x \\
    w \\
    y
  \end{array} \\
  \begin{array}{c}
    \\ \\
    \\ \\
    \\ \\
  \end{array}
\end{array} \quad \begin{array}{c}
  \begin{array}{c}
    w \\
    x \\
    y
  \end{array} \\
  \begin{array}{c}
    \\ \\
    \\ \\
    \\ \\
  \end{array}
\end{array} \]

Delete \( x, y \) & the path from them to \( w \).
By ind. remaining trees homeomorphic.

Adding \( x, y \) to both does not change homeomorphism.
Delete $y$ and its path to $w$.

By induction remaining trees homeomorphic.

Now add $y$ and its path.